Numerical convergence and stability of mixed formulation with X-FEM cut-off

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RÉSUMÉ. *On s'interresse dans ce papier à l'analyse mathématique et numérique de la convergence et de la stabilité de la formulation mixte d'un problème d'élasticité incompressible dans un domaine fissuré. L'objectif est d'étendre l'étude faite sur la variante X-FEM cut-off, dans le cas de l'élasticité compressible, au comportement incompressible. Une preuve mathématique de la condition inf-sup de la formulation mixte discrète avec X-FEM est établie pour certains champs enrichis. Nous donnons également un résultat mathématique de la quasi-optimalité de l'estimation d'erreur. Enfin, nous validons ces résultats avec des tests numériques.*

ABSTRACT. *In this paper we are concerned with the mathematical and numerical analysis of convergence and stability of the mixed formulation for incompressible elasticity in cracked domains. The objective is to extend the X-FEM cut-off analysis done in the case of compressible elasticity to the incompressible one. A mathematical proof of the inf-sup condition of the discrete mixed formulation with X-FEM is established for some enriched fields. We also give a mathematical result of quasi-optimal error estimate. Finally, we validate these results with numerical tests.*

MOTS-CLÉS : *X-FEM cut-off, Formulation mixte, Élasticité lineaire, Estimation d'erreur.*

KEYWORDS: *X-FEM cut-off, Mixed formulation, Linear elasticity, Error estimate.*

1. Introduction

The presence of a crack in a structure reveals two types of discontinuities : a strong discontinuity that requires an adapted mesh to the shape of the crack, hence the domain is meshed at each time step ; and a weak discontinuity that requires refinement at the crack tip. These two operations lead to a huge computational cost. In order to overcome these difficulties we use the eXtended Finite Element Method (X-FEM). This method allows to model cracks, material inclusions and holes on nonconforming meshes. It was introduced by Moës et al. (Moës *et al.*, 1999). It consists in enriching the basis of the classical finite element method by a step function along the crack line and by some non-smooth functions representing the asymptotic displacement around the crack tip. To obtain an optimal accuracy, Chahine et al. introduced a new enrichment strategy (Chahine *et al.*, 2008) : the so called X-FEM cut-off. This enrichment strategy uses a cut-off function to locate the crack tip surface. In their work, Chahine et al. have shown that the X-FEM cut-off has an optimal convergence rate of order h and that the conditioning of the stiffness matrix does not deteriorate. In this work, we extend the numerical results given by Chahine et al. (Chahine *et al.*, 2008) to an incompressible isotropic linear plane elasticity problem in fracture mechanics. In particular, this formulation must satisfy the so-called inf-sup or "Ladyzhenskaya-Brezzi-Babuška condition" (LBB) condition.

2. Model problem and discretization

Let Ω be a two-dimensional cracked domain, Γ_c denotes the crack and Γ the boundary of Ω. We assume that $\Gamma \setminus \Gamma_c$ is partitioned into two parts : Γ_N where a Neumann surface force $\mathbf{t}$ is applied and Γ_D where a Dirichlet condition $\mathbf{u} = \mathbf{0}$ is prescribed (see Fig. 1). We assume that we have a traction-free condition on Γ_c. Let $\mathbf{f}$ be the body force applied on Ω. The equilibrium equation, constitutive law and boundary conditions are given by

$$-\operatorname{div} \sigma(\mathbf{u}) = \mathbf{f}, \quad \text{in } \Omega, \qquad [1]$$

$$\sigma(\mathbf{u}) = \lambda \operatorname{tr} \varepsilon(\mathbf{u}) I + 2\mu\, \varepsilon(\mathbf{u}), \quad \text{in } \Omega, \qquad [2]$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_D, \qquad [3]$$

$$\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}, \quad \text{on } \Gamma_N, \qquad [4]$$

$$\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0}, \quad \text{on } \Gamma_c. \qquad [5]$$

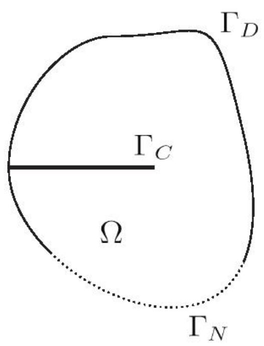


Figure 1. *Cracked domain*

with $\varepsilon(\mathbf{u}) = \dfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ and $\mathbf{n}$ is the outside normal to the domain Ω.
Let $V = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \text{with } \mathbf{u} = 0 \text{ on } \Gamma_D \right\}$, $Q = \mathbf{L}^2(\Omega)$, σ^d the deviatoric part of σ and p the hydrostatic pressure. By a classical way we find the weak mixed formulation (Brezzi *et al.*, New York, 1991)

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in (V, Q) \text{ such that :} \\ a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = 0, & \forall q \in Q, \end{cases} \qquad [6]$$

with $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sigma^d(\mathbf{u}) : \varepsilon(\mathbf{v})\, d\Omega,\ \ b(\mathbf{v}, p) = \int_\Omega p \operatorname{div} \mathbf{v}\, d\Omega,\ \ L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v}\, d\Gamma$. Discretization of the elasticity problem follows the usual steps. Let τ_h an affine mesh of the non cracked domain $\overline{\Omega}$. We approximate $(\mathbf{u}, p)$ by $(\mathbf{u}_h, p_h) \in V_h \times Q_h$. The subspaces V_h and Q_h are finite dimensional spaces that will be defined later. The discretized problem is then :

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in (V_h, Q_h) \text{ such that} \\ a(\mathbf{u}_h, v_h) - b(\mathbf{v}_h, p_h) = L(\mathbf{v}_h), & \forall \mathbf{v}_h \in V_h, \\ b(\mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \qquad [7]$$

The existence of a stable finite element approximate solution $(\mathbf{u}_h, p_h)$ depends on choosing a pair of spaces V_h and Q_h such that the following LBB condition holds :

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, q_h)}{\| q_h \|_{0,\Omega} \| \mathbf{v}_h \|_{1,\Omega}} \geq \beta_0,$$

where $\beta_0 > 0$ is independent of h (Brezzi *et al.*, New York, 1991). The satisfaction of this condition for a couple (V_h, Q_h) is very difficult to prove in practical situations. Therefore, the numerical evaluation of the inf-sup has been widely used (Chapelle *et al.*, 1993). It gives an indication of the verification of the LBB condition for a given finite element discretization.

3. X-FEM cut off approximation spaces

The idea of X-FEM is to use a classical finite element space enriched by some additional functions. These functions result from the product of global enrichment functions and some classical finite element functions. we consider the variant of X-FEM which uses a cut-off function to define the singular enrichment surface. The classical enrichment strategy for this problem is to use the asymptotic expansion of the displacement and pressure fields at the crack tip area. Indeed, the displacement is enriched by the Westergaard functions :

$$F^u = \left\{F_j^u(x), 1 \leq j \leq 4\right\} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\},$$

where (r, θ) are polar coordinates around the crack's tip. These functions allow to generate the asymptotic non-smooth function at the crack's tip (Laborde *et al.*, 2005). For the pressure, the asymptotic expansion at the crack tip is given by $p(r, \theta) = \frac{2K_I}{3\sqrt{2\pi r}} \cos\frac{\theta}{2} + \frac{2K_{II}}{3\sqrt{2\pi r}} \sin\frac{\theta}{2}$ where K_I and K_{II} are the stress intensity factors. This expression is used to obtain the basis of enrichment of the pressure in the area of the crack's tip (Legrain *et al.*, 2008) :

$$F^p = \left\{F_j^p(x), 1 \leq j \leq 2\right\} = \left\{\frac{1}{\sqrt{r}}\cos\frac{\theta}{2}; \frac{1}{\sqrt{r}}\sin\frac{\theta}{2}\right\}.$$

The displacement and pressure are also enriched with a Heaviside-type function at the nodes for which the support of their shape functions is totally cut by the crack. Using this enrichment strategy, the discretisation spaces V_h and Q_h take the following forms :

$$V_h = \left\{ \mathbf{v}_h = \sum_{i \in I} \alpha_k \boldsymbol{\psi}_{u,k} + \sum_{i \in I_H} \beta_k H \boldsymbol{\psi}_{u,k} + \sum_{j=1}^{4} \gamma_j F_j^u \chi; \quad \alpha_k, \beta_k, \gamma_j \in \mathbb{R} \right\},$$

$$Q_h = \left\{ p_h = \sum_{i \in I} p_i \, \varphi_{p,i} + \sum_{i \in I_H} b_i^p H \varphi_{p,i} + \sum_{j=1}^{2} c_j^p F_j^p \chi; \quad p_i, b_i^p, c_j^p \in \mathbb{R} \right\},$$

with I the set of node indices of τ_h, I_H the set of node indices of τ_h for which the supports of their shape functions are totally cut by the crack, $\varphi_{u,i}$ (resp. $\varphi_{p,i}$) are the scalar schape functions for displacement (resp. for pressure), $\boldsymbol{\psi}_{u,k}$ are the vector shape functions defined by $\boldsymbol{\psi}_{u,k} = \begin{cases} \begin{pmatrix} \varphi_{u,i} \\ 0 \end{pmatrix} & \text{if } i = \dfrac{k+1}{2} \\ \begin{pmatrix} 0 \\ \varphi_{u,i} \end{pmatrix} & \text{if } i = \dfrac{k}{2} \end{cases}$ and χ is a $\mathscr{C}^1$-piecewise function which is polynomial of degree 3 in the annular region $r_0 \leq r \leq r_1$, and satisfies $\chi(r) = 1$ if $r < r_0$ and $\chi(r) = 0$ if $r > r_1$. In our case we take $\chi(r) = \dfrac{2r^3 - 3(r_0 + r_1)r^2 + 6r_1 r_0 r + (r_0 - 3r_1)r_0^2}{(r_0 - r_1)^3}$ if $r_0 \leq r \leq r_1$ with $r_0 = 0.01$ and $r_1 = 0.49$.

4. Proof of inf-sup condition and error analysis

In this section we prove that the LBB condition holds for the P_2/P_0 element without the singular enrichment of the pressure. In order to simplify the presentation we assume that the crack cuts the mesh far enough from the vertices. We use a general technique introduced by Brezzi and Fortin in their book (Brezzi *et al.*, New York, 1991).

4.1. *Construction of a H_1-stable interpolation operator*

The proof of the LBB condition requires the definition of an interpolation operator adapted to the proposed method. Since the displacement field is discontinuous across the crack on Ω, we divide Ω into Ω_1 and Ω_2 according to the crack and a straight extension of it (Fig. 2). Let $\mathbf{u}^k$ be the restriction of $\mathbf{u}$ to Ω_k, $k \in \{1, 2\}$. As $\mathbf{u} \in \mathbf{H}^1(\Omega)$ then there exists an extension $\tilde{\mathbf{u}}^k$ in $\mathbf{H}^1(\Omega)$ of $\mathbf{u}^k$ across the crack on Ω such that :

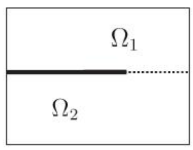


Figure 2. *Domain decomposition*

$$\| \tilde{\mathbf{u}}^k \|_{1,\Omega} \leq C_k \| \mathbf{u}^k \|_{1,\Omega_k}, \qquad [8]$$

where C_k is independent of $\mathbf{u}$ (Adams, 1975).

Definition 1. *Given a displacement field* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and two extensions* $\tilde{\mathbf{u}}^1$ *and* $\tilde{\mathbf{u}}^2$ *of* $\mathbf{u}^1$ *and* $\mathbf{u}^2$ *in* $\mathbf{H}_0^1(\Omega)$*, respectively, we define* $\Pi_1\mathbf{u}$ *as the element of* V_h *such that :*

$$\Pi_1\mathbf{u} = \sum_{j\in I\setminus I_H} \boldsymbol{\alpha}_j\varphi_j + \sum_{j\in I_H} \big[\boldsymbol{\beta}_j\varphi_j H_1 + \boldsymbol{\gamma}_i\varphi_j H_2\big], \qquad [9]$$

with

$$H_1(\mathbf{x}) = \begin{cases} 1 & \textit{if} \;\; \mathbf{x}\in\Omega_1, \\ 0 & \textit{if} \;\; \mathbf{x}\in\Omega_2, \end{cases} \qquad H_2(\mathbf{x}) = 1 - H_1(\mathbf{x}),$$

$$\boldsymbol{\alpha}_i = \frac{1}{|\,\Delta_i\,|}\int_{\Delta_i} \tilde{\mathbf{u}}^k \mathrm{d}x \;\; \text{if}\, \mathbf{x}_\mathrm{i}\in\Omega_\mathrm{k}, \quad \boldsymbol{\beta}_\mathrm{i} = \frac{1}{|\,\Delta_\mathrm{i}\,|}\int_{\Delta_\mathrm{i}} \tilde{\mathbf{u}}^1 \mathrm{dx},$$

$$\boldsymbol{\gamma}_i = \frac{1}{|\,\Delta_i\,|}\int_{\Delta_i} \tilde{\mathbf{u}}^2 \mathrm{d}x, \;\; S_j := \bigcup\{S\in\tau_h : \mathrm{supp}(\varphi_j)\cap S \neq \emptyset\},$$

where Δ_j *is the maximal ball centered at* x_j *such that* $\Delta_j \subset S_j$ *and* $\{\mathbf{x}_j\}_{j=1}^J$ *are the interior nodes of mesh* τ_h*.*

This definition is inspired by the work of Chen and Nochetto (Chen *et al.*, 2000).

Lemma 1. *The interpolation operator defined by [9] satisfies* $\forall\mathbf{u}\in\mathbf{H}_0^1(\Omega)$

$$\|\, \Pi_1\mathbf{u} \,\|_{1,\Omega} \leqslant C \,\|\, \mathbf{u} \,\|_{1,\Omega}, \qquad [10]$$

$$\|\, \mathbf{u} - \Pi_1\mathbf{u} \,\|_{r,\Omega} \leqslant Ch^{1-r} \;\; \|\, \mathbf{u} \,\|_{1,\Omega}, \quad r = \{0,1\}. \qquad [11]$$

Proof : In the proof we take $i\in\{1,2\}$, $k = 3-i$ and $\tilde{s}$ the union of all elements surrounding the elment s of τ_h.

In order to prove this Lemma, we calculate the above estimates locally on every different type of triangles : non-enriched triangles, triangles cut by the straight extension of the crack, triangles partially enriched by the discontinuous functions, triangles containing the crack tip and triangles totally enriched by the discontinuous functions. Before, let us establish the following intermediary result :

Lemma 2. *Let* δ *be a square of size* h *centered at the crack tip (see Fig. 3) and* $f \in H_0^1(\delta\setminus\Gamma_c)$ *with* $f(\boldsymbol{x}) = 0,\ \forall x \in\ \tilde{\Gamma}_c\bigcap\delta$ *(where* $\tilde{\Gamma}_c$ *is the extension of the crack* Γ_c*). Then, there exists* $c > 0$*, independent of* h *such that :*

$$\|\, f \,\|_{0,\delta} \leq ch \,\|\, \nabla f \,\|_{0,\delta\setminus\Gamma_c}\,. \qquad [12]$$

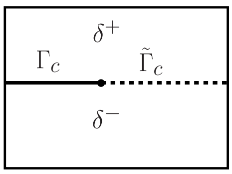
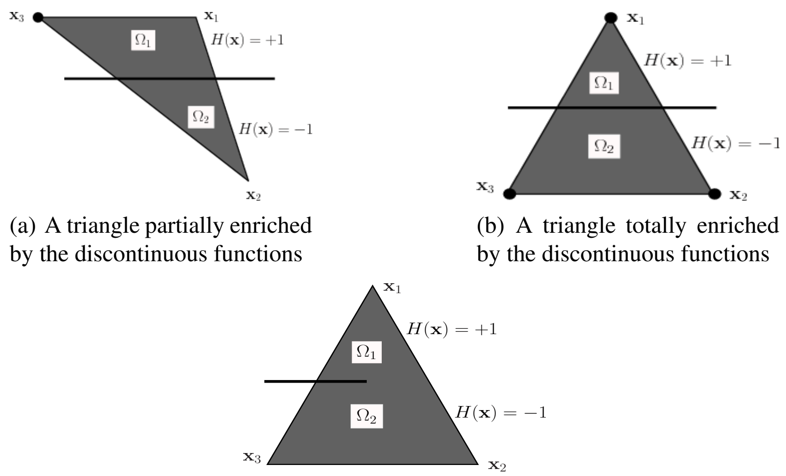


Figure 3. *Centered domain on the crack tip*



(a) A triangle partially enriched by the discontinuous functions

(b) A triangle totally enriched by the discontinuous functions

(c) A triangle containing the crack tip

Figure 4. *Triangles partially and totally enriched by the discontinuous functions*

Proof : Dividing the square into two parts δ^+ (above the crack) and δ^- (below the crack). Let $\hat{f^+} = f \circ T_k$ defined on the reference rectangle $\hat{\delta^+}$ (assumed of size 1) obtained by an affine transformation T_K of the rectangle δ^+. Then, by construction, $\hat{f^+}(\boldsymbol{x}) = 0 \ \forall \boldsymbol{x} \in \{x_1 \geq 0\} \times \{x_2 = 0\}$ which implies the following Poincaré inequality :

$$\| \hat{f^+} \|_{0,\hat{\delta^+}} \leq c \, \| \nabla \hat{f^+} \|_{0,\hat{\delta^+}} \ . \qquad [13]$$

Using inequality [13] and the fact that the mesh is affine we obtain :

$$\begin{aligned} \| f \|_{0,\delta^+} &\leq c \mid \det(J_K) \mid^{1/2} \| \hat{f^+} \|_{0,\hat{\delta^+}} \leq c \mid \det(J_K) \mid^{1/2} \| \nabla \hat{f^+} \|_{0,\hat{\delta^+}} \\ &\leq c \mid \det(J_K) \mid^{-1/2} \| J_K \|_2 \mid \det(J_K) \mid^{1/2} \mid f \mid_{1,\delta^+} \leq c \, h \ \mid f \mid_{1,\delta^+} \end{aligned}$$

where $| \cdot |_{1,\delta^+}$ the $\mathbf{H}^1$ semi-norm on δ^+. Thus

$$\| f \|_{0,\delta^+} \leq c \, h \ \mid f \mid_{1,\delta^+}, \qquad [14]$$

Similarly we prove the same result for δ^- which finish the proof of Lemma 2.

Non-enriched triangles :

Let s be a non-enriched triangle in Ω_i. In this case we have $\Pi_1 \mathbf{u} = \Pi_1 \tilde{\mathbf{u}}^i$ on Ω_i. Because $\tilde{\mathbf{u}}^i$ is continuous over Ω this operator is equivalent to the classical operator of Chen and Nochetto (Chen *et al.*, 2000). Then we have

$$\| \Pi_1 \mathbf{u} \|_{1,s} = \| \Pi_1 \tilde{\mathbf{u}}^i \|_{1,s} \leqslant c \, \| \tilde{\mathbf{u}}^i \|_{1,\tilde{s}} \qquad [15]$$

and

$$\| \mathbf{u} - \Pi_1 \mathbf{u} \|_{r,s} = \| \mathbf{u}^i - \Pi_1 \tilde{\mathbf{u}}^i \|_{r,s} = \| \tilde{\mathbf{u}}^i - \Pi_1 \tilde{\mathbf{u}}^i \|_{r,s} \leqslant ch^{1-r} \| \tilde{\mathbf{u}}^i \|_{1,\tilde{s}}, \quad [16]$$

Triangles cut by the straight extension of the crack or containing the crack tip : Let s be a triangle cut by the straight extension of the crack or containing the crack tip (see Fig. 4(c)). Then $\Pi_1 \mathbf{u} = \boldsymbol{\alpha}_1 \varphi_1 + \boldsymbol{\alpha}_2 \varphi_2 + \boldsymbol{\alpha}_3 \varphi_3$ on s, with : $\boldsymbol{\alpha}_1 = \dfrac{1}{\mid \Delta_1 \mid} \int_{\Delta_1} \tilde{\mathbf{u}}^1 \, \mathrm{d}x$, $\boldsymbol{\alpha}_2 = \dfrac{1}{\mid \Delta_2 \mid} \int_{\Delta_2} \tilde{\mathbf{u}}^2 \, \mathrm{d}x$ and $\boldsymbol{\alpha}_3 = \dfrac{1}{\mid \Delta_3 \mid} \int_{\Delta_3} \tilde{\mathbf{u}}^2 \, \mathrm{d}x$.
We remark that :

$$\Pi_1 \mathbf{u} = \tilde{\boldsymbol{\alpha}}_1 \varphi_1 + \boldsymbol{\alpha}_2 \varphi_2 + \boldsymbol{\alpha}_3 \varphi_3 + (\boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1)\varphi_1 = \Pi_1 \tilde{\mathbf{u}}^2 + (\boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1)\varphi_1,$$

with $\tilde{\boldsymbol{\alpha}}_1 = \dfrac{1}{\mid \Delta_1 \mid} \int_{\Delta_1} \tilde{\mathbf{u}}^2 \, \mathrm{d}x$. By the triangle inequality, we may write

$$\| \Pi_1 \mathbf{u} \|_{1,s} \leqslant \| \Pi_1 \tilde{\mathbf{u}}^2 \|_{1,s} + \mid \boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1 \mid \| \varphi_1 \|_{1,s},$$

$$\begin{aligned} \| \mathbf{u} - \Pi_1 \mathbf{u} \|_{r,s} &\leqslant \| \mathbf{u} - \Pi_1 \tilde{\mathbf{u}}^2 \|_{r,s} + \mid \boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1 \mid \| \varphi_1 \|_{r,s} \\ &\leqslant \| \mathbf{u} - \tilde{\mathbf{u}}^2 \|_{r,s} + \| \tilde{\mathbf{u}}^2 - \Pi_1 \tilde{\mathbf{u}}^2 \|_{r,s} + \mid \boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1 \mid \| \varphi_1 \|_{r,s}, \end{aligned}$$

where $\| \varphi_1 \|_{r,s} \leqslant ch^{1-r}$ because φ_1 is the piecewise P_1 basis function,

$\| \Pi_1 \tilde{\mathbf{u}}^2 \|_{r,s} \leqslant ch^{1-r} \| \tilde{\mathbf{u}}^2 \|_{1,\tilde{s}}$ Because $\tilde{\mathbf{u}}^k$ is continuous over Ω ,

$\| \mathbf{u} - \tilde{\mathbf{u}}^2 \|_{0,s} \leqslant c \, h \, \| \mathbf{u} - \tilde{\mathbf{u}}^2 \|_{1,\delta},$

and if we use Cauchy-Schwartz inequality and Lemma 2 we obtain

$$\mid \boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1 \mid \leqslant \frac{h}{\mid \Delta_1 \mid} \| \tilde{\mathbf{u}}^1 - \tilde{\mathbf{u}}^2 \|_{0,\Delta_1} \leqslant c \frac{h^2}{\mid \Delta_1 \mid} \| \nabla(\tilde{\mathbf{u}}^1 - \tilde{\mathbf{u}}^2) \|_{0,\delta} \, .$$

Therefore

$$\| \Pi_1 \mathbf{u} \|_{1,s} \leqslant c\big(\| \tilde{\mathbf{u}}^2 \|_{1,\tilde{s}} + \| \tilde{\mathbf{u}}^1 - \tilde{\mathbf{u}}^2 \|_{1,\delta} \big) \quad [17]$$

$$\| \mathbf{u} - \Pi_1 \mathbf{u} \|_{r,s} \leqslant ch^{1-r} \big(\| \mathbf{u} - \tilde{\mathbf{u}}^2 \|_{1,s} + \| \tilde{\mathbf{u}}^2 \|_{1,\tilde{s}} + \| \tilde{\mathbf{u}}^1 - \tilde{\mathbf{u}}^2 \|_{1,\delta} \big), \quad [18]$$

Triangles partially enriched by the discontinuous functions : Let s be a triangle partially enriched by the discontinuous functions (see Fig. 4(a)). In this case we have $\Pi_1 \mathbf{u} = \Pi_1 \tilde{\mathbf{u}}_1 + (\boldsymbol{\alpha}_2 - \tilde{\boldsymbol{\alpha}}_2)\varphi_2$ on $s \cap \Omega_1$ and $\Pi_1 \mathbf{u} = \Pi_1 \tilde{\mathbf{u}}_2 + (\boldsymbol{\alpha}_1 - \tilde{\boldsymbol{\alpha}}_1)\varphi_1$ on $s \cap \Omega_2$ with $\tilde{\boldsymbol{\alpha}}_1 = \dfrac{1}{\mid \Delta_1 \mid} \int_{\Delta_1} \tilde{\mathbf{u}}^2 \, \mathrm{d}x$ and $\tilde{\boldsymbol{\alpha}}_2 = \dfrac{1}{\mid \Delta_2 \mid} \int_{\Delta_2} \tilde{\mathbf{u}}^1 \, \mathrm{d}x$.
In the same manner we prove that

$$\| \Pi_1 \mathbf{u} \|_{1,s\cap\Omega_i} \leqslant c \, \big(\| \tilde{\mathbf{u}}_i \|_{1,\tilde{s}} + \| \tilde{\mathbf{u}}^k - \tilde{\mathbf{u}}^i \|_{1,\delta} \big), \quad [19]$$

$$\| \mathbf{u} - \Pi_1 \mathbf{u} \|_{r,s\cap\Omega_i} \leqslant ch^{1-r} \big(\| \tilde{\mathbf{u}}^i \|_{1,\tilde{s}} + \| \tilde{\mathbf{u}}^k - \tilde{\mathbf{u}}^i \|_{1,\delta} \big). \quad [20]$$

Triangles totally enriched by the discontinuous functions
Let s be the triangle totally enriched by the discontinuous functions (see Fig. 4(b)). In this case we have : $\Pi_1 \mathbf{u} = \Pi_1 \tilde{\mathbf{u}}^i$ on $s \cap \Omega_i$. Then we have

$$\| \Pi_1 \mathbf{u} \|_{1,s\cap\Omega_i} \leqslant \| \Pi_1 \tilde{\mathbf{u}}^i \|_{1,s}, \quad [21]$$

$$\| \mathbf{u} - \Pi_1 \mathbf{u} \|_{r,s\cap\Omega_i} \leqslant \| \tilde{\mathbf{u}}^i - \Pi \tilde{\mathbf{u}}^i \|_{1,s} \leqslant c \, h^{1-r} \, \| \tilde{\mathbf{u}}^i \|_{r,\tilde{s}} \quad [22]$$

Inequalities [15], [17], [19], [21] imply the first inequality of Lemma 1. Inequalities [16], [18], [20], [22], imply the second and third inequalities of Lemma 1.

4.2. *Construction of a local interpolation operator*

In this subsection we prove the discrete inf-sup condition for the P_2/P_0 element with the additional assumption, that the crack cuts the mesh far enough from the nodes.

Definition 2. *Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$. We define $\Pi_2\mathbf{u}$ as the element of V_h such that*

$$\Pi_2\mathbf{u} = \sum_{k\in\tau_h/\tau_H} \sum_{i=1}^{3} \boldsymbol{\alpha}_i\varphi_i + \sum_{k\in\tau_H} \sum_{i=1}^{3} \big(\boldsymbol{\beta}_i\varphi_i H_1 + \boldsymbol{\gamma}_i\varphi_i H_2\big), \qquad [23]$$

where τ_H is the set of triangle totally cut by the crack, φ_i is the classical finite element shape function of order 2 associated to node i being the center of the edge e_i of the element K and with

$$\boldsymbol{\alpha}_i = \frac{\int_{e_i}\mathbf{u}}{\int_{e_i}\varphi_i}, \quad \boldsymbol{\beta}_i = \frac{\int_{e_i\cap\Omega_1}\mathbf{u}}{\int_{e_i\cap\Omega_1}\varphi_i}, \quad \boldsymbol{\gamma}_i = \frac{\int_{e_i\cap\Omega_2}\mathbf{u}}{\int_{e_i\cap\Omega_2}\varphi_i}.$$

Lemma 3. *Suppose that the crack cuts the mesh far enough from the nodes then the interpolation operator defined by [23] satisfies $\forall\mathbf{u} \in V_h$*

$$\int_{s\backslash\Gamma_c} \operatorname{div}(\mathbf{u} - \mathit{\Pi}_2\mathbf{u}) = 0 \qquad \forall s \in \tau_h$$

$$\| \mathit{\Pi}_2\mathbf{u} \|_{1,s\cap\Omega_i} \leq c\,\big(h^{-1} \| \tilde{\mathbf{u}}^i \|_{0,s} + | \tilde{\mathbf{u}}^i |_{1,s}\,\big) \qquad \forall s \in \tau_h.$$

Therefore, the discrete inf-sup condition for the P_2/P_0 element holds.

Proof : The first equation is obvious. Now let s be a triangle totaly cut by the crack. Then by using triangle inequality, the hypothesis “crack far enough from nodes” and Cauchy-Schwarz inequality we have :

$$| \mathit{\Pi}_2\mathbf{u} |_{1,\widehat{s\cap\Omega_i}} \leq c\ | \widehat{\mathit{\Pi}_2\mathbf{u}} |_{1,\widehat{s\cap\Omega_i}} \leq c \sum_{j=1}^{3} \Big| \int_{\widehat{e_j\cap\Omega_i}} \hat{\mathbf{u}} \Big| \frac{| \hat{\varphi}_j |_{1,\widehat{s\cap\Omega_i}}}{| \int_{\widehat{e_j\cap\Omega_i}} \hat{\varphi}_j |}$$

$$\leq c \sum_{j=1}^{3} \int_{\widehat{e_j\cap\Omega_i}} | \hat{\mathbf{u}} | \leq c \sum_{j=1}^{3} \int_{\hat{e}_j} | \hat{\tilde{\mathbf{u}}}^i | \leq c\ \| \hat{\tilde{\mathbf{u}}}^i \|_{1,\hat{s}}$$

and by a scaling argument we have :

$$\| \mathit{\Pi}_2\mathbf{u} \|_{1,s\cap\Omega_i} \leq c\,(h^{-1} \| \tilde{\mathbf{u}}^i \|_{0,s} + | \tilde{\mathbf{u}}^i |_{1,s}). \qquad [24]$$

Now for non-enriched triangle we use the same argument to prove :

$$\| \mathit{\Pi}_2\mathbf{u} \|_{1,s} \leq c\,(h^{-1} \| \mathbf{u} \|_{0,s} + | \mathbf{u} |_{1,s}), \qquad [25]$$

which finishes the proof of Lemma 3.

4.3. *Error analysis*

We suppose in this section that the non-cracked domain $\bar{\Omega}$ has a regular boundary, and that $\mathbf{f}, \mathbf{t}$ are smooth enough, for the solution $(\mathbf{u}, p)$ of the mixed elasticity problem to be written as a sum of a singular part $(\mathbf{u}_s, p_s)$ and a regular part $(\mathbf{u} - \mathbf{u}_s\ ,\ p - p_s)$ in Ω satisfying $\mathbf{u} - \mathbf{u}_s \in \mathbf{H}^2$ and $p - p_s \in \mathbf{H}^1$.

Proposition 1. *Under the assumption of existence and uniqueness of solutions* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *of the continuous [6] and discrete [7] mixed elasticity problems, and if the LBB condition is satisfied, then :*

$$\| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} + \| p - p_h \|_{0,\Omega} \leq c\, h \left[\| \mathbf{u} - \chi \mathbf{u}_s \|_{2,\Omega} + \| p - \chi p_s \|_{1,\Omega} \right],$$

where χ *is the cut-off function.*

Proof By using the equivalent Céa lemma (see (Brezzi *et al.*, New York, 1991)) we have $\forall \mathbf{v}_h \in V^h$ and $q_h \in Q^h$:

$$\| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} + \| p - p_h \|_{0,\Omega} \leq c \left[\| \mathbf{u} - \mathbf{v}_h \|_{1,\Omega} + \| p - q_h \|_{0,\Omega} \right]. \quad [26]$$

Now let $\mathit{\Pi}_h \mathbf{u}$ be the classical interpolation operator introduced by (Nicaise *et al.*, 2011) then we have :

$$\| \mathbf{u} - \mathit{\Pi}_h \mathbf{u} \|_{1,\Omega} \leq c\, h\ \| \mathbf{u} - \chi \mathbf{u}_s \|_{2,\Omega}\ . \quad [27]$$

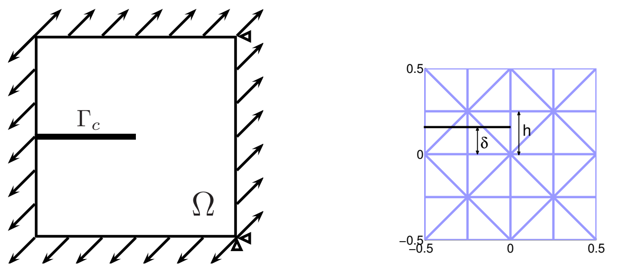
Let $\Pi_h p = \mathit{\Pi}_1 p + \sum_{i=1}^{2} c_i F_{ip}\, \chi = \mathit{\Pi}_1 p + \chi\, p_s$, where $\mathit{\Pi}_1$ is the interpolation operator defined in Section 4.1. Then :

$$\| p - \Pi_h p \|_{0,\Omega} = \| p_r - \mathit{\Pi}_1 p_r \|_{0,\Omega} \leq c\, h\ \| p_r \|_{1,\Omega}\ . \quad [28]$$

Finally, the result of Proposition 1 can be obtained by choosing $\mathbf{v}_h = \mathit{\Pi}_h \mathbf{u}$ and $q_h = \Pi_h p$ in [26] and by using equations [27] and [28].

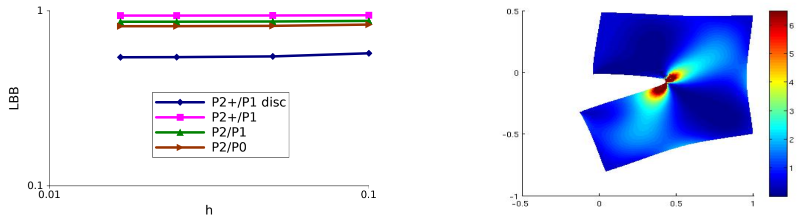
5. Numerical study

The numerical tests are made on a non-cracked domain defined by $\bar{\Omega} =]-0.5, 0.5[\times]-0.5, 0.5[$, and the considered crack is the line segment $\Gamma_c =]-0.5; 0[\times\{0\}$ (see Fig. 5(a)). To remove rigid body motions, we eliminate three degrees of freedom (see Fig. 5(a)). In this numerical test, we impose only a boundary condition of Neumann type (see Fig. 5(a)), in order to avoid possibility of singular stress for mixed Dirichlet-Neumann condition at transition points. The finite element method is defined on a structured triangulation of $\bar{\Omega}$. The von Mises stress for this test is presented in Fig. 6(b). As expected the von Mises stress is concentrated at the crack tip. The notation P_i (resp. P_i^+) means that we use an extended finite-element method of order i (resp. with an additional cubic buble function) and P_j disc means that we use a discontinuous extended finite-element method. The reference solution is obtained with a structured $P_2/P1$ method and $h = 1/160$.



(a) Cracked specimen (b) Position of the crack.

Figure 5. *Cracked specimen and position of the crack*



(a) Evaluation of the inf-sup condition (b) von Mises stress

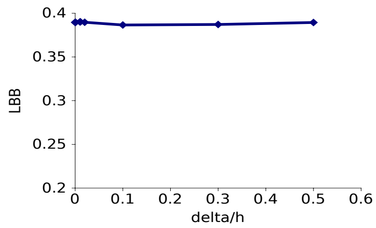
Figure 6. *Evolution of the inf-sup condition for mixed problem and von Mises stress ($\delta = 0$)*

5.1. *Numerical inf-sup test*

In this section we numerically study the inf-sup condition and its dependence on the position of the crack. First, the inf-sup condition is evaluated using gradually refined structured triangulation meshes. The evolution of the numerical inf-sup value is plotted in Fig. 6(a) with respect to the element size. From this figure we can conclude that the numerical inf-sup value is stable for all studied formulations. Let δ be the crack position as shown in Fig. 5(b). To test the influence of the position of the crack on the inf-sup condition, we check the LBB condition by decreasing δ. The tests are made, on a P_1^+/P_1 formulation, with $h = 1/100$ (see Fig. 7(a)) and $h = 1/10$ (see Fig. 7(b)). The results presented in Figs. 7(a) and 7(b) show that the inf-sup condition remains bounded regardless of the position of the crack. Hence, one can conclude that the formulation is stable independently of the position of the crack.

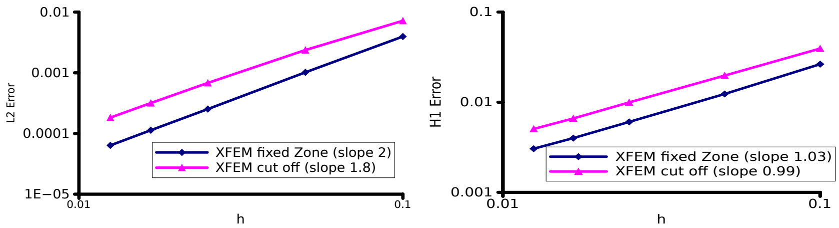
5.2. *Convergence rate and the computational cost*

Figures 8(a), 8(b) and 8(c) show a comparison between the convergence rates of the X-FEM fixed area and X-FEM cut off for the $\mathbf{L}^2$-norm and $\mathbf{H}^1$-norm ($P1^+/P_1$ element are used). These errors were obtained by running the test problem for some values of the parameter n_s, where n_s is the number of subdivision (number of cells) in each direction $h = \dfrac{1}{n_s}$. Figure 8(b) confirms that the convergence rate for the energy norm is of order h for both variants of the X-FEM : with fixed area and cut-off. Figure 8(a)

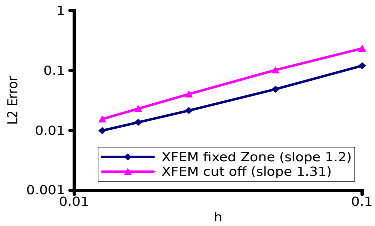


(a) $h = 1/10$ (b) $h = 1/100$

Figure 7. *Evolution of the inf-sup condition as a function of the position of the crack*



(a) $\mathbf{L}^2$-displacement error (b) $\mathbf{H}^1$-displacement error



(c) L^2-pressure error

Figure 8. *Errors for the mixed problem with enriched P_1^+/P_1 elements.*

shows that the convergence rates for the $\mathbf{L}^2$-norm in displacement is of order h^2 for both variants. Figure 8(c) shows that the convergence rates for the $\mathbf{L}^2$-norm in pressure is h for both variants. Compared to the X-FEM method with a fixed enrichment area, the convergence rate for X-FEM cut-off is very close but the error values are a bit larger. In order to test the computational cost of X-FEM cut-off, Table (1) shows a comparison between the number of degrees of freedom for different refinements of the classical method X-FEM with fixed enrichment area and the cut-off method. This

Number of cells in each direction	Number of degrees of freedom	
	X-FEM fixed enrichment area	X-FEM Cut Off
40	13456	11516
60	30046	25666
80	53376	45416

Tableau 1. *Number of degrees of freedom for enriched $P_2/P1$ element*

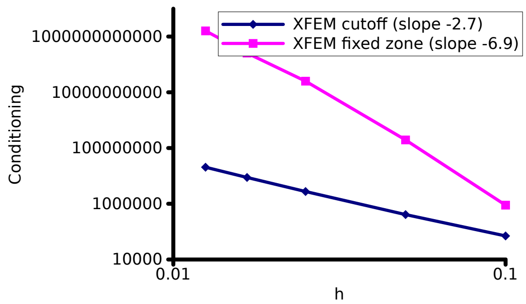
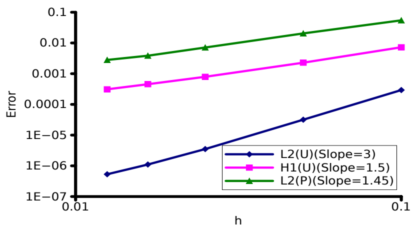


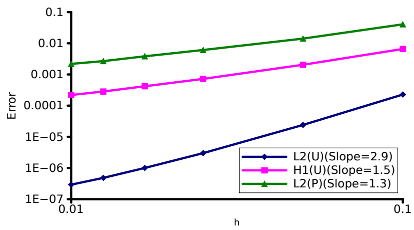
Figure 9. *Conditioning number of the stiffness matrix for the mixed problem*

latter enrichment leads to a significant decrease in the number of degrees of freedom. The condition number of the linear system associated to the cut-off enrichment is much better than the one associated with the X-FEM with a fixed enrichment area (see Fig. 9). We can conclude that, similarly to the X-FEM with fixed enrichment area, the X-FEM cut-off leads to an optimal convergence rate and also reduces the approximation errors but without significant additional costs.

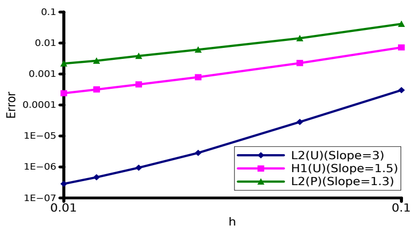
The numerical tests of the higher order X-FEM method (P_2^+/P_1 disc, P_2^+/P_1, P_2/P_1 and P_2/P_0) do not give an optimal order of convergence (see Figs. 10(a), 10(b), 10(c) and 10(d)). This means that the enrichment function does not capture the behavior of the solution at the crack's tip. This result was expected as the asymp-



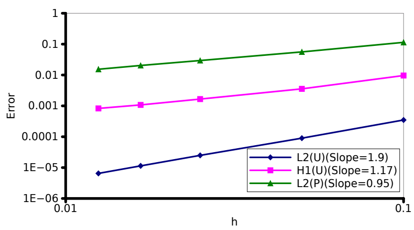
(a) Enriched P_2^+/P_1 disc element



(b) Enriched P_2^+/P_1 element



(c) Enriched P_2/P_1 element



(d) Enriched P_2/P_0 element

Figure 10. *Convergence rate for the high-order elements (logarithmic scales)*

totic displacement at the crack tip belongs to $\mathbf{H}^{3/2-\eta}(\Omega)$ for all $\eta > 0$. Then, for the X-FEM cut-off, the convergence rate remains limited to $h^{3/2}$ with high order polynomials. To have an optimal convergence rate, one must make an asymptotic expansion of order 2 to find the correct expression of the enrichment basis for the displacement and pressure.

6. Conclusion

From this study we can conclude that the X-FEM cut-off mixed formulation is stable, regardless of the position of the crack. Similarly to the X-FEM with fixed enrichment area, the X-FEM cut-off gives an optimal convergence rate but without significant additional costs. For shape functions of higher order, the convergence rate is limited to $h^{3/2}$. This result was expected as the main singularity belongs to $\mathbf{H}^{5/2-\eta}(\Omega)$ for all $\eta > 0$.

7. Bibliographie

Adams R., *Sobolev spaces*, Academic Press, 1975.

Brezzi F., Fortin M., *Mixed and Hybrid Finite Element Methods*, vol. 15, Springer Series in Computational Mathematics, New York, 1991.

Chahine E., Labrode P., Renard Y., « Crack tip enrichment in the X-FEM method using a cut-off function », *Int. J. Numer. Meth. Engng*, vol. 75, n° 6, p. 629-646, 2008.

Chapelle D., Bathe K. J., « The inf-sup test », *Computers and Structures*, vol. 47, n° 4/5, p. 537-545, 1993.

Chen Z., Nochetto. R. H., « Residual-type a posteriori error estimates for elliptic obstacle problems », *Numerische Mathematik*, vol. 84, p. 527-548, 2000.

Laborde P., Renard Y., Pommier J., Salaun M., « High order extended finite element method for cracked domains », *Int. J. Numer. Meth. Engng.*, vol. 64, p. 354-381, 2005.

Legrain G., Moes N., Huerta A., « Stability of incompressible formulations enriched with X-FEM », *Comput. Meth. Appl. Mech. Engrg.*, vol. 197, p. 1835-1849, 2008.

Moës N., Dolbow J., Belytschko T., « A finite element method for crack growth without remeshing », *Int. J. Numer. Meth. Engng.*, vol. 46, p. 131-150, 1999.

Nicaise S., Renard Y., Chahine E., « Optimal convergence analysis for the extended finite element method », *Int. J. Numer. Meth. Engng.*, vol. 86, p. 528-548, 2011.

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