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A METHOD OF PIECEWISE-SMOOTH NUMERICAL BRANCHING

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Abstract

A method of numerical branching is proposed for piecewise-smooth steady-state problems when any analytical expressions are not known for the regions of smoothness of the functions involved. It is shown in model examples of contact problems that the method can reliably discover all solution branches around a known solution if its input parameters are set up properly. It is also suggested how to choose the optimum parameter settings.

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1. Introduction

The steady-state bifurcation problem:

$$\left.\begin{array}{l}\textit{Find } \boldsymbol{y} \in U \textit{ such that} \\ \boldsymbol{H}(\boldsymbol{y}) = \boldsymbol{0},\end{array}\right\} \qquad (\mathscr{P})$$

where $U \subset \mathbb{R}^{N+1}$ and $\boldsymbol{H}\colon U \to \mathbb{R}^N$, has been the subject of large number of studies in the last decades. In particular, a variety of numerical bifurcation methods has been constructed if $\boldsymbol{H}$ is smooth, say continuously differentiable (see, e.g., [2, 6, 9] and the references therein).

However, there are many equilibrium problems in economics and diverse engineering fields whose models lead to a system of non-smooth equations [1, 5, 13]. For instance, let us mention frictional and frictionless contact problems in solid mechanics, which are of our specific interest. All the same, methods of numerical branching of solutions of non-smooth problems when they depend on a parameter are still very little explored to our knowledge: Only branching of static equilibrium curves of discretised frictionless contact problems were treated in [3, 14], where tangential directions of

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curves emanating from points of non-smoothness were determined by a certain linear complementarity problem, and a method based on resolution of this problem was suggested for branch switching during numerical continuation.

In our recent paper [11], we developed a restarted predictor-corrector method for numerical continuation of solution curves of Problem ($\mathscr{P}$) provided that $\boldsymbol{H}$ is piecewise C^1 (PC^1). The method consists in continuing smooth solution branches by a standard predictor-corrector method and joining the smooth branches continuously.

More precisely, our method computes a sequence of points $\{\boldsymbol{y}_k\}$ lying approximately on selected solution branches, and a sequence of the corresponding tangent vectors $\{\boldsymbol{t}_k\}$ with a unit weighted norm $\|\boldsymbol{t}_k\|_w$. Starting from a couple $(\boldsymbol{y}_k, \boldsymbol{t}_k)$, one predictor-corrector step yields a new couple $(\boldsymbol{y}_{k+1}, \boldsymbol{t}_{k+1})$. Firstly, an initial approximation $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{t}})$ of the new couple is generated in the direction of $\boldsymbol{t}_k$ in the predictor step:

$$\tilde{\boldsymbol{y}} := \boldsymbol{y}_k + h\boldsymbol{t}_k, \quad \tilde{\boldsymbol{t}} := \boldsymbol{t}_k,$$

where h is a step size. Secondly, the corrector steps, which are iterative steps of Newton's type, are run with the initial approximation $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{t}})$. If they succeed in bringing the predicted point back to the currently approximated branch, the final couple is accepted, the step size h is adapted for the next predictor-corrector step and the current step is done. Otherwise, h is reduced and the predictor and the corrector steps are repeated. If the standard predictor-corrector method fails in computing a new couple, a special procedure is carried out for locating a new smooth branch (simple tangent switch). Afterwards, the predictor-corrector method is restarted for continuation of the branch found. The whole method does not obey any analytical expressions of the regions of smoothness of $\boldsymbol{H}$ and is implementable generically.

Similar continuation techniques constructed specially for static plane frictional contact problems can be found in [7, 8]. Nevertheless, all the methods were proposed for finding solely one continuing branch when an end point of the most recently traced branch is encountered.

The present paper deals with purely non-smooth numerical branching of solutions of ($\mathscr{P}$) when $\boldsymbol{H}$ is PC^1. We develop an approach that does not rely on any analytical expressions of the regions of smoothness of $\boldsymbol{H}$ in Section 2, and we demonstrate its performance for model examples of contact problems in Section 3.

Throughout the paper, we employ the following definition of a PC^1-function [13]:

Definition 1.1. A function $\boldsymbol{H}\colon U \to \mathbb{R}^N$, $U \subset \mathbb{R}^M$, is PC^1 if it is continuous and for every $\bar{\boldsymbol{y}} \in U$, there exist an open neighbourhood $O \subset U$ of $\bar{\boldsymbol{y}}$ and a finite family of C^1-functions $\boldsymbol{H}^{(i)}\colon O \to \mathbb{R}^N$, $i \in \mathcal{I}(\bar{\boldsymbol{y}})$, such that

$$\forall \boldsymbol{y} \in O\colon\ \boldsymbol{H}(\boldsymbol{y}) \in \{\boldsymbol{H}^{(i)}(\boldsymbol{y});\ i \in \mathcal{I}(\bar{\boldsymbol{y}})\}.$$

The functions $\boldsymbol{H}^{(i)}$ are termed *selections* of $\boldsymbol{H}$ at $\bar{\boldsymbol{y}}$.

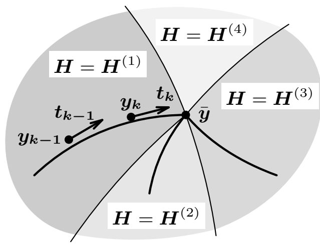


FIGURE 1. Example of the considered situation.

2. Numerical Branching

Consider that an approximation of a smooth solution branch of ($\mathscr{P}$) has been obtained by the restarted predictor-corrector method from [11], and either the method has failed in locating a new smooth branch or it has located one but we want to find other ones because a bifurcation is expected by virtue of the criterion from [10], for example. Restricting ourselves to purely non-smooth branching (Assumptions (I)–(III) below), we shall describe a method designed for discovering potentially all continuing branches.

Let $\bar{\boldsymbol{y}}$ be the corresponding end point of the recovered branch. We shall assume the following:

(I) The whole solution set of ($\mathscr{P}$) in a vicinity of $\bar{\boldsymbol{y}}$ is formed by one-sided smooth solution branches emanating from $\bar{\boldsymbol{y}}$ into mutually distinct regions of smoothness $\{\boldsymbol{y} \in O;\ \boldsymbol{H}(\boldsymbol{y}) = \boldsymbol{H}^{(i)}(\boldsymbol{y})\}$ for some $i \in \mathcal{I}(\bar{\boldsymbol{y}})$.

(II) The gradients $\nabla \boldsymbol{H}^{(i)}(\bar{\boldsymbol{y}})$, $i \in \mathcal{I}(\bar{\boldsymbol{y}})$, have the full rank.

(III) $\operatorname{Ker} \nabla \boldsymbol{H}^{(i)}(\bar{\boldsymbol{y}}) \cap \operatorname{Ker} \nabla \boldsymbol{H}^{(j)}(\bar{\boldsymbol{y}}) = \{\boldsymbol{0}\}, \quad \forall i, j \in \mathcal{I}(\bar{\boldsymbol{y}}),\ i \neq j.$

Particularly, these assumptions guarantee that tangent vectors at $\bar{\boldsymbol{y}}$ to any two different branches are linearly independent (see Figure 1 for an illustration of branching under our consideration).

Since it seems to be hardly possible to encounter a point of non-smoothness in practical computations, we shall suppose in addition that $\boldsymbol{H}$ is smooth in any point considered in the procedures hereinafter. Then, a unit tangent vector $\boldsymbol{t}$ at such a point $\boldsymbol{y}$ lying in a vicinity of $\bar{\boldsymbol{y}}$ and belonging to a solution branch in the region $\{z \in U;\ \boldsymbol{H}(z) = \boldsymbol{H}^{(i)}(z)\}$ is uniquely determined up to a direction by the conditions:

$$\nabla \boldsymbol{H}^{(i)}(\boldsymbol{y})\boldsymbol{t} = \boldsymbol{0}, \quad \|\boldsymbol{t}\|_w = 1.$$

Our branching method consists of two steps: approximation of the end point $\bar{\boldsymbol{y}}$ and subsequent location of new branches from a neighbourhood of the approximate end point.

2.1. Approximation of the end point The end point is approximated by a bisection-like procedure that is based on the predictor-corrector method used in the process of numerical continuation of smooth branches. This procedure relies on the fact that individual branches have numerically distinguishable tangents, which is justified by Assumptions (I)–(III). The algorithm can be sketched as follows:

Algorithm 2.1.

Input: $c_{\mathrm{diff}} \in (0,1)$, $h > h_{\min} > 0$, $\boldsymbol{y}_0$, $\boldsymbol{t}_0 \in \mathbb{R}^{N+1}$ such that

$$\boldsymbol{H}(\boldsymbol{y}_0) \approx \boldsymbol{0}, \quad \nabla\boldsymbol{H}(\boldsymbol{y}_0)\boldsymbol{t}_0 = \boldsymbol{0}, \ \|\boldsymbol{t}_0\|_w = 1,$$

and the corrector steps for the prediction $\tilde{\boldsymbol{y}} = \boldsymbol{y}_0 + h\boldsymbol{t}_0$, $\tilde{\boldsymbol{t}} = \boldsymbol{t}_0$ do not converge or they converge to a couple $(\boldsymbol{y}, \boldsymbol{t})$ with $\boldsymbol{t}^\top \boldsymbol{t}_0/(\|\boldsymbol{t}\|\|\boldsymbol{t}_0\|) < c_{\mathrm{diff}}$.

Step 1: Set $h := h/2$.

Step 2: Do the prediction $\tilde{\boldsymbol{y}} := \boldsymbol{y}_0 + h\boldsymbol{t}_0$, $\tilde{\boldsymbol{t}} := \boldsymbol{t}_0$.

Step 3: Start the correction with $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{t}})$.

Step 4: If the correction converges to a new couple $(\boldsymbol{y}, \boldsymbol{t})$ and $\boldsymbol{t}^\top \boldsymbol{t}_0/(\|\boldsymbol{t}\|\|\boldsymbol{t}_0\|) \geq c_{\mathrm{diff}}$, then set $\boldsymbol{y}_0 := \boldsymbol{y}$, $\boldsymbol{t}_0 := \boldsymbol{t}$.

Step 5: If $h < h_{\min}$, break. Otherwise, go to Step 1.

Output: The point $\boldsymbol{y}_0$ with the corresponding unit tangent $\boldsymbol{t}_0$.

The couple $(\boldsymbol{y}_0, \boldsymbol{t}_0)$ in the input is the last couple obtained by the numerical continuation of the most recent smooth branch (the couple $(\boldsymbol{y}_k, \boldsymbol{t}_k)$ in Figure 1) and h equals the minimal step size from the predictor-corrector method employed during the continuation. We denote the Euclidean norm by $\|.\|$ and c_{diff} is the minimal value of the cosine of the angle of two tangent vectors considered to correspond to the same branch. Hence, if the conditions in Step 4 are satisfied, the newly found point $\boldsymbol{y}$ is supposed to be from the same branch as the one stored in $\boldsymbol{y}_0$, and it is accepted as a better approximation of $\bar{\boldsymbol{y}}$. The minimal step size $h_{\min}$ determines the precision required for the resulting approximation of $\bar{\boldsymbol{y}}$.

2.2. Location of new branches Suppose that $\boldsymbol{y}_0$ calculated by Algorithm 2.1 is a good approximation of $\bar{\boldsymbol{y}}$ belonging to one smooth branch, and $\boldsymbol{t}_0$ is the corresponding unit tangent pointing out from the region of smoothness with $\boldsymbol{y}_0$ (see Figure 2). Inspired by the methods for branch switching at a smooth simple bifurcation point [6, 9], we shall propose a heuristic technique composed of a sequence of predictor-corrector steps and yielding points on other smooth branches.

Our heuristic stems from the idea used in the simple tangent switch [11]: Let $\tilde{\boldsymbol{t}}$ be a vector satisfying

$$\nabla\boldsymbol{H}(\tilde{\boldsymbol{y}})\tilde{\boldsymbol{t}} = \boldsymbol{0}, \quad \|\tilde{\boldsymbol{t}}\|_w = 1, \tag{2.1}$$

where $\tilde{\boldsymbol{y}}$ is a point close to $\bar{\boldsymbol{y}}$ and lies in the region of smoothness containing a new smooth branch (see Figure 3). Then, a suitable prediction for a predictor-corrector step giving a point on this branch should be either of the ones in the directions of $\pm\tilde{\boldsymbol{t}}$.

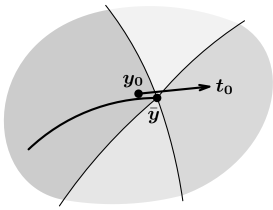


FIGURE 2. Output of Algorithm 2.1.

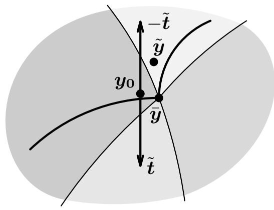


FIGURE 3. Direction of a prediction for finding a new branch.

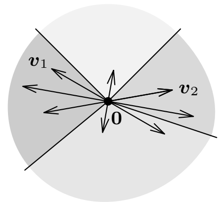


FIGURE 4. Generation of directions into all regions.

Imagine for a while that you have a sequence of points $\tilde{\boldsymbol{y}}$ that pass through all regions of smoothness intersecting at $\bar{\boldsymbol{y}}$ at your disposal. Under Assumptions (I)–(III), you could discover all solution branches by computing the sequence of vectors $\tilde{\boldsymbol{t}}$ from (2.1) and trying the corresponding predictor-corrector steps with $\tilde{\boldsymbol{t}}$ and $-\tilde{\boldsymbol{t}}$ successively. Therefore, we would like to *guess* such a sequence of points from all the regions intersecting at $\bar{\boldsymbol{y}}$.

For this purpose, consider the most probable branching scenarios described in [10] next: By linearisation and projection, the corresponding regions of smoothness can be represented by cones with vertices at $\boldsymbol{0}$ in a two-dimensional vector space. This representative space can be spanned by any two linearly independent vectors, say $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. As a consequence, a sequence of linear combinations of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ whose directions are deployed densely enough comprises directions pointing into representatives of all regions of smoothness (Figure 4). From here, one can conclude that if $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ are vectors represented by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then the corresponding sequence of linear combinations of $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ contains directions pointing into all regions and it can be used for calculating a sequence of points as desired.

In the general setting with $(\boldsymbol{y}_0, \boldsymbol{t}_0)$ from the output of Algorithm 2.1, choose a step length h sufficiently large in comparison with the precision of the approximation of $\bar{\boldsymbol{y}}$ by $\boldsymbol{y}_0$, in particular so large that $\boldsymbol{y}_0 + h\boldsymbol{t}_0$ leaves the region with $\boldsymbol{y}_0$. Now, take $\tilde{\boldsymbol{t}}_1 := -\boldsymbol{t}_0$ and calculate a vector $\tilde{\boldsymbol{t}}_2$ satisfying

$$\nabla \boldsymbol{H}(\boldsymbol{y}_0 + h\boldsymbol{t}_0)\tilde{\boldsymbol{t}}_2 = \boldsymbol{0}, \ \|\tilde{\boldsymbol{t}}_2\|_w = 1.$$

According to the imposed assumptions, $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ are linearly independent. So, choose a set $\mathcal{B} \subset \mathbb{R}^2$ so that $\{\beta_1\tilde{\boldsymbol{t}}_1 + \beta_2\tilde{\boldsymbol{t}}_2\}_{(\beta_1,\beta_2)\in\mathcal{B}}$ is a sequence of densely deployed directions. With regard to the argumentation for the most probable branching scenarios, one can expect that the sequence $\{\boldsymbol{y}_0 + h(\beta_1\tilde{\boldsymbol{t}}_1 + \beta_2\tilde{\boldsymbol{t}}_2)\}_{(\beta_1,\beta_2)\in\mathcal{B}}$ might be the desired one containing points from all regions intersecting at $\bar{\boldsymbol{y}}$.

This leads us to the following algorithm:

ALGORITHM 2.2.

Input: $c_{\text{diff}} \in (0, 1)$, $n_{\text{dir}} \in \mathbb{N}$, $h, h_{\text{init}} > 0$, $\boldsymbol{y}_0$, $\boldsymbol{t}_0 \in \mathbb{R}^{N+1}$ such that

$$\boldsymbol{H}(\boldsymbol{y}_0) \approx \boldsymbol{0}, \quad \nabla \boldsymbol{H}(\boldsymbol{y}_0)\boldsymbol{t}_0 = \boldsymbol{0}, \ \|\boldsymbol{t}_0\|_w = 1,$$

and $\boldsymbol{y}_0 + h\boldsymbol{t}_0$ leaves the region of smoothness with $\boldsymbol{y}_0$.

Step 1: Take $\tilde{\boldsymbol{t}}_1 := -\boldsymbol{t}_0$ and compute $\tilde{\boldsymbol{t}}_2$ such that

$$\nabla \boldsymbol{H}(\boldsymbol{y}_0 + h\boldsymbol{t}_0)\tilde{\boldsymbol{t}}_2 = \boldsymbol{0}, \quad \|\tilde{\boldsymbol{t}}_2\|_w = 1.$$

Set $n_{\mathrm{br}} := 1$, $\boldsymbol{y}_{n_{\mathrm{br}}} := \boldsymbol{y}_0$, $\boldsymbol{t}_{n_{\mathrm{br}}} := \tilde{\boldsymbol{t}}_1$.

Step 2: Set $i := 0$.

Step 3: Set

$$\alpha := \frac{2\pi i}{n_{\mathrm{dir}}}, \quad \boldsymbol{v} := (\sin\alpha)\tilde{\boldsymbol{t}}_1 + (\cos\alpha)\tilde{\boldsymbol{t}}_2, \quad \boldsymbol{v} := \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_w},$$

and compute $\tilde{\boldsymbol{t}}$ such that

$$\nabla \boldsymbol{H}(\boldsymbol{y}_0 + h\boldsymbol{v})\tilde{\boldsymbol{t}} = \boldsymbol{0}, \quad \|\tilde{\boldsymbol{t}}\|_w = 1.$$

Step 4: If $i = 0$ or $|\tilde{\boldsymbol{t}}^\top \boldsymbol{t}_0|/(\|\tilde{\boldsymbol{t}}\|\|\boldsymbol{t}_0\|) < c_{\mathrm{diff}}$, set $\boldsymbol{t}_0 := \tilde{\boldsymbol{t}}$. Otherwise, go to Step 7.

Step 5: Try one predictor-corrector step with the couple $(\boldsymbol{y}_0, \tilde{\boldsymbol{t}})$ and the initial step size h_{init}. If it converges to a new couple $(\boldsymbol{y}, \boldsymbol{t})$, then

(i) if $\boldsymbol{t}^\top(\boldsymbol{y} - \boldsymbol{y}_0) < 0$, take $\boldsymbol{t} := -\boldsymbol{t}$;

(ii) if $\boldsymbol{t}^\top \boldsymbol{t}_j/(\|\boldsymbol{t}\|\|\boldsymbol{t}_j\|) < c_{\mathrm{diff}}$, $j = 1, \ldots, n_{\mathrm{br}}$, then set $n_{\mathrm{br}} := n_{\mathrm{br}} + 1$, $\boldsymbol{y}_{n_{\mathrm{br}}} := \boldsymbol{y}$, $\boldsymbol{t}_{n_{\mathrm{br}}} := \boldsymbol{t}$.

Step 6: Repeat the lines in Step 5 with $\tilde{\boldsymbol{t}} := -\tilde{\boldsymbol{t}}$.

Step 7: Set $i := i + 1$. If $i < n_{\mathrm{dir}}$, go to Step 3. Otherwise, break.

Output: Points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n_{\mathrm{br}}}$ with the corresponding unit tangents $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n_{\mathrm{br}}}$.

Here, c_{diff} is the minimal absolute value of the cosine of the angle of two vectors from the kernels of gradients considered to correspond to the same selection similarly as in Algorithm 2.1. Further, n_{dir} is the total number of linear combinations of $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ for seeking points from distinct regions. The step sizes h and h_{init} should be appropriately large in comparison with the expected precision of the approximation of $\bar{\boldsymbol{y}}$ by $\boldsymbol{y}_0$ so that the corresponding steps fall into the desired regions.

The conditions in Step 4 serve for determining whether $\tilde{\boldsymbol{t}}$ has been computed from a gradient from the same region as previously. If it is so, its testing is superfluous, hence skipped. In Step 5(ii), one verifies whether $(\boldsymbol{y}, \boldsymbol{t})$ does not belong to any of the branches already found before. This ensures that the points in the output are from different smooth branches. The corresponding tangents are oriented in the directions of branching from $\bar{\boldsymbol{y}}$, which is guaranteed by Step 5(i). Subsequently, any of the branches located in this way can be traced by predictor-corrector steps.

Remark 2.3. *To increase the probability of recovering all branches, one can restart Algorithm 2.2 several times from Step 2 with other choices of* $\tilde{\boldsymbol{t}}_1$ *and* $\tilde{\boldsymbol{t}}_2$. *We propose the following possibilities:*

(i) *Take* $\tilde{\boldsymbol{t}}_i$ *from the set* $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n_{\rm br}}\}$ *at your current disposal.*
(ii) *Take* $\tilde{\boldsymbol{t}}_i := \tilde{\boldsymbol{t}}_i^+$ *with* $\tilde{\boldsymbol{t}}_i^+$ *satisfying*

$$\nabla \boldsymbol{H}\Big(\boldsymbol{y}_0 + h\Big(\tilde{\boldsymbol{t}}_i^- + 0.1\frac{\tilde{\boldsymbol{t}}}{\|\tilde{\boldsymbol{t}}\|_w}\Big)\Big)\tilde{\boldsymbol{t}}_i^+ = \boldsymbol{0}, \quad \|\tilde{\boldsymbol{t}}_i^+\|_w = 1,$$

where the value of h *is the same as in Algorithm 2.2,* $\tilde{\boldsymbol{t}}_i^-$ *is equal to the vector* $\tilde{\boldsymbol{t}}_i$ *employed in the previous run of Algorithm 2.2 and* $\tilde{\boldsymbol{t}}$ *is chosen randomly.*

3. Numerical Tests

We have tested our technique on examples of bifurcations in discretised plane contact problems with friction described in detail in [10]. We have chosen the following strategy based on Remark 2.3 for restarting Algorithm 2.2:

1. Pass with $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ successively through combinations from $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{n_{\rm br}}\}$ at your disposal.
2. If all combinations which are available so far have already been employed, try to let $\tilde{\boldsymbol{t}}_1$ be the same and choose $\tilde{\boldsymbol{t}}_2$ according to Remark 2.3(ii).

In the following tests, we denote the total number of selections of $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ by $n_{\rm span}$, that is, the total number of restarts of Algorithm 2.2 equals $n_{\rm span} - 1$. Further, $n_{\rm pc}$ stands for the total number of the predictor-corrector steps, which constitute the most expensive part of our technique. Hence, this number characterises the computational cost of each test.

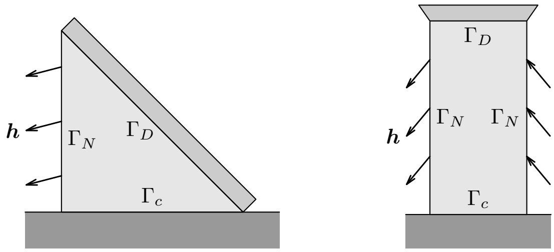
Our calculations were performed with the aid of the finite-element library GetFEM++ [12].

3.1. Triangular Body Consider static deformation of an elastic isosceles triangle with legs 1 m long in the framework of small-deformation elasticity with Lamé constants $\lambda = 100$ GN/m^2 and $\mu = 82$ GN/m^2. The boundary of the triangle being split into three parts Γ_D, Γ_c and Γ_N, the triangle is fixed along Γ_D, and points from Γ_c may come into contact with a flat rigid foundation. This contact is described by unilateral conditions and the Coulomb friction law. Finally, the triangle is subject to surface forces of the density $\boldsymbol{h}$ on Γ_N, and the density depends on a real parameter γ, that is, $\boldsymbol{h} = \boldsymbol{h}(\gamma)$ (Figure 5(a)).

Discretisation of the problem has been done by the finite-element method for two different meshes with nodal approximation of the contact conditions. The settings of Algorithms 2.1 and 2.2 have been the same in both cases: $c_{\rm diff} = 0.99999$, $h = 5\mathrm{e}{-8}$, $h_{\rm min} = h/2^{14} \doteq 3.1\mathrm{e}{-12}$ and $h_{\rm init} = 5\mathrm{e}{-4}$.

1. We have taken a model formed by a single linear triangular finite element where the lower left vertex of the triangle is the only free node. We have chosen $\boldsymbol{h}(\gamma) = \gamma(-14.5\ \mathrm{GN/m^2}, -7.5\ \mathrm{GN/m^2})$ and the friction coefficient $\mathscr{F}$ has been set to 2.

There are four solution branches intersecting at $\gamma = 0$ in this case. They correspond to loss of contact, contact-slip to the right, contact-stick and contact-slip to the left of



(a) Triangular body. (b) Rectangular body.

FIGURE 5. Geometries of the model problems.

TABLE 1. Single triangle.

Input	n_{dir}	n_{span}	n_{pc}
Branch 1	10	2	8
	20	2	8
	30	1	8
Branch 2	10	2	8
	20	2	8
	30	2	8
	40	2	8
	50	1	8
Branch 3	10	1	8
Branch 4	10	1	8

TABLE 2. Refined triangle.

Input	n_{dir}	n_{span}	n_{pc}
Branch 1	10	2	36
	20	1	32
Branch 2	10	1	18
Branch 3	10	2	34
	20	2	52
	30	2	72
	40	1	56
Branch 4	10	1	20

the lower left vertex of the triangle and will be denoted respectively by numbers 1 to 4 in what follows.

2. A uniform mesh with 4096 linear triangles and 64 contact nodes has been used for the discretisation, and we have prescribed $\boldsymbol{h}(\gamma) = \gamma(-26\ \text{GN/m}^2, -7.5\ \text{GN/m}^2)$ and $\mathscr{F} = 1.7$.

There are four solution branches intersecting at $\gamma = 0$, again. In this case, they correspond to a partial contact and slip of the triangle to the right, and to no contact, contact-stick and contact-slip to the left of the lower left vertex of the triangle with pulling the whole triangle to the left. These branches will be referred to as Branch 2, Branch 1, Branch 3 and Branch 4, respectively, in what follows.

We have investigated sufficient values of n_{span} and the corresponding values of n_{pc} for finding all four branches for given n_{dir} for both meshes. One gets by with selecting $\tilde{\boldsymbol{t}}_1$ and $\tilde{\boldsymbol{t}}_2$ solely according to Remark 2.3(i), no random generations are necessary in any of the tests performed. Our observations are summarised in Tables 1 and 2. In the tables, the entry input means the branch from which $\boldsymbol{y}_0$ is chosen on the input of Algorithm 2.1. One can conclude that the setting with $n_{\mathrm{dir}} = 10$ and $n_{\mathrm{span}} = 2$ seems to be the most suitable in these examples with four branches.

3.2. Rectangular Body Next, consider contact of an elastic rectangular block that is 40 mm wide and 80 mm high with a flat foundation (Figure 5(b)). A plane-strain approximation of the nonlinear Ciarlet-Geymonat constitutive law [4, Chapter 4] is used for the material of the block. Namely, the response function $\hat{\boldsymbol{\sigma}}$ of the first Piola-Kirchhoff stress tensor is defined by

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{F}) = (\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{F}}))_{1\le i,j\le 2}, \quad \tilde{\boldsymbol{F}} = \begin{pmatrix} \boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}, \quad \boldsymbol{F} \in \mathbb{R}^{2\times 2},$$

$$\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{F}}) = 2b(\mathrm{tr}(\tilde{\boldsymbol{F}}^\top \tilde{\boldsymbol{F}}))\boldsymbol{I} + 2(a - b\tilde{\boldsymbol{F}}\tilde{\boldsymbol{F}}^\top)\tilde{\boldsymbol{F}} + (2c\det(\tilde{\boldsymbol{F}}^\top \tilde{\boldsymbol{F}}) - d)\tilde{\boldsymbol{F}}^{-\top}, \quad \tilde{\boldsymbol{F}} \in \mathbb{R}^{3\times 3},$$

where

$$\lambda = 4000\ \mathrm{N/mm^2}, \quad \mu = 120\ \mathrm{N/mm^2}, \quad a = 30\ \mathrm{N/mm^2}$$

and

$$b = \frac{\mu}{2} - a, \quad c = \frac{\lambda}{4} - \frac{\mu}{2} + a, \quad d = \frac{\lambda}{2} + \mu.$$

The rectangle is fixed along Γ_D, $\mathscr{F} = 1$ on Γ_c, and the parametrised surface forces of the density $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{x}, \gamma) = \gamma(-2, 0.12(x_1 - 20))$ (in $\mathrm{N/mm^2}$) act on both parts of Γ_N. Discretisation is done by the finite-element method with a uniform mesh with 800 bilinear squares and 21 contact nodes, and nodal approximation of the contact conditions.

There are six solution branches intersecting at $\gamma = 0$. Branches 1–3 correspond to forcing the rectangle to the right with no contact, contact-stick and contact-slip to the right of the lower right vertex of the triangle. Branches 4–6, which are symmetric, correspond to forcing the rectangle to the left with no contact, contact-stick and contact-slip to the left of the lower left vertex.

We have set $c_{\mathrm{diff}} = 0.99999$, $h = 5\mathrm{e}{-7}$, $h_{\min} = h/2^{14} \doteq 3.1\mathrm{e}{-11}$, $h_{\mathrm{init}} = 5\mathrm{e}{-4}$ in Algorithms 2.1 and 2.2. As previously, we have started our testing by looking into sufficient values of n_{span} and the corresponding values of n_{pc} for finding all six branches for n_{dir} fixed. In this problem, random generations according to Remark 2.3(ii) are needed for some initial branches and some values of n_{dir}. In such cases, we have repeated the corresponding test two times, and the behaviour of the whole computational process has varied from one test to another; see Table 3 for the results (the behaviour for Branches 4–6 is symmetric). We denote the total number of restarts of Algorithm 2.2 following a random generation by n_{rand} there.

Afterwards, we have prescribed n_{dir} and n_{span}, and we have compared the obtained values of n_{pc} to get an idea about the cheapest but quiet reliable strategy. The summary of our observations is shown in Table 4. All six branches have been recovered in all the tests. According to the results, we can propose to set n_{dir} to about 20 and n_{span} to about 15 in this example with six branches. In addition, we can suggest to increase rather n_{dir} than n_{span} to increase the probability of finding all branches at the lowest price, in general.

TABLE 3. Rectangular body, n_{dir} given a priori.

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 1	20	8	168	1
	20	10	214	3
	20	13	266	6
	40	7	292	2
	40	8	330	3
	40	14	478	7
	50	3	162	0
	60	5	232	0
	100	3	206	0
Branch 3	20	5	128	0
	40	3	134	0
	50	1	72	0
	60	4	202	0
	100	1	102	0

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 2	20	4	136	1
	20	7	220	3
	20	8	240	4
	40	4	260	1
	40	5	294	2
	40	5	316	2
	50	4	274	1
	50	5	342	2
	50	7	490	4
	60	5	374	2
	60	8	620	5
	60	9	654	6
	100	4	372	1
	100	4	406	1
	100	7	662	4

4. Conclusion

We have developed a method of numerical branching for piecewise-smooth problems, and we have demonstrated that our method can reliably discover all solution branches in two model situations if it is set up properly. We have suggested how to choose the optimum parameter settings, as well. As far as we know, this is the first attempt to devise a numerical method of piecewise-smooth branching that does not rely on any analytical expressions of the regions of smoothness of the functions involved.

References

[1] J. C. Alexander, T.-Y. Li, and J. A. Yorke. 'Piecewise smooth homotopies'. In B. C. Eaves, F. J. Gould, H.-O. Peitgen, and M. J. Todd, editors, *Homotopy methods and global convergence* (Plenum Press, New York, 1983), 1–14.

[2] E. L. Allgower and K. Georg. 'Numerical path following'. In *Handbook of numerical analysis*, Volume V (North-Holland, Amsterdam, 1997), 3–207.

[3] G. Björkman. 'Path following and critical points for contact problems'. *Comput. Mech.* **10** (3–4) (1992), 231–246.

[4] P. G. Ciarlet. *Mathematical elasticity. Vol. I: Three-dimensional elasticity*, *Studies in Mathematics and its Applications*, Volume 20 (North-Holland, Amsterdam, 1988).

[5] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems. Vol. I*. Springer Series in Operations Research (Springer-Verlag, New York, 2003).

TABLE 4. Rectangular body, n_{dir} and n_{span} given a priori.

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 1	20	15	310	3
	20	15	316	5
	20	15	322	1
	20	16	326	3
	20	16	344	5
	20	16	346	1
	50	10	400	0
	60	10	416	0
	100	8	474	0
Branch 3	20	15	318	0
	20	15	318	0
	20	16	334	1
	20	16	334	1
	50	10	396	0
	60	10	414	0
	100	8	456	0

Input	n_{dir}	n_{span}	n_{pc}	n_{rand}
Branch 2	20	15	350	2
	20	15	360	4
	20	15	416	8
	20	16	374	2
	20	16	386	4
	20	16	434	8
	50	10	518	1
	50	10	532	1
	50	10	562	2
	60	10	618	2
	60	10	642	3
	60	10	734	6
	100	8	642	1
	100	8	684	2
	100	8	720	3

[6] K. Georg. ‘Matrix-free numerical continuation and bifurcation’. *Numer. Funct. Anal. Optim.* **22** (3–4) (2001), 303–320.

[7] J. Haslinger, V. Janovský, and R. Kučera. ‘Path-following the static contact problem with Coulomb friction’. In J. Brandts, S. Korotov, M. Křížek, J. Šístek, and T. Vejchodský, editors, *Proceedings of the International Conference Applications of Mathematics 2013* (Institute of Mathematics, Academy of Sciences of the Czech Republic, 2013), 104–116.

[8] J. Haslinger, V. Janovský, and T. Ligurský. ‘Qualitative analysis of solutions to discrete static contact problems with Coulomb friction’. *Comput. Methods Appl. Mech. Engrg.* **205–208** (2012), 149–161.

[9] J. Hughes and M. Friedman. ‘A bisection-like algorithm for branch switching at a simple branch point’. *J. Sci. Comput.* **41** (1) (2009), 62–69.

[10] T. Ligurský and Y. Renard. ‘Bifurcations in piecewise-smooth steady-state problems: Abstract study and application to plane contact problems with friction’, <hal-01077161>, 2014.

[11] T. Ligurský and Y. Renard. ‘A continuation problem for computing solutions of discretised evolution problems with application to plane quasi-static contact problems with friction’. *Comput. Methods Appl. Mech. Engrg.* **280** (2014), 222–262.

[12] Y. Renard and J. Pommier. *GetFEM++*. An open source generic C++ library for finite element methods, `http://home.gna.org/getfem`.

[13] S. Scholtes. ‘Introduction to piecewise differentiable equations’. Preprint No. 53/1994, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, 1994.

[14] M. Schulz and S. Pellegrino. ‘Equilibrium paths of mechanical systems with unilateral constraints

I. Theory'. *Proc. R. Soc. Lond. A* **456** (2001) (2000), 2223–2242.

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