Lagrangian and Nitsche methods for frictional contact

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1 Introduction

Augmented Lagrangians and Lagrangians are constrained optimization tools that very early have naturally been applied to contact problems with deformable solids (see for example (Rockafellar 1974, 1976)). The augmented Lagrangian has since quite widely become established for the approximation and resolution of contact problems in small and large strains, mainly following the research of (Curnier and Alart 1988 ; Alart and Curnier 1991 ; Simo and Laursen 1992). The method by Nitsche (1971) was originally proposed to allow a Dirichlet-type boundary condition to be weakly taken into account, precisely avoiding the use of Lagrange multipliers. Only recently has it been extended to contact conditions with or without friction in (Chouly and Hild 2013*a* ; Annavarapu *et al.* 2014 ; Chouly 2014 ; Chouly *et al.* 2015). The close connection between Nitsche and Lagrangian methods is however quite clear and it is the objective of this chapter to shed some light on this relationship. This is achieved namely by looking into the mechanisms underlying these methods, and also by way of presenting some recent developments within the framework of small and large elastic strains.

Section 2 first presents the continuous problem of frictional contact between two elastic solids, within the framework of small strains. Section 3 is dedicated to finite element approximation within the framework of small strains, where mathematical analysis of numerical methods is possible. Section 4 finally presents the extension of the methods described in previous sections to the regime of large elastic transformations, as well as numerical results related to this context.

2 Small-strains frictional contact between two elastic bodies

The problem of frictional contact between two elastic solids is first described in Section 2.1, then in Section 2.2, this problem is reformulated as a quasi variational inequality. Then, Section 2.3 introduces the weak multiplier form and Section 2.4, the proximal augmented Lagrangian formulation; these reformulations are the basis of the numerical approximations presented in Sections 3 and 4.

2.1 Contact between two elastic bodies

We consider two elastic solids whose respective reference configurations are denoted by Ω^1 and Ω^2 corresponding to two domains of $\mathbb{R}^d$ ($d = 2$ or 3) of regular boundaries (piecewise of class $\mathscr{C}^1$), as shown in Figure 1. At the boundaries $\partial\Omega^1$ and $\partial\Omega^2$ of Ω^1 and Ω^2 we can identify: the boundaries Γ_D^1 and Γ_D^2 (with non-empty interiors) on which the elastic bodies are clamped, the boundaries Γ_N^1 and Γ_N^2 with an imposed force density and Γ_C^1 and Γ_C^2 which are the potential contact boundaries, slave and master, respectively. We assume that these boundaries form a partition without boundaries overlapping of $\partial\Omega^1$ and $\partial\Omega^2$.

The two elastic bodies are subjected to force densities (volumic forces if $d = 3$) denoted f^1 and f^2 and on Γ_N^1 and Γ_N^2 to force densities (surface forces if $d = 3$) denoted ℓ^1 and ℓ^2. The focus is now on expressing the contact condition with Coulomb friction. To this end, we consider the slave surface Γ_C^1. For a point $x \in \Gamma_C^1$, the point $y \in \Gamma_C^2$ that potentially comes into contact therewith must be determined. This is called contact pairing. In the contact condition small-strain approximation, this correspondence is determined on the reference configuration and is not questioned during deformation. In general, a projection is used, but it is not the only possible choice. Let this correspondence:

$$\begin{array}{rcl} \Pi : \Gamma_C^1 & \to & \Gamma_C^2 \\ x & \mapsto & y = \Pi(x) \end{array}$$

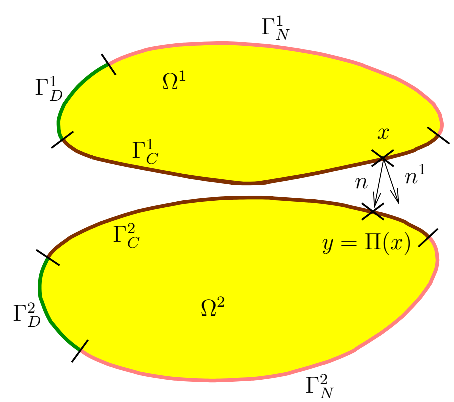

Figure 1: Two bodies with their respective potential contact boundaries

There are then two outward vectors of interest at point x (see Figure 1): the outward unit normal vector of Ω^1, which we will denote by n^1 and the outward unit vector in the direction of $y = \Pi(x)$, which we will denote by n and which can be defined by:

$$n = \begin{cases} \dfrac{y-x}{|y-x|} & \text{if } (y-x)\cdot n^1 > 0 \\ n^1 & \text{if } (y-x)\cdot n^1 = 0 \\ -\dfrac{y-x}{|y-x|} & \text{if } (y-x)\cdot n^1 < 0 \end{cases}$$

since the last two cases (that is, for $(y-x)\cdot n^1 \leq 0$) are expected, either when contact is established in the reference configuration, or if both domains are overlapping, which is *a priori* not prohibited. There is no reason that these two vectors n^1 and n should be equal, in general, except by using the "ray tracing" strategy exposed in 4.1. The vector n is usually called the contact normal. To express the contact condition, one should determine what is the normal component of the stress. Let $u^1 : \Omega^1 \to \mathbb{R}^d$ be the displacement of the first body and $\sigma(u^1)$ its Cauchy stress tensor. So, we will denote

$$\sigma_n = (\sigma(u^1)n^1)\cdot n, \;\; \sigma_t = (I - n\otimes n)(\sigma(u^1)n^1)$$

the normal and tangential component decomposition of the stress on the slave contact boundary. We will also denote:

$$g_0 = (y-x)\cdot n$$

the initial *gap* between the two potential contact surfaces as well as:

$$[\![u]\!] = (u^1 - u^2\circ\Pi), \;\; [\![u_n]\!] = (u^1 - u^2\circ\Pi)\cdot n$$

the jumps of the displacements and of the normal displacements. Therefrom, the non-interpenetration condition, or Signorini condition, can be written on Γ_C^1 as the following complementarity relation:

$$[\![u_n]\!] \leq g_0, \;\; \sigma_n \leq 0, \;\; ([\![u_n]\!] - g_0)\sigma_n = 0 \tag{1}$$

To write the friction condition, a coefficient of friction is of course needed, which will be denoted by $\mathscr{F} \geq 0$ and rigorously a notion of sliding velocity. Here, in a supposedly quasi-static evolution, we will not use a sliding velocity but a tangential displacement increment that we will denote by d_t. (Duvaut and Lions 1972 ; Kikuchi and Oden 1988) use the expression $d_t(u_1,u_2) = (I - n\otimes n)[\![u]\!]$, which leads to a problem, although artificial, which exhibits the same characteristics as that obtained for an expression of d_t that would derive from a time discretization that can be written as:

$$d_t(u_1,u_2) = (I - n\otimes n)([\![u]\!] - [\![u^0]\!])$$

where $\llbracket u^0 \rrbracket$ is the displacement jump at the previous time step. The friction condition is then written as:

$$|\sigma_t| \leq -\mathscr{F}\sigma_n, \quad \text{if } d_t \neq 0 \text{ then } \sigma_t = \mathscr{F}\sigma_n \frac{d_t}{|d_t|} \tag{2}$$

The second Newton law, or action-reaction principle, imposes that:

$$\sigma(u^1)n^1 = \sigma(u^2 \circ \Pi)n^2 \circ \Pi J_\Pi \tag{3}$$

where n^2 is the outward unit normal to Γ_C^2 at point $y = \Pi(x)$ and J_Π is the Jacobian of the transformation Π between the two surfaces Γ_C^1 and Γ_C^2.

The description of the linarized elasticity law is carried out by the intermediate of the small strain tensor $\varepsilon(u) = (\nabla u + \nabla u^T)/2$. The Cauchy stress tensor is then connected to the strain tensor by the 4th-order elasticity tensor $\mathcal{A}$ with the usual symmetry and coerciveness properties. This relationship is written as $\sigma(u) = A\,\varepsilon(u)$. The displacements u^1, u^2 of the two elastic bodies are then subjected to the following equations on Ω^i, $i = 1, 2$ in addition to the contact and friction equations (1), (2) and (3):

$$\left\{ \begin{array}{r} -\text{div}\,\sigma(u^i) = f^i \text{ in } \Omega^i \\ \sigma(u^i) = A\,\varepsilon(u^i) \text{ in } \Omega^i \\ u^i = 0 \text{ on } \Gamma_D^i \\ \sigma(u^i)n^i = \ell^i \text{ on } \Gamma_N^i \end{array} \right. \tag{4}$$

2.2 The classical weak inequality form

The weak formulation in the form of inequality that can be found in (Duvaut and Lions 1972 ; Kikuchi and Oden 1988) for example, can be constructed by introducing the following spaces:

$$V = H^1(\Omega^1; \mathbb{R}^d) \times H^1(\Omega^2; \mathbb{R}^d)$$

$$V_0 = \{v = (v^1, v^2) \in V : v^1 = 0 \text{ on } \Gamma_D^1 \text{ and } v^2 = 0 \text{ on } \Gamma_D^2\}$$

and the set of admissible displacements:

$$K = \left\{v = (v^1, v^2) \in V_0 : \llbracket v_n \rrbracket - g_0 \in K_0\right\}$$
$$K_0 = \{v \in L^2(\Gamma_C^1) : v \leq 0\}$$

The spaces of normal and tangential traces on Γ_C^1 are also introduced as:

$$X_N = \left\{w \in L^2(\Gamma_C^1; \mathbb{R}) : \exists v \in H^1(\Omega^1; \mathbb{R}^d), v{=}0 \text{ on } \Gamma_D^1, w{=}v_{|\Gamma_C^1} \cdot n\right\}$$

$$X_T = \left\{w \in L^2(\Gamma_C^1; \mathbb{R}^{d-1}) : \exists v \in H^1(\Omega^1; \mathbb{R}^d), v{=}0 \text{ on } \Gamma_D^1, w{=}(I{-}n{\otimes}n)v_{|\Gamma_C^1}\right\}$$

as well as their respective topological duals X_N' et X_T'. Let (f_1, f_2) be in $L^2(\Omega^1; \mathbb{R}^d) \times L^2(\Omega^2; \mathbb{R}^d)$, (ℓ^1, ℓ^2) in $L^2(\Gamma_N^1; \mathbb{R}^d) \times L^2(\Gamma_N^2; \mathbb{R}^d)$, the following bilinear et linear forms on V are defined:

$$a(u, v) = \sum_{i=1,2} \int_{\Omega^i} \sigma(u^i) : \varepsilon(v^i)\, d\Omega$$
$$L(v) = \sum_{i=1,2} \int_{\Omega^i} f^i \cdot v^i\, d\Omega + \sum_{i=1,2} \int_{\Gamma_N^i} \ell^i \cdot v^i\, d\Gamma$$

as well as the functional corresponding to the virtual work of the frictional force:

$$j(s,v) = \langle s, |d_t(v)| \rangle_{X'_N, X_N}$$

where $s = -\mathscr{F}\sigma_n(u)$ is the friction threshold. The notation $\langle \cdot, \cdot \rangle_{X'_N, X_N}$ denotes the product of duality between the spaces X'_N and X_N. When s is regular, this product is reduced to the integral $\displaystyle\int_{\Gamma_C^1} s|d_t(v)|d\Gamma$. Therefore, the classical weak form associated with (1)–(2)–(3)–(4) is written as:

$$\left\{ \begin{array}{l} \text{Find } u \in K \text{ such that :} \\ a(u, v-u) - j(\mathscr{F}\sigma_n(u), v) + j(\mathscr{F}\sigma_n(u), u) \geq L(v-u), \qquad \forall v \in K \end{array} \right. \tag{5}$$

When there is no friction ($\mathscr{F} = 0$), the weak problem (5) is a variational inequality of the first kind. Then, the Stampacchia theorem allows us to conclude that it admits a unique solution, which moreover is the only minimizer of the functional $\mathcal{J}(u) = \frac{1}{2}a(u,u) - L(u)$ on the convex set K. In the presence of friction, results of existence could be shown, for example in (Eck *et al.* 2005), given that the coefficient of friction $\mathscr{F}$ is small. A recent result in (Ballard and Iurlano 2023), claims that existence holds for any friction coefficient. Regarding the uniqueness of the solution, counterexamples were presented for large friction coefficients in (Hild 2003, 2004) and a criterion for characterizing the uniqueness of the solution was presented in (Renard 2006). The uniqueness of the solution for a sufficiently small coefficient is still an open problem.

It should be noted that when the friction threshold $s \in X'_N$ is known (this is then referred to as Tresca friction), the weak problem, which is then a variational inequality of the second kind, is equivalent to the minimization of the non-regular functional $\mathcal{J}_s(u) = \mathcal{J}(u) + j(s,u)$ on the convex K. The existence and uniqueness of the solution are also guaranteed, since the functional $j(\cdot,\cdot)$ has the property of being convex and lower semi-continuous compared to its second argument (see for example (Glowinski 1984 ; Kikuchi and Oden 1988)).

2.3 The principle of duality and the weak form with multipliers

The use of Lagrange multipliers leads to transforming a constrained minimization problem into a problem with simple constraints on multipliers. In this case, it also allows the non-regular nature of the norm involved in the expression of the functional $j(\cdot,\cdot)$ to be taken into consideration. Given that Coulomb friction does not naturally derive from a potential, we look into this matter with the problem involving Tresca friction. By using $I_{K_0}(\cdot)$ the characteristic function for the set K_0, a convex function equal to 0 in K_0 and $+\infty$ elsewhere, the solution to the Tresca problem minimizes on V_0 the functional:

$$\tilde{\mathcal{J}}_s(u) = \mathcal{J}(u) + j(s,u) + I_{K_0}(\llbracket u_n \rrbracket - g_0)$$

The principle of dualization consists in introducing an auxiliary variable, here $\mu = (\mu_n, \mu_t) \in X_N \times X_T$, and set:

$$\mathcal{W}(u,\mu) = \mathcal{J}(u) + \langle s, |d_t(u) - \mu_t| \rangle_{X'_N, X_N} + I_{K_0}(\llbracket u_n \rrbracket - g_0 - \mu_n) \tag{6}$$

so that $\mathcal{W}(u,0) = \tilde{\mathcal{J}}_s(u)$ and that $\mathcal{W}(u,\cdot)$ be convex. Therefore, the solution to the Tresca problem is the minimizer of $\mathcal{W}(u,0)$ on V_0 and applying the Fenchel-Legendre conjugate to the functional $\mathcal{W}(u,\cdot)$ (see for instance, (Rockafellar and Wets 1998)), this will be the saddle point of the Lagrangian:

$$\mathscr{L}(u,\lambda) = -\sup_{\mu}\left(\int_{\Gamma_C^1} \lambda\cdot\mu\, d\Gamma - W(u,\mu)\right).$$

In practice, the computation of this conjugate yields the sets:

$$\Lambda_N = \{\lambda_n \in X_N' \,:\, \langle \lambda_n, v_n\rangle_{X_N',X_N} \geq 0,\, \forall v_n \in X_N,\, v_n \leq 0\}$$

$$\Lambda_T(s) = \{\lambda_t \in X_T' \,:\, -\langle \lambda_t, v_t\rangle_{X_T',X_T} - \langle s, |v_t|\rangle_{X_N',X_N} \leq 0,\, \forall v_t \in X_T\}$$

which are respectively the sets of admissible normal and tangential stresses. This leads to obtaining the Lagrangian expression:

$$\begin{aligned}\mathscr{L}(u,\lambda) \;&=\; \mathcal{J}(u) - \langle \lambda_n, [\![u_n]\!] - g_0\rangle_{X_N',X_N} \\ &\quad -\langle \lambda_t, d_t(u)\rangle_{X_T',X_T} - I_{\Lambda_N}(\lambda_n) - I_{\Lambda_T(s)}(\lambda_t)\end{aligned}$$

which is a Lagrangian with constraints on the multiplier that are represented by the presence of characteristic functions $I_{\Lambda_N}(\lambda_n)$ and $I_{\Lambda_T(s)}(\lambda_t)$. This Lagrangian has then simply to be differentiated to obtain the optimality system. Bearing in mind the fact that the subdifferential of the characteristic function of a set is its normal cone (we still refer to (Rockafellar and Wets 1998)), the formulation of the Tresca friction problem can be derived as follows (see also (Laborde and Renard 2008)):

$$\begin{cases} \text{Find } u \in V_0,\ \lambda_n \in \Lambda_N \text{ and } \lambda_t \in \Lambda_T(s) \text{ such that:} \\ a(u,v) = L(v) + \langle \lambda_n, [\![v_n]\!]\rangle_{X_N',X_N} + \langle \lambda_t, [\![v_t]\!]\rangle_{X_T',X_T}, \quad \forall v \in V_0 \\ \langle \lambda_n - \mu_n, [\![u_n]\!] - g_0\rangle \leq 0, \quad \forall \mu_n \in \Lambda_N \\ \langle \lambda_t - \mu_t, d_t(u)\rangle \leq 0, \quad \forall \mu_t \in \Lambda_T(s). \end{cases} \tag{7}$$

A weak form of the problem with Coulomb friction can be then easily obtained by replacing $\Lambda_T(s)$ by $\Lambda_T(-\mathscr{F}\sigma_n(u))$.

2.4 Proximal augmented Lagrangian: principle and use

A drawback of the simple Lagrangian presented in Section 1.2.3 is that removing constraints and non-regular terms on u induces some constraints on the Lagrange multipliers. An augmented Lagrangian makes it possible to obtain an optimality system without constraints. To this end, the formula in (6) has to be modified keeping in mind that any convex quantity can be added to the variable μ provided that the latter cancels out for $\mu = 0$. The proximal augmented

Lagrangian (see (Rockafellar and Wets 1998)) consists in introducing a norm of the additional variable μ, that is, in this case:

$$\mathcal{W}_\gamma(u,\mu) = \mathcal{W}(u,\mu) + \frac{\gamma}{2}\int_{\Gamma_C^1} |\mu|^2 d\Gamma$$

where $\gamma > 0$ is the augmentation parameter. It should be noted that by using a norm in $L^2(\Gamma_C^1)$, only the cases where the multiplier is in this space are taken into consideration. This will be assumed hereafter, and will make it possible to replace duality products by integrals. Using again the Fenchel-Legendre conjugate with respect to the variable μ, the augmented Lagrangian is obtained:

$$\mathscr{L}_\gamma(u,\lambda) = -\sup_{\mu}\left(\int_{\Gamma_C^1} \lambda\cdot\mu \, d\Gamma - W_\gamma(u,\mu)\right)$$

which gives after computation:

$$\begin{aligned}
\mathscr{L}_\gamma(u,\lambda) \qquad &= \mathcal{J}(u) - \int_{\Gamma_C^1} \lambda_n(\llbracket u_n \rrbracket - g_0) d\Gamma - \int_{\Gamma_C^1} \lambda_t \cdot d_t(u) d\Gamma \\
&- \int_{\Gamma_C^1} \frac{1}{2\gamma}\left[\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0) + (\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_-\right]^2 d\Gamma \\
&+ \int_{\Gamma_C^1} \frac{\gamma}{2}(\llbracket u_n \rrbracket - g_0)^2 d\Gamma \\
&- \int_{\Gamma_C^1} \frac{1}{2\gamma}\left|\lambda_t - \gamma d_t(u) - P_{B(s)}(\lambda_t - \gamma d_t(u))\right|^2 d\Gamma \\
&+ \int_{\Gamma_C^1} \frac{\gamma}{2}|d_t(u)|^2 d\Gamma
\end{aligned} \tag{8}$$

where $(x)_- = (|x| - x)/2$ designates the negative part of x and $P_{B(s)}$ is the projection onto the ball of center 0 and radius s. The optimality system of this augmented Lagrangian is the weak form of the following Tresca friction contact problem:

$$\left\{\begin{aligned}
&\text{Find } u \in V_0,\ \lambda \in L^2(\Gamma_C^1;\mathbb{R}^d) \text{ such that for any } v \in V_0 \\
&\ \text{and any } \mu \in L^2(\Gamma_C^1;\mathbb{R}^d) \text{ one has:} \\
&a(u,v) = L(v) - \int_{\Gamma_C^1} (\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_- \llbracket v_n \rrbracket d\Gamma \\
&\qquad + \int_{\Gamma_C^1} P_{B(s)}(\lambda_t - \gamma d_t(u)) \llbracket v_t \rrbracket d\Gamma \\
&\qquad - \frac{1}{\gamma}\int_{\Gamma_C^1} \left[\lambda_n + (\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_-\right] \mu_n d\Gamma \\
&\qquad - \frac{1}{\gamma}\int_{\Gamma_C^1} \left[\lambda_t - P_{B(s)}(\lambda_t - \gamma d_t(u))\right] \cdot \mu_t \, d\Gamma.
\end{aligned}\right. \tag{9}$$

A weak form of the Coulomb friction problem can then be obtained by replacing the threshold s by the friction threshold from Coulomb's law, that is, either with $-\mathscr{F}\lambda_n$, or with $\mathscr{F}(\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_-$.

The mechanism by which the augmented Lagrangian provides an optimality condition that is not a constrained condition is related to the Moreau-Yosida regularization. So to quickly introduce the principle, we examine the optimality condition on the multiplier λ which is obtained by canceling out

$$\partial_\lambda \mathscr{L}_\gamma(u,\lambda) = \partial_\lambda(\mathcal{W}_\gamma^{*\mu}(u,\lambda))$$

where $\mathcal{W}_\gamma^{*\mu}(u,\lambda)$ is the Fenchel-Legendre conjugate of $\mathcal{W}_\gamma(u,\mu)$ with respect to the variable μ in $L^2(\Gamma_C^1)$. Now, for $\varphi : X \to \mathbb{R}$ a convex functional on a Hilbert space X, from the definitions of the conjugate and the subderivative, it follows that: $x \in \partial\varphi(u) \Leftrightarrow -u \in \partial\varphi^*(x)$. Here, for $X = L^2(\Gamma_C^1)$, this gives:

$$0 \in \partial_\lambda \mathscr{L}_\gamma(u,\lambda) \Leftrightarrow -\lambda \in \partial_\mu \mathcal{W}_\gamma(u,0) \Leftrightarrow 0 \in (\partial_\mu \mathcal{W}(u,\cdot) + \gamma I)^{-1}(-\lambda)$$

where I is the identity. The term $(\partial_\mu \mathcal{W}(u,\cdot)+\gamma I)^{-1}$ is called the Moreau-Yosida resolvent of $\partial_\mu \mathcal{W}(u,\cdot)$. If $\partial_\mu \mathcal{W}(u,\cdot)$ is indeed maximal monotone, that is, if $\mathcal{W}(u,\mu)$ is indeed semi-continuous convex with respect to μ, then the resolvent will be single valued and a contraction for all $\gamma > 0$ (see (Brézis 1973)). This implies at least a Lipschitz-continuous regularity to the optimality condition on the multiplier.

As an example, consider the normal part $\mathcal{W}^n(u,\mu_n) = I_{K_0}(\llbracket u_n \rrbracket - g_0 + \mu_n)$. The subderivative of a characteristic function I_C of a convex subset C of a Hilbert space X is $\partial I_C = N_C$ where N_C, the normal cone to C is defined by:

$$N_C(x) = \begin{cases} \{w \in X : \langle w, v - x\rangle \leq 0 \quad \forall v \in C\} & \text{if } x \in \overline{C} \\ \emptyset & \text{otherwise} \end{cases}$$

and in particular satisfies $N_C(x) = \{0\}$ for x in the interior of C. This set is nontrivial only for $x \in \partial C$. It thus follows here that $\partial_{\mu_n} \mathcal{W}^n(u,\mu_n) = N_{K_0}(\llbracket u_n \rrbracket - g_0 + \mu_n)$, and for the standard Lagrangian, we then have:

$$0 \in (\partial_\mu \mathcal{W}^n(u,\cdot))^{-1}(-\lambda_n) \Leftrightarrow g_0 - \llbracket u_n \rrbracket \in N_{\Lambda_N}(\lambda_n)$$

because it can be easily verified that $N_{K_0}^{-1} = N_{\Lambda_N}$ (the two sets K_0 and Λ_N being two mutually polar cones). The condition $g_0 - \llbracket u_n \rrbracket \in N_{\Lambda_N}(\lambda_n)$ is indeed the one represented in the second line of the optimality system of the simple Lagrangian (7) and it can be seen that this condition remains constrained because subjected to a normal cone. Conversely, for the augmented Lagrangian, we take $\mathcal{W}_\gamma^n(u,\mu_n) = \mathcal{W}^n(u,\mu_n) + \dfrac{\gamma}{2}\displaystyle\int_{\Gamma_C^1} |\mu_n|^2 d\Gamma$, and it follows that:

$$0 \in (N_{K_0}(\llbracket u_n \rrbracket - g_0 + \cdot) + \gamma I)^{-1}(-\lambda_n).$$

As the resolvent of a normal cone can be written as a projection (see (Rockafellar and Wets 1998)), it yields here:

$$(N_{K_0}(\llbracket u_n \rrbracket - g_0 + \cdot) + \gamma I)^{-1}(-\lambda_n) = \frac{1}{\gamma}(\lambda_n + (\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_-)$$

and the normal part of the second equation in (9), whose regularity is Lipschitz, finally can be found. In Section 3, it will also be shown how to more directly make use of the relation $\lambda_n + (\lambda_n - \gamma(\llbracket u_n \rrbracket - g_0))_- = 0$, which is a reformulation of the contact conditions (1), weakly imposed in (9), in order to obtain a Nitsche-type formulation.

3 Finite element approximation in small deformations

To achieve a finite element approximation of the contact problem of two elastic bodies, consider two meshes $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$, regular in the sense of Ciarlet (1991), for the two domains Ω^1 and Ω^2. These meshes are made of geometric elements of desired shapes and sizes, such that:

$$\Omega^i = \bigcup_{T \in \mathcal{T}_h^i} T.$$

For the sake of simplicity, it is therefore considered that the domains Ω^1 and Ω^2 are exactly covered, which might not be the case when these domains present curved boundaries that are impossible to cover with the geometric elements under consideration. Moreover, it is also assumed that the meshes comply with the partition of the boundaries into Γ_D^i, Γ_N^i and Γ_C^i.

By V_0^h, one will denote the subspace of V_0 of piecewise regular functions (usually polynomial) on the two meshes $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$. For instance, if the choice is a P_1 Lagrange element, the meshes will consist of triangles in dimension $d = 2$ and tetrahedra in dimension $d = 3$ and the unknowns will be continuous piecewise affine functions.

There are many different ways for approaching the frictional contact problem. An important feature from the practical point of view is whether we are considering matching (or compatible) meshes or not. The meshes will be said to be coinciding (or compatible) if for any finite element node on the slave boundary $a_i \in \Gamma_C^1$ then $\Pi(a_i)$ is also a finite element node of the mesh $\mathcal{T}_h^2$. In the latter case, more direct approximations can be considered, see for example (Wriggers 2002).

It should also be noted that the application of the techniques presented here for the account of the contact condition is not restricted to finite element approximations only, despite that this choice remains one of the most widespread in structural mechanics. They can actually be combined with any other type of discretization based on the weak formulation of the problem (Galerkin methods) and which then allows for an integral formulation of the contact condition. It is particularly possible to use recent NURBS-based (*Non Uniform Rational B-Splines*) isogeometric-type discretizations, which make it easier to link with computer-aided design (see for example (Temizer *et al.* 2011, 2012 ; De Lorenzis *et al.* 2012 ; Kim and Youn 2012 ; Seitz *et al.* 2016 ; Hu *et al.* 2018 ; Antolin *et al.* 2019)) or even polytopal methods, which allow meshes consisting of polygons or polyhedra (see for example (Wriggers *et al.* 2016 ; Cascavita *et al.* 2020 ; Chouly *et al.* 2020)).

This section first presents a state of the art in 3.1 with regard to the results of existence-uniqueness of discrete solutions and their convergence, and the multiplier methods seen in 2.3 are revisited. Then, 3.2 presents a stabilized method that allows more flexibility in choosing the finite element spaces for the displacement fields and the multiplier. From this stabilized method, a primal (without multiplier) and consistent method can be derived, the so called Nitsche's method, which is covered in 3.3. The existing relationship between this method and the augmented Lagrangian presented in 2.4 is detailed in 3.4, then the penalty method is finally found in 3.5 as an approximation of Nitsche's method.

3.1 State of the art, methods with multipliers

First, it should be understood that each continuous weak formulation, such as those described in Section 2 or as penalized or regularized formulations, has its finite element discretized analogue. The state of the art regarding these discrete versions is described below:

1. for frictionless contact, the existence and uniqueness of solutions are guaranteed for any type of "reasonable" discretization. The convergence of discrete solutions with an optimal rate remained an open problem since the 1970s and was only established recently, first for the Nitsche method in 2013 (Chouly and Hild 2013*a*) then for the weak inequality in 2015 (Drouet and Hild 2015) (the latter result could then be applied to other discretizations). The analyses published earlier were in fact suboptimal or otherwise involved additional artificial hypotheses concerning the behavior of the solution on the contact boundary (see in particular (Wohlmuth 2011 ; Drouet and Hild 2015 ; Chouly *et al.* 2017, 2023) for a detailed review of the results related to this matter);

2. for Tresca friction, the existence and uniqueness of solutions is also ensured for any type of "reasonable" discretization. As for the convergence of discrete solutions, it is generally established with suboptimal convergence rates, without knowing whether this suboptimality is due to analysis defect or not, see (Wohlmuth 2011 ; Chouly and Hild 2013*a* ; Chouly 2014 ; Chouly *et al.* 2017, 2023) for a detailed state of the art. In dimension two, there is an optimal result established in (Wohlmuth 2011) but which involves a technical assumption on the behavior of the solution in the contact and friction zone. In dimension three, there is no optimal result for most methods to our knowledge. However, the recent introduction of the Nitsche method (see Section 3.3) has made it possible to produce discrete solutions that optimally converge towards the solution of the continuous problem, both in dimensions two and three (see (Chouly 2014)). Later on an optimal result has been obtained for the penalty formulation in (Chouly

et al. 2023), for low order Lagrange finite elements, and in dimension two and three, that combines both the techniques of (Drouet and Hild 2015) and of (Chouly and Hild 2013*b*);

3. with regard to Coulomb friction, independently of the discretization chosen (weak inequality, multiplier-based method, Nitsche method, etc.) it can generally be shown with fixed-point arguments that the discrete problem admits (at least) one solution, regardless of the value of the coefficient of friction $\mathscr{F}$. In addition, it can be shown that the solution is unique if the coefficient of friction is less than $Ch^{1/2}$, where $C > 0$ is a constant depending on the geometry but not on the mesh used. It is still unclear whether the term in $Ch^{1/2}$ corresponds either to real cases (with multiple solutions that appear when the mesh is refined) or if it is simply a defect of mathematical analysis. Studies concerning the non-uniqueness of discrete solutions were conducted and simple explicit examples of non-uniqueness obtained (see for example (Hild 2002) for examples of multiple solutions with a coefficient of friction greater than one). In addition, examples of non-uniqueness were obtained numerically (since finite element solutions cannot be explained by hand) in (Hassani *et al.* 2003) for given and arbitrarily small friction coefficients. The more sensitive issue of finding a discrete (explicit or not) example with which multiple solutions are obtained for any arbitrarily small coefficient of friction remains open.

To date, the additional results (compared to the above) have mainly been obtained for multiplier methods which have been the subject of most publications in recent decades. These methods consist of a discretization of (7) which, in the case of Coulomb friction, is written as

$$\begin{cases} \text{Find } u^h \in V_0^h,\ \lambda_n^h \in \Lambda_N^h \text{ and } \lambda_t^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h) \text{ such that:} \\ a(u^h, v^h) = L(v^h) + \displaystyle\int_{\Gamma_C^1} \lambda_n^h [\![v_n^h]\!] + \lambda_t^h \cdot [\![v_t^h]\!] d\Gamma, \quad \forall v^h \in V_0^h \\ \displaystyle\int_{\Gamma_C^1} (\lambda_n^h - \mu_n^h)([\![u_n^h]\!] - g_0^h) d\Gamma \le 0, \quad \forall \mu_n^h \in \Lambda_N^h \\ \displaystyle\int_{\Gamma_C^1} (\lambda_t^h - \mu_t^h) \cdot d_t(u^h) d\Gamma \le 0, \quad \forall \mu_t^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h) \end{cases} \tag{10}$$

where $V_0^h, \Lambda_N^h, \Lambda_T^h$ are discrete versions of $V_0, \Lambda_N, \Lambda_T$ (several choices are possible). Historically, the first convergence results were obtained for the formulation type (10) (with displacements approximated with a P_1-Lagrange and piecewise constant multipliers) in (Haslinger 1983). In this reference, convergence is proved under the assumption of the existence of a solution to the continuous problem (that is, if the coefficient of friction is sufficiently small) and the author establishes the existence of a subsequence of discrete solutions converging towards a solution to the continuous problem. Using piecewise linear continuous displacements and multipliers, the reference (Hild and Renard 2007) obtains the convergence of solutions with a velocity in $Ch^{1/2}$ under the assumptions ensuring the existence of a unique solution established in (Renard 2006).

The authors further assume that the solution of the continuous problem is in $H^{3/2+\varepsilon}(\Omega)$ ($\varepsilon > 0$) (see also (Chouly *et al.* 2023) for a similar result, but with a method that involves different discrete convex sets). All previous results were obtained in space dimension two and some extend to dimension three. In addition, almost of all these results were obtained for compatible meshes, and it is reasonable to consider generalizing them to non-compatible meshes. Within this context, there are two effective approaches that have already proven successful for frictionless problems with non-compatible meshes: the first approach is the standard "mortar" approach in which the displacements (in each solid) are continuous and piecewise P_k ($k = 1.2$ in practice) and the multipliers are chosen in the trace space of one of the two meshes and thereby are continuous and piecewise P_k just as displacements (see (Ben Belgacem *et al.* 1999 ; Hild 2000)). The second, more recent approach, called LAC (*Local Average Contact*) differs from the first only in the multipliers that are chosen piecewise P_0 (independently of the degree k chosen for displacements) on macro-meshes (see (Drouet and Hild 2017 ; Abbas *et al.* 2018)). Displacement-multiplier pairs satisfy an independent inf-sup condition of h for both mortar and LAC approaches.

3.2 Absence of inf-sup condition and stabilized methods

An important aspect in the formulation (10) is the need to have an inf-sup condition between the displacement space and the multiplier space in order for the problem to be well posed (at least in the frictionless case). If this condition is not available (not satisfied or too difficult to demonstrate), it is possible to add a term to the previous formulation and this is then referred to as a stabilized multiplier method. For the frictional problem, such a method is written as:

$$\left\{\begin{aligned}
&\text{Find } u^h \in V_0^h,\ \lambda_n^h \in \Lambda_N^h \text{ and } \lambda_t^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h) \text{ such that:}\\
&a(u^h, v^h) - \int_{\Gamma_C^1} \frac{1}{\gamma}(\sigma(u^h)n)\cdot(\sigma(v^h)n)d\Gamma\\
&= L(v^h) + \int_{\Gamma_C^1} \lambda_n^h \left([\![v_n^h]\!] - \frac{1}{\gamma}\sigma_n(v^h)\right) d\Gamma\\
&\quad + \int_{\Gamma_C^1} \lambda_t^h \cdot \left([\![v_t^h]\!] - \frac{1}{\gamma}\sigma_t(v^h)\right) d\Gamma, \quad \forall v^h \in V_0^h\\
&\int_{\Gamma_C^1} (\lambda_n^h - \mu_n^h)([\![u_n^h]\!] - g_0^h)d\Gamma\\
&\quad + \int_{\Gamma_C^1} \frac{1}{\gamma}(\lambda_n^h - \mu_n^h)(\lambda_n^h - \sigma_n(u^h))d\Gamma \le 0, \quad \forall \mu_n^h \in \Lambda_N^h\\
&\int_{\Gamma_C^1} (\lambda_t^h - \mu_t^h)\cdot d_t(u^h)d\Gamma\\
&\quad + \int_{\Gamma_C^1} \frac{1}{\gamma}(\lambda_t^h - \mu_t^h)\cdot(\lambda_t^h - \sigma_t(u^h))d\Gamma \le 0, \qquad \forall \mu_t^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h)
\end{aligned}\right. \tag{11}$$

where $\gamma = \gamma_0/h_T$ is the stabilization parameter in which $\gamma_0 > 0$ is chosen large enough and h_T denotes the diameter of the element T. This is the stabilization originally proposed by Barbosa and Hughes (1992*b*) for variational inequalities, and adapted to the contact framework in (Hild and Renard 2010). Although numerical analysis was only carried out for frictionless contact and with compatible meshes (see (Hild and Renard 2010)), it is reasonable to assume that all the results obtained in unstabilized cases with an inf-sup condition can be redemonstrated for stabilized problems. Recently in (Beaude *et al.* 2023) (suboptimal) error estimates have been derived for Tresca friction, but that are robust with respect to the value of the stabilization parameter for infsup compatible pairs. Still in the case of infsup compatible pairs, (Beaude *et al.* 2023) establish the convergence to the mixed formulation when the stabilization parameter is large. Extension for Coulomb friction have also been studied in (Beaude *et al.* 2023 ; Lleras 2009).

3.3 Nitsche's method seen as a limit stabilized method model

In (Stenberg 1995) (see also (Juntunen 2015 ; Chouly 2023)), for Dirichlet boundary conditions, one can find an explanation of the relationship between the stabilized method of Barbosa and Hughes (1992*a*) and that previously proposed by Nitsche (1971), which can be found from that of Barbosa and Hughes through the local elimination of the multiplier. A similar approach can be followed within the framework of contact and the Nitsche method can be obtained from the previous stabilized formulation. The last two equalities in (11) can indeed be expressed as follows:

$$\left\{\begin{array}{l} \displaystyle\int_{\Gamma_C^1} (\lambda_n^h - \mu_n^h)(\lambda_n^h - \sigma_n(u^h) + \gamma([\![u_n^h]\!] - g_0^h))d\Gamma \leq 0, \quad \forall \mu_n^h \in \Lambda_N^h \\ \displaystyle\int_{\Gamma_C^1} (\lambda_t^h - \mu_t^h)\cdot(\lambda_t^h - \sigma_t(u^h) + \gamma d_t(u^h))d\Gamma \leq 0, \quad \forall \mu_t^h \in \Lambda_T^h(-\mathscr{F}\lambda_N^h) \end{array}\right.$$

or equivalently using projection operators in terms of $L^2(\Gamma_C^1)$:

$$\left\{\begin{array}{l} \lambda_n^h = \mathrm{Proj}_{\Lambda_N^h}\left(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h)\right) \\ \lambda_t^h = \mathrm{Proj}_{\Lambda_T^h(-\mathscr{F}\lambda_N^h)}\left(\sigma_t(u^h) - \gamma d_t(u^h)\right) \end{array}\right.$$

Since the formulation in (11) does not require an inf-sup condition and is well posed irrespective of the choice of the multiplier space (at least for frictionless contact), it can be considered that the discrete space of the multipliers should "tend" to the continuous, that is, $\Lambda_N^h \to \Lambda_N$ and $\Lambda_T^h(-\mathscr{F}\lambda_N^h) \to \Lambda_T(-\mathscr{F}\lambda_N)$, which is equivalent to looking for a first multiplier $\lambda_n^h \leq 0$ and a second multiplier $\lambda_t^h \in B(-\mathscr{F}\lambda_n^h)$. It is therefore formally obtained that:

$$\left\{\begin{array}{l} \lambda_n^h = -\left(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h)\right)_- \\ \lambda_t^h = \mathrm{P}_{B(-\mathscr{F}\lambda_n^h)}\left(\sigma_t(u^h) - \gamma d_t(u^h)\right) \end{array}\right. \tag{12}$$

One should note that the formal expressions of (12) are the discrete counterparts of the relations:

$$\begin{aligned} \sigma_n(u) &= -\big(\sigma_n(u) - \gamma([\![u_n]\!] - g_0)\big)_- \\ \sigma_t(u) &= \mathrm{P}_{B(-\mathscr{F}\sigma_n(u))}\,(\sigma_t(u) - \gamma d_t(u)) \end{aligned} \tag{13}$$

which are strictly equivalent to the contact (1) and friction (2) conditions reformulated as equations (see a formal proof in (Chouly 2014 ; Chouly *et al.* 2023), for example). These reformulations of contact and friction conditions can be directly obtained by deriving the proximal augmented Lagrangian associated with the contact problem, as discussed in Section 2.4. Moreover, it should be noted that the discrete character of multipliers in (12) no longer results from belonging to a finite-dimensional space but from the fact that they can be expressed according to discrete unknowns u^h. The two expressions of (12) are then replaced in the first equation of [11] to obtain Nitsche's formulation. By setting the approximate friction threshold as:

$$s(u^h) = \mathscr{F}(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h))_-$$

one gets:

$$\begin{cases} \text{Find } u^h \in V_0^h \text{ such that for all } v^h \in V_0^h \text{ one has:} \\ a(u^h, v^h) - \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}(\sigma(u^h)n)\cdot(\sigma(v^h)n)d\Gamma \\ = L(v^h) \\ \qquad + \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}\big(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h)\big)_-\big(\sigma_n(v^h) - \gamma[\![v_n^h]\!]\big)\,d\Gamma \\ \qquad - \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}P_{B(s(u^h))}\big(\sigma_t(u^h) - \gamma d_t(u^h)\big)\cdot\big(\sigma_t(v^h) - \gamma[\![v_t^h]\!]\big)\,d\Gamma \end{cases} \tag{14}$$

The formulation in [14] admits a solution if γ_0 is large enough. In addition, this solution is unique if $\mathscr{F}^2\gamma_0 h^{-1}$ is sufficiently small (see (Chouly *et al.* 2019a, 2019b)). An alternative formulation, called mean-Nitsche, can be obtained using static condensation and elimination of the multiplier, when spaces of piecewise constant multipliers are considered (Beaude *et al.* 2023). This formulation is primal, but involves local projection operators of the contact and friction condition on each contact edge or face.

Further analysis in the frictionless case (see (Chouly *et al.* 2015)), or with Tresca friction (see (Chouly 2014)), made it possible to generalize the Nitsche method by introducing an additional parameter denoted by $\theta \in \mathbb{R}$ as Formulation [14] corresponds to $\theta = 1$. This generalization has resulted in distinguishing three cases of interest: when $\theta = 1$ which corresponds on the one hand to the natural case resulting from the stabilized method and which, by its symmetry properties, derives from the energy function:

$$
\begin{aligned}
\mathcal{J}_{\mathrm{N}}(u^h) = \quad & \mathcal{J}(u^h) - \int_{\Gamma_C^1} \frac{1}{2\gamma}\sigma_n(u^h)^2 \; d\Gamma \\
& + \int_{\Gamma_C^1} \frac{1}{2\gamma}(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h))_-^2 \; d\Gamma
\end{aligned}
\tag{15}
$$

(see (Chouly *et al.* 2017) for proof and extension to Tresca friction). The second case of interest corresponds to $\theta = 0$ which leads to a particularly simple formulation, close to an augmented Lagrangian formulation (see Section 3.4) or also to a penalized formulation (see Section 3.5). As for the third case obtained for $\theta = -1$, the condition on γ_0 (quite large) vanishes (see (Chouly *et al.* 2015)), and the method obtained is more robust (the reader can also refer to (Burman *et al.* 2017) for a "penalty free" variant inspired by the method proposed in (Burman 2012) for the Dirichlet condition). When Coulomb friction is added, the analyses become more complicated: these generalized formulations are being studied in (Chouly *et al.* 2022) (see (Chouly *et al.* 2019) for a summary of the results) and are written as:

$$
\left\{
\begin{aligned}
& \text{Find } u^h \in V_0^h \text{ such that } v^h \in V_0^h \text{ one has:} \\
& a(u^h, v^h) - \int_{\Gamma_C^1} \frac{\theta}{\gamma}(\sigma(u^h)n)\cdot(\sigma(v^h)n) d\Gamma \\
& = L(v^h) \\
& \quad + \int_{\Gamma_C^1} \frac{1}{\gamma}\left(\sigma_n(u^h) - \gamma([\![u_n^h]\!] - g_0^h)\right)_- \left(\theta\sigma_n(v^h) - \gamma[\![v_n^h]\!]\right) d\Gamma \\
& \quad - \int_{\Gamma_C^1} \frac{1}{\gamma} P_{B(s(u^h))}\left(\sigma_t(u^h) - \gamma d_t(u^h)\right)\cdot\left(\theta\sigma_t(v^h) - \gamma[\![v_t^h]\!]\right) d\Gamma
\end{aligned}
\right.
\tag{16}
$$

Finally, it should be noted that Nitsche's method, similarly to Barbosa and Hughes' stabilization, only makes sense for the discretized problem and does not admit a counterpart at the continuous level, other than purely formal, unlike most other methods (multipliers, augmented Lagrangian, penalty) that can be written for the continuous problem (see Section 2). On the other hand, it is a consistent method, which does not induce an additional approximation of contact and friction conditions, and it is also a primal method, where the only unknown is the discrete displacement field.

3.4 Relationship between Nitsche and proximal augmented Lagrangian

For problems, where $\theta = 0$, Nitsche's method [16] can be rewritten more simply as:

$$\left\{\begin{aligned} &\text{Find } u^h \in V_0^h \text{ such that:}\\ &a(u^h, v^h) = L(v^h) - \int_{\Gamma_C^1} \left(\sigma_n(u^h) - \gamma(\llbracket u_n^h \rrbracket - g_0^h)\right)_- \llbracket v_n^h \rrbracket d\Gamma\\ &\qquad\qquad + \int_{\Gamma_C^1} P_{B(s(u^h))}\left(\sigma_t(u^h) - \gamma d_t(u^h)\right) \cdot \llbracket v_t^h \rrbracket d\Gamma \end{aligned}\right.$$

Through the introduction of multipliers, as new unknowns, instead of normal and tangential stresses, and by weakly placing contact and friction conditions (13) on these multipliers, we obtain from the previous formulation:

$$\left\{\begin{aligned} &\text{Find } (u^h, \lambda_n^h, \lambda_t^h) \in V_0^h \times X_N^h \times X_T^h\\ &\text{such that for } (v^h, \mu_n^h, \mu_t^h) \in V_0^h \times X_N^h \times X_T^h \text{ we have:}\\ &a(u^h, v^h) = L(v^h) - \int_{\Gamma_C^1} (\lambda_n^h - \gamma(\llbracket u_n^h \rrbracket - g_0^h))_- \llbracket v_n^h \rrbracket d\Gamma\\ &\qquad\qquad + \int_{\Gamma_C^1} P_{B(s(u^h))}(\lambda_t^h - \gamma d_t(u^h)) \cdot \llbracket v_t^h \rrbracket d\Gamma\\ &\qquad\qquad -\frac{1}{\gamma} \int_{\Gamma_C^1} (\lambda_n^h + (\lambda_n^h - \gamma(\llbracket u_n^h \rrbracket - g_0^h))_-)\, \mu_n^h\\ &\qquad\qquad +(\lambda_t^h - P_{B(s(u^h))}(\lambda_t^h - \gamma d_t(u^h))) \cdot \mu_t^h \, d\Gamma = 0 \end{aligned}\right. \tag{17}$$

Here, unlike the previous formulations, the discrete spaces for multipliers, X_N^h, respectively X_T^h, are vector subspaces (without inequality constrains) of X_N, respectively X_T. We then find the discrete version of the proximal augmented Lagrangian Formulation (9) presented in Section 2.4. The use of augmented Lagrangian as an approximation technique has been revisited recently (Burman *et al.* 2019 ; Burman and Hansbo 2017 ; Burman *et al.* 2023), by establishing the close relationship with the Nitsche method (see also (Chouly *et al.* 2017, 2023)). For frictionless contact, in particular, in (Burman *et al.* 2019), the identification of the close connection with the Nitsche method resulted in establishing an optimal convergence result for formulations such as (17).

3.5 The connection between Nitsche and penalty

Still for $\theta = 0$, Nitsche's method [16] can also be written as:

$$\left\{\begin{array}{l} \text{Find } u^h \in V_0^h \text{ such that for any } v^h \in V_0^h \text{ one has:} \\ a(u^h, v^h) = L(v^h) - \displaystyle\int_{\Gamma_C^1} \gamma \left(\frac{1}{\gamma}\sigma_n(u^h) - (\llbracket u_n^h \rrbracket - g_0^h)\right)_- \llbracket v_n^h \rrbracket d\Gamma \\ \qquad\qquad + \displaystyle\int_{\Gamma_C^1} \gamma\, P_{B\left(\frac{1}{\gamma}s(u^h)\right)} \left(\frac{1}{\gamma}\sigma_t(u^h) - d_t(u^h)\right) \cdot \llbracket v_t^h \rrbracket d\Gamma \end{array}\right. \tag{18}$$

Assuming that γ is large enough, the terms in $1/\gamma$ are neglected in the previous formulation (18), which gives the following penalized model:

$$\left\{\begin{array}{l} \text{Find } u^h \in V_0^h \text{ such that } \forall\, v^h \in V_0^h \\ a(u^h, v^h) = L(v^h) - \displaystyle\int_{\Gamma_C^1} \gamma \left(\llbracket u_n^h \rrbracket - g_0^h\right)_+ \llbracket v_n^h \rrbracket d\Gamma \\ \qquad\qquad - \displaystyle\int_{\Gamma_C^1} \gamma\, P_{B\left(\mathscr{F}(\llbracket u_n^h \rrbracket - g_0^h)_+\right)} \left(d_t(u^h)\right) \cdot \llbracket v_t^h \rrbracket d\Gamma \end{array}\right.$$

where $(x)_+ = (|x| + x)/2$ is the positive part of x. It can be verified that this formulation is not fully consistent. This penalized formulation admits a continuous counterpart (see for example (Kikuchi and Oden 1988)) which can be interpreted as a regularization of the contact and friction conditions, whose approximation is more accurate for large γ. This formulation allows in particular an interpenetration proportional to the contact pressure and inversely proportional to the value of γ (see among others the numerical examples presented in (Wohlmuth 2011)). The numerical analysis of this formulation has apparently not been carried out in cases with Coulomb friction. For frictionless problems or involving Tresca friction, the initial analyses of (Kikuchi and Oden 1988) were improved in (Chouly and Hild 2013*b*) and more recently in (Chouly *et al.* 2023 ; Dione 2020).

4 Large strain finite element approximation

We now consider how the augmented Lagrangian and Nitsche methods can be extended within the framework of large elastic strains, and also examine the challenges that this implies. Consider the Lagrangian description, with materials supposed to follow a hyperelastic law (the extension to other constitutive laws can be taken into account as for example in (Seitz 2019 ; Seitz *et al.* 2019) for plasticity and thermoelasticity). We will denote by Ω^1 and Ω^2 the reference configurations of the two elastic solids and $u^i : \Omega^i \to \mathbb{R}^d$, $i = 1, 2$ their displacements. To differentiate between deformed and reference configurations, we adopt the notation conventions shown in Figure 2. Let:

$$\begin{array}{rl} \varphi^i : & \Omega^i \longrightarrow \mathbb{R}^d \\ & X \longmapsto x = X + u^i(X) \end{array}$$

the transformation associated with the elastic solid of index i. We denote also Γ_D^1 and Γ_D^2, the respective boundaries where a clamped condition is prescribed, as well as Γ_N^1 and Γ_N^2 the free boundaries.

The mapping:

$$\begin{aligned} \Pi : \quad & \Gamma_C^1 \longrightarrow \Gamma_C^2 \\ & X \longmapsto Y = \Pi(X) \end{aligned}$$

which connects the points on the slave surface Γ_C^1 to their contact candidate on the master surface Γ_C^2 is obviously no longer a given problem as in the case of small strains, but a mapping that depends on the two displacements u^1 and u^2. We give in Section 4.1 two conventional strategies for defining Π.

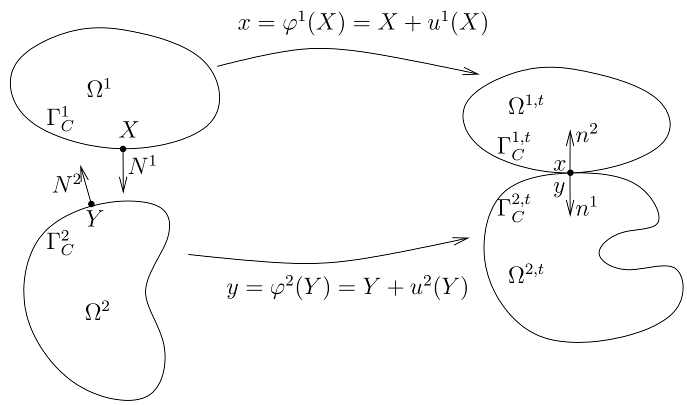

Figure 2: Large-strain Lagrangian description

For each solid i, the strains are described by the Cauchy-Green tensor $C^i = (F^i)^T F^i$ where $F^i = \nabla \varphi^i = I + \nabla_X u^i$ is the strain gradient, and $J^i = \det(F^i)$ the associated Jacobian. The Green-Lagrange strain tensor $E^i = (C^i - I)/2$ will also be used. The Cauchy stress tensor is always denoted by σ^i, and $\hat{\sigma}^i = J^i \sigma (F^i)^{-T}$, $S^i = J^i (F^i)^{-1} \sigma (F^i)^{-T}$ will be the first and second Piola-Kirchhoff stress tensor, respectively. For a hyperelastic law, there exists a potential W^i which depends on the strain through E^i or C^i (see for example (Gurtin 1981 ; Ogden 1984 ; Ciarlet 1988)), such as:

$$S^i = \frac{\partial W^i}{\partial E}(E^i) = 2\frac{\partial W^i}{\partial C}(C^i)$$

To focus on the description of the contact and friction terms, in the following the potential energy of the system $\mathcal{J}(u)$ will be adjusted, and is written for example, in the presence of gravitational forces:

$$\mathcal{J}(u) = \sum_{i=1}^{2} \left(\int_{\Omega^i} W^i(E^i)\, dX - \int_{\Omega^i} \rho_0^i\, \mathcal{g} \cdot u^i\, dX \right)$$

where ρ_0^i is the density in the reference configuration of the body i and $\mathcal{g}$ is the gravitational acceleration vector. Depending on the models, this potential energy may contain other terms, such as terms representing the potential energies of forces at the boundaries Γ_N^1 and Γ_N^2.

The directional derivative of a quantity A with respect to a displacement $u = (u^1, u^2)$ and in the direction $\delta u = (\delta u^2, \delta u^2)$ will be denoted by:

$$\mathcal{D}A(u)[\delta u]$$

where even $\mathcal{D}A[\delta u]$ if there is no ambiguity about the argument of the quantity A. This directional derivative is defined by:

$$\mathcal{D}A(u)[\delta u] = \lim_{\varepsilon \to 0} \frac{A(u^1 + \varepsilon\, \delta u^1, u^2 + \varepsilon\, \delta u^2) - A(u^1, u^2)}{\varepsilon}$$

when this limit exists.

We will first see how can be defined the *gap* function in Section 4.1, then explain the formulation of the contact conditions in Section 4.2. We will then show in Section 4.3 and 4.4 how the augmented Lagrangian and Nitsche methods seen previously can be adapted to the framework of large strains. We will conclude with practical considerations in Section 4.5 about the choice of the numerical parameter γ that can be seen in the different methods, and then by presenting some numerical examples in Section 4.6.

4.1 About contact pairing and *gap* function

The function Π associates a point X on the slave surface with a point Y on the master surface, facing each other. This association can be achieved in different ways. The most conventional strategy consists in using the orthogonal projection of $x = \varphi(X)$ onto the deformed master surface as illustrated in Figure 3a (see for example (Laursen 2002)).

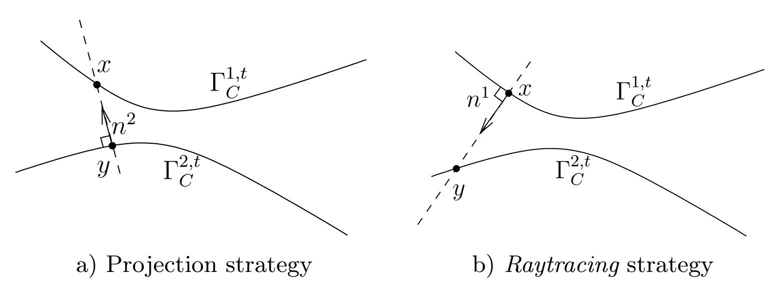

Figure 3: Illustration of projection and raytracing strategies

An alternative strategy, called *raytracing*, corresponding to the description of Figure 3b, consists in defining y as the closest intersection between the master surface and the straight line passing through the point x and carried by the normal n^1 (see for example (Poulios and Renard 2015)). The *gap* functions that correspond to these two policies are then defined by:

$$g = n^1 \cdot (y - x) \;\text{ for raytracing}$$
$$\text{and } g = n^2 \cdot (x - y) \;\text{ for projection}$$

This also gives:

$$y = x + g\ n^1 \text{ for raytracing} \quad (19)$$
$$\text{and } y = x - g\ n^2 \text{ for projection} \quad (20)$$

It is useful for what follows to get the directional derivative of the gap. First of all, we get (see (Poulios and Renard 2015)):

$$\mathcal{D}y[\delta u] = \delta u^2(Y) + F^2\ \mathcal{D}Y[\delta u]. \quad (21)$$

Since $\mathcal{D}Y[\delta u]$ is tangent to Γ_C^2, the vector $F^2\ \mathcal{D}Y[\delta u]$ is tangent to $\Gamma_C^{2;t}$ and thus:

$$n^2 \cdot F^2\ \mathcal{D}Y[\delta u] = 0. \quad (22)$$

Using (19) and (20), it can also be rewritten that:

$$\mathcal{D}y[\delta u] = \delta u^1(X) + \mathcal{D}g[\delta u]\ n^1 + g\ \mathcal{D}n^1[\delta u] \text{ for raytracing} \quad (23)$$

$$\mathcal{D}y[\delta u] = \delta u^1(X) - \mathcal{D}g[\delta u]\ n^2 - g\ \mathcal{D}n^2[\delta u] \text{ for projection} \quad (24)$$

and by combining the expressions (21)–(22)–(23)–(24), the following derivatives of the gap can be obtained:

$$\mathcal{D}g[\delta u] = -\frac{n^2}{n^1 \cdot n^2} \cdot \left(\delta u^1(X) - \delta u^2(Y) + g\ \mathcal{D}n^1[\delta u]\right) \text{ for raytracing} \quad (25)$$
$$\mathcal{D}g[\delta u] = n^2 \cdot \left(\delta u^1(X) - \delta u^2(Y)\right) \text{ for projection.} \quad (26)$$

Computing the derivative of the normal n^1 does not raise any particular problem and gives:

$$\mathcal{D}n^1[\delta u] = -(I - n^1 \otimes n^1)(F^1)^{-T}\, (\nabla \delta u^1(X))^T\, n^1.$$

On the other hand, the derivative of the normal n^2 is much more complex because it depends on the variation of the point y. The computation of the derivative $\mathcal{D}Y[\delta u]$ for the projection gives according to $\mathcal{D}n^2[\delta u]$:

$$\mathcal{D}Y[\delta u] = (F^2)^{-1}\ \left(I - n^2 \otimes n^2\right) \left(\delta u^1(X) - \delta u^2(Y) - g\mathcal{D}n^2[\delta u]\right)$$

whereas for raytracing, with the use of the expression of $\mathcal{D}n^1[\delta u]$, it follows that:

$$\mathcal{D}Y[\delta u] = (F^2)^{-1} \left(I - \tfrac{n^1 \otimes n^2}{n^1 \cdot n^2}\right) \left(\delta u^1(X) - \delta u^2(Y) - g\,(F^1)^{-T}\,(\nabla \delta u^1(X))^T\, n^1\right).$$

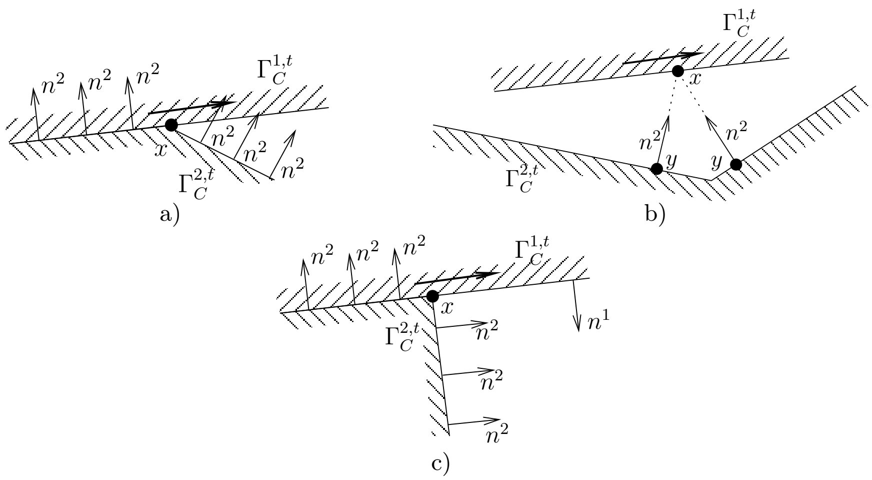

Figure 4: Example of discontinuity of the normal n^2 with respect to x: a) when crossing an element boundary whether for projection or raytracing, b) in the presence of concave boundaries for projection (in this case, even y is discontinuous), c) when raytracing fails (in this case, y becomes undefined)

It can thus be seen that for the projection, it will be necessary to have an expression of the derivative of the normal n^2 with respect to displacement, which, on the one hand, is very complex because these derivatives depend in particular on the curvature of the master surface (see the expressions in (Laursen 2002) for example). On the other hand, it is challenging since the normal n^2 is not continuous with respect to displacement in a number of situations and particularly in the projection case. Non-continuity examples can be seen described in Figure 4. These discontinuities create issues when attempting to make a Newton method converge. One can avoid problem a) of Figure 4 by using finite elements $\mathscr{C}^1$ which ensure the continuity of the normal when crossing the boundary between the elements as in (Padmanabhan and Laursen 2001) with Hermite elements, or as in (Krstulović-Opara *et al.* 2002 ; Stadler *et al.* 2003 ; Lengiewicz *et al.* 2011) with splines, Bézier curves or NURBS, respectively. However, the difficulty remains when modeling angular surfaces. Problem b) is more difficult to avoid and can cause Newton's method to oscillate between two possible positions y. Strategies are then sometimes employed to avoid these oscillations (see for example (Alart 1997)). When using the raytracing strategy, priority is given to the use of the normal n^1, which generally does not raise any discontinuity problem. A example of discontinuity of point y with raytracing is however depicted in illustration c) of Figure 4.

4.2 Formulation of contact and friction conditions

The non-interpenetration condition is simply expressed as $g(u) \geq 0$. The frictional contact condition can either be described with the Cauchy stress or in Piola stress. We choose here a constrained Piola description. The Piola contact stress is therefore decomposed at point X (slave boundary) into normal and tangential part as follows:

$$\hat{\sigma}^1(u^1)N^1 = \underbrace{(\hat{\sigma}^1(u^1)N^1 \cdot n)}_{=\hat{\sigma}_n(u)} n + \underbrace{(I - n \otimes n)\, \hat{\sigma}^1(u^1)N^1}_{=\hat{\sigma}_t(u)}$$

where n is either n^1 for raytracing or $-n^2$ for the projection such that:

$$y = x + gn. \tag{27}$$

It should be noted that when the contact is effective ($\hat{\sigma}_n < 0$), the outward unit normal vectors n^1 and n^2 are opposite. The quantity $\hat{\sigma}_n$ represents the contact pressure (represented in the reference configuration) in X, and must be nonpositive. Contact conditions can be written as:

$$g(u) \geq 0 \tag{28a}$$
$$\hat{\sigma}_n(u) \leq 0 \quad \text{on } \Gamma_C^1 \tag{28b}$$
$$\hat{\sigma}_n(u)g(u) = 0. \tag{28c}$$

For any $\gamma > 0$, these contact conditions (28)–(28a)–(28b) can be rewritten using the following reformulation that is derived from the proximal augmented Lagrangian (see Section 2.4, and see also (13) in Section 3.3):

$$\hat{\sigma}_n(u) = -(\hat{\sigma}_n(u) + \gamma g(u))_-$$

With friction, normal and tangential stresses are coupled to the relative sliding velocity. Relative velocity can be simply defined as:

$$v_r(X) = \dot{\varphi}^1(X) - \dot{\varphi}^2(Y)$$

but this relative velocity does not satisfy the principle of objectivity when the gap is non-zero (see (Curnier *et al.* 1995)). As in (Curnier *et al.* 1995), we can then use the definition:

$$v_r(X) = \dot{\varphi}^1(X) - \dot{\varphi}^2(Y) + g\,\dot{n}$$

which is an objective quantity and which coincides with the sliding velocity when contact is established. This, of course, indicates that we are dealing with a evolution problem. It is possible to revert to a problem without time derivative by achieving a temporal discretization, we will then set:

$$v_r(X) = \frac{1}{\Delta t}(\varphi^1(X) - \varphi^2(Y)) - \frac{1}{\Delta t}(\varphi_0^1(X) - \varphi_0^2(Y))$$

with simple relative velocity and:

$$\begin{aligned} v_r(X) &= \frac{1}{\Delta t}(\varphi^1(X) - \varphi^2(Y) + g\,n\,) - \frac{1}{\Delta t}(\varphi_0^1(X) - \varphi_0^2(Y) + g\,n_0\,) \\ &= -\frac{1}{\Delta t}(\varphi_0^1(X) - \varphi_0^2(Y) + g\,n_0\,) \end{aligned}$$

for objective relative velocity, where the formula in (27) was used to obtain the second line. The notations φ_0^1, φ_0^2 and n_0 respectively denote the deformed and the normal to the previous time step (always depending on the chosen strategy), and Δt is the time step associated with discretization.

Despite that this relative velocity is *a priori* tangential when the contact is persistent in the continuous problem, this will not exactly be the case in the approximate problem, and it might seems more intuitive to define the tangential displacement increment by $d_t = \Delta t(I - n \otimes n)v_r$. For more simplicity, we will hereafter choose:

$$d_t = \Delta t \, v_r$$

The Coulomb friction conditions are then written as ($\mathscr{F}$ still denotes the coefficient of friction):

$$\begin{cases} \|\hat{\sigma}_t(u)\| \leq -\mathscr{F}\hat{\sigma}_n(u) \text{ if } d_t = 0 \\ \hat{\sigma}_t(u) = \mathscr{F}\hat{\sigma}_n(u)\dfrac{d_t}{\|d_t\|} \text{ otherwise.} \end{cases} \tag{29}$$

Using expressions derived from the augmented Lagrangian (see also (13)), the Coulomb friction condition can be reformulated based on the projection $P_{B(\tau)}$. Still for $\gamma > 0$, the friction condition is then equivalent to the non-regular equation:

$$\hat{\sigma}_t(u) = P_{B(-\mathscr{F}\hat{\sigma}_n(u))}(\hat{\sigma}_t(u) - \gamma d_t)$$

4.3 Augmented Lagrangian and penalization

We present here the augmented Lagrangian technique for the contact problem in large transformations previously described. An initial formulation thereof will be first given. It will then be seen that a simpler formulation can be reached if we accept to lose symmetry. The associated finite element formulation will then be described, as well as the Uzawa algorithm for the effective resolution of the problem obtained. Finally, it will be seen how it is possible to find the penalized formulation from the Uzawa algorithm.

4.3.1 Augmented Lagrangian and weak form of the friction problem

It is less direct to obtain a large-strain augmented Lagrangian type formulation for the frictional problem than in the framework of small strains. It is possible to give an analogue of the proximal augmented Lagrangian (8) for large strains that would be for a given frictional threshold problem s:

$$\begin{aligned} &\mathscr{L}_\gamma(u, \lambda_n, \lambda_t) = \mathcal{J}(u) \\ &+ \int_{\Gamma_C^1} \frac{1}{2\gamma}\left((\lambda_n + \gamma g(u))_-^2 - \lambda_n^2\right) d\Gamma - \int_{\Gamma_C^1} \lambda_t \cdot d_t(u) d\Gamma \\ &- \int_{\Gamma_C^1} \frac{1}{2\gamma}\left|\lambda_t - \gamma d_t(u) - P_{B(s)}(\lambda_t - \gamma d_t(u))\right|^2 d\Gamma + \int_{\Gamma_C^1} \frac{\gamma}{2}|d_t(u)|^2 d\Gamma \end{aligned}$$

Here, compared to (8), the property $x(x)_- = (x)_-^2$ was used to simplify the expression associated with the normal stress. Large-strain Tresca friction may seem even more artificial than in small strains since not only is friction allowed when there is no contact but the relative velocity between two potentially very distant points cannot be interpreted as a sliding velocity and highly depends on the chosen contact pairing strategy. If we write the optimality system of this Lagrangian, it follows that:

$$\left\{\begin{array}{l} \mathcal{D}\mathcal{J}(u)[\delta u] - \displaystyle\int_{\Gamma_C^1} (\lambda_n + \gamma g(u))_- \mathcal{D}g(u)[\delta u]\, d\Gamma \\ - \displaystyle\int_{\Gamma_C^1} P_{B(s)}(\lambda_t - \gamma d_t(u)) \cdot \mathcal{D}d_t(u)[\delta u]\, d\Gamma \\ = 0, \quad \forall\, \delta u,\ (\delta u = 0 \text{ on } \Gamma_D^1 \cup \Gamma_D^2) \\ - \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma} \left(\lambda + (\lambda_n + \gamma g)_- n - P_{B(s)}(\lambda_t - \gamma d_t(u))\right) \cdot \delta\lambda\, d\Gamma \\ = 0, \quad \forall\, \delta\lambda \end{array}\right. \tag{30}$$

with $\lambda = \lambda_n n + \lambda_t$. If the second equation of (30) is indeed a weak equivalent of the contact (28) and friction (29) conditions, the interpretation of λ_n and λ_t as a constraint is ensured only if the terms $\mathcal{D}g(u)[\delta u]$ and $\mathcal{D}d_t(u)[\delta u]$ are expressed in terms of relative displacement. This is the case for $\mathcal{D}g(u)[\delta u]$ when using the projection (see Formula (26)) but additional terms are already present when using raytracing (see Formula (25)). Regarding the derivative $\mathcal{D}d_t(u)[\delta u]$, it is written as:

$$\mathcal{D}d_t(u)[\delta u] = \delta u^1(X) - \delta u^2(Y) + (F_0^2 - F^2)\mathcal{D}Y[\delta u]$$

with simple relative velocity and:

$$\mathcal{D}d_t(u)[\delta u] = F_0^2 \mathcal{D}Y[\delta u] - \mathcal{D}g[\delta u] n_0$$

with objective relative velocity. In both cases, the computation involves $\mathcal{D}Y[\delta u]$, which supposes the computation of the second derivative of Y to obtain the tangent problem, as part of Newton's method solution.

4.3.2 Simplified weak form and non-preservation of symmetry

To simplify the expression of (30) one be replace $\mathcal{D}g(u)[\delta u]$ with $-n \cdot (\delta u^1(X) - \delta u^1(Y))$ and $\mathcal{D}d_t(u)[\delta u]$ with $(\delta u^1(X) - \delta u^2(Y))$. This changes the interpretation of the multiplier λ, but not the fact that a unilateral friction contact condition is correctly imposed. To take into account Coulomb friction, we replace the threshold s either by $-\mathscr{F}\lambda_n$, or by $\mathscr{F}(\lambda_n + \gamma g)_-$. The second expression is preferred for example in (Curnier and Alart 1988) and its main advantage is that its sign is always positive irrespective of the value of λ_n during solver iterations. If this second expression is adopted, the weak form of the Coulomb friction problem is written as:

$$\left\{\begin{array}{l} \mathcal{D}\mathcal{J}(u)[\delta u] + \displaystyle\int_{\Gamma_C^1} \big((\lambda_n + \gamma g(u))_- n \\ -P_{B(\mathscr{F}(\lambda_n+\gamma g(u))_-)}(\lambda_t - \gamma d_t(u)\big) \cdot (\delta u^1(X) - \delta u^2(Y)) d\Gamma \\ = 0, \quad \forall\, \delta u, \ (\delta u = 0 \text{ on } \Gamma_D^1 \cup \Gamma_D^2) \\ - \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}\big(\lambda + (\lambda_n + \gamma g(u))_- n \\ -P_{B(\mathscr{F}(\lambda_n+\gamma g(u))_-)}(\lambda_t - \gamma d_t(u))\big) \cdot \delta\lambda \, d\Gamma = 0, \quad \forall\, \delta\lambda \end{array}\right. \tag{31}$$

However, the modification carried out makes that the tangent problem loses symmetry, even when the coefficient of friction $\mathscr{F}$ is zero.

The conservation of symmetry can be a challenge to use potentially more efficient linear solvers. In our case, the dilemma is between using Formulation (30) (previously adapted to Coulomb friction) and using the simplified formulation in (31). The formulation in (31), in addition to its simplicity, has the advantage of providing a simple interpretation of multipliers in terms of contact force density and its redifferentiation is also simpler for obtaining the tangent problem. Nonetheless, its disadvantage resides in the loss of symmetry of the tangent problem. This is not a disadvantage if the loss of symmetry has already been identified, for example due to intrinsically non-symmetric constitutive laws (non-associated plasticity for example), but it can be a disadvantage if it is the only cause of non-symmetry and that symmetric linear solvers are to be used.

In the latter case, the use of Formulation (30) may seem more appealing. However, as mentioned in Section 4.1, in addition to its additional complexity, the derivative of the normal n^2 and the second derivative of Y are also extremely difficult to obtain, because it depends on the curvature of the master surface, and often impossible to compute because of the discontinuities in n^2 and Y. This is the reason why many implementations that use the projection strategy adopt simplified versions of the tangent problem that do not involve the curvature of the master surface (see for example (Konyukhov and Schweizerhof 2012)), at the cost of possibly losing the second-order convergence of the Newton method.

Moreover, because of the unassociated nature of Coulomb's law of friction, the global tangent system will be nonsymmetric. Relatively to the Uzawa algorithm (see Section 4.3.4), strategies have been developed, for example in (Laursen and Simo 1993 ; Konyukhov and Schweizerhof 2012), to recover the symmetry of the tangent problem in displacement. These strategies mainly consist of taking the non-symmetrical parts to the previous iteration of Uzawa's algorithm.

The method chosen in this chapter is to use the simplified formulation of [1.31], even if it implies losing the symmetry of the tangent problem. This choice, which can also be found in (Puso and Laursen 2004 ; Popp *et al.* 2013 ; Poulios and Renard 2015), is driven by the concern to provide a weak formulation as regular as possible in relation to the displacement. Actually, the non-continuity cases of the normal n^2 and the position Y are potentially problematic when using the Newton method.

4.3.3 Finite element approximation

In order to write the finite element approximation of Problem (31), we consider V_0^h and X^h as two finite element spaces; the first space V_0^h is intended to approximate the displacement (as in the previous sections, the index 0 means that the clamped conditions are integrated therein) and the second X^h the multiplier. The latter is therefore restricted to the slave boundary Γ_C^1. As in small strains, the relative choice of these two spaces is subject to whether an inf-sup condition is satisfied (see Section 3.1) in order to guarantee the uniqueness of the multiplier and not to over-constrain the problem. After finite element discretization, the problem (31) is written as:

$$\left\{\begin{array}{l}
\text{Find } u^h = (u^{1,h}, u^{2,h}) \in V_0^h \text{ and } \lambda^h \in X^h \\
\text{such that } \forall\, \delta u^h \in V_0^h \text{ and } \forall\, \delta\lambda^h \in X^h \\
\mathcal{D}\mathcal{J}(u^h)[\delta u^h] \\
+ \displaystyle\int_{\Gamma_C^1} (\lambda_n^h + \gamma g(u^h))_- n \cdot (\delta u^{1,h}(X) - \delta u^{2,h}(Y))\, d\Gamma \\
- \displaystyle\int_{\Gamma_C^1} P_{B(\mathscr{F}(\lambda_n^h + \gamma g(u^h))_-)}(\lambda_t^h - \gamma d_t(u^h)) \\
\cdot(\delta u^{1,h}(X) - \delta u^{2,h}(Y))\, d\Gamma = 0 \\
- \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}\big(\lambda^h + (\lambda_n^h + \gamma g(u^h))_- n \\
P_{B(\mathscr{F}(\lambda_n^h + \gamma g(u^h))_-)}(\lambda_t^h - \gamma d_t(u^h))\big) \cdot \delta\lambda^h\, d\Gamma = 0
\end{array}\right. \tag{32}$$

The numerical resolution of this highly nonlinear discrete problem is usually done using a generalized Newton method (Curnier and Alart 1988 ; Renard 2013) (also called non-regular Newton's method (Christensen 2002)). This requires calculating the tangent problem associated with the previous problem.

The tangent problem for solving by a Newton method is obtained by redifferentiating (32) with respect to displacement and multiplier. This results in a rather complex tangent system due to the large number of terms, including when using the simplifications introduced above. A computer implementation can quickly prove difficult to be realized, the slightest omission or error can degrade the convergence speed of Newton's method. An alternative to direct implementation in a compiled language is to use automation strategies such as those found in the AceGen (Lengiewicz *et al.* 2011) or GetFEM (Renard and Poulios 2020) software program for example, which makes it possible to automatically derive the tangent problem and obtain an efficient computation from a description of the weak form (32).

4.3.4 Numerical resolution using Uzawa's algorithm

For solving Problem (32) by a Newton method, a simultaneous displacement and Lagrange multiplier solution has to be performed. An alternative widely employed in structural mechanics softwares is based on Uzawa's algorithm (see for example (Simo and Laursen 1992), in this context). We refer to (Bertsekas

1982 ; Kunisch and Stadler 2005 ; Laborde and Renard 2008) for the principle of the Uzawa algorithm for a Lagrangian or an augmented Lagrangian. In this case, the algorithm can be written as:

0) choose an initial value λ_0^h for the multiplier and a forward step $r > 0$;

1) solve the first equation of (32) in displacement, for the multiplier set to the value $\lambda^{h,k}$. Let $u^{h,k}$ be the solution obtained;

2) calculate $\lambda^{h,k+1}$ based on the following updates:

$$\lambda_n^{h,k+1} = \lambda_n^{h,k} + \frac{r}{\gamma}\left(-(\lambda_n^{h,k} + \gamma g(u^{h,k}))_- - \lambda_n^{h,k}\right) \tag{33}$$

$$\lambda_t^{h,k+1} = \lambda_t^{h,k} + \frac{r}{\gamma}\left(P_{B(\mathscr{F}(\lambda_n^{h,k} + \gamma g(u^{h,k}))_-)}(\lambda_t^{h,k} - \gamma d_t(u^{h,k})) - \lambda_t^{h,k}\right) \tag{34}$$

3) loop in step 1 by incrementing k until a convergence criterion is satisfied.

Algorithm 1.1. *The Uzawa algorithm*

Since step 1 of the algorithm consists of solving a nonlinear problem, it is usually performed using a Newton method. Step 2 corresponds to a fixed-step gradient method $r > 0$ on the multiplier λ^h. For the continuous problem, not approximated by finite elements, the update given by Equations (33) and (34) gives a fixed point in $L^2(\Gamma_C^1)$ which in (Stadler 2004 ; Kunisch and Stadler 2005) was shown to be always convergent for the Tresca friction problem (in small strains), when $r = \gamma$. In practice, this choice is the one that is always made. Moreover, the fixed point admits a contraction constant that is all the smaller as $r = \gamma$ is chosen large (and therefore converges all the faster). On the other hand, it is a 1st-order convergence to be compared with the 2nd-order convergence of Newton's method.

With regard to the finite element problem, Equations (33) and (34) are obviously not satisfactory as such because the result of the right-hand side calculation are not necessarily in the finite element space X^h. A first solution is to project them again, which yields (for $r = \gamma$): find $\lambda^{h,k+1} \in X^h$ such that for all $\delta\lambda^h \in X^h$ we have:

$$\int_{\Gamma_C^1} \big(\lambda^{h,k+1} + (\lambda_n^{h,k} + \gamma g(u^{h,k})_- n \\ - P_{B(\mathscr{F}(\lambda_n^{h,k} + \gamma g(u^{h,k}))_-)}(\lambda_t^{h,k} - \gamma d_t(u^{h,k}))\big) \cdot \delta\lambda^h \, d\Gamma = 0$$

This is equivalent to considering the second equation of Problem (32) by taking all the terms at iteration k except the first λ^h, taken at iteration $k+1$. The price to pay is an inversion of an additional linear system reduced to the slave contact boundary. The cost of this resolution, however, is mostly negligible compared to the Newton method when solving the first equation.

The additional cost of this resolution is avoided in many references by updating the degrees of freedom of the multiplier (see for example (Simo and Laursen 1992)). Denoting by $\lambda_i^{h,k+1}$ the contact constraint at the finite element node $a_i \in \Gamma_C^1$ and $\lambda_{t,i}^{h,k+1}$, $\lambda_{n,i}^{h,k+1}$ the normal and tangential parts, this update is written as:

$$\begin{aligned}
\lambda_{n,i}^{h,k+1} &= -(\lambda_{n,i}^{h,k} + \gamma g(u^{h,k}(a_i)))_- \\
\lambda_{t,i}^{h,k+1} &= P_{B(\mathscr{F}(\lambda_{n,i}^{h,k}+\gamma g(u^{h,k}(a_i)))_-)}(\lambda_{t,i}^{h,k} - \gamma d_t(u^{h,k}(a_i))).
\end{aligned}$$

This usually represent an additional approximation and means that the contact and friction condition is locally imposed on the finite element node of the multiplier. The use of bi-orthogonal bases in (Popp *et al.* 2012, 2013) makes it possible to perform this operation without additional approximation.

4.3.5 Connection to penalty approximation

In addition to its good convergence properties, one of the reasons for using the Uzawa algorithm for augmented Lagrangian resides in its proximity to the problem with penalization. Indeed, if we take $\lambda^{h,0} = 0$, the first iteration of the Uzawa algorithm consists in finding $u^h = (u^{1,h}, u^{2,h})$ as a solution of:

$$\left\{ \begin{array}{l}
\mathcal{DJ}(u^h)[\delta u^h] + \displaystyle\int_{\Gamma_C^1} (\gamma g(u^h))_- n \cdot (\delta u^{1,h}(X) - \delta u^{2,h}(Y))\, d\Gamma \\
+ \displaystyle\int_{\Gamma_C^1} P_{B(\mathscr{F}(\gamma g(u^h))_-)}(\gamma d_t(u^h)) \cdot (\delta u^{1,h}(X) - \delta u^{2,h}(Y))\, d\Gamma \\
= 0, \quad \forall\, \delta u^h \in V_0^h
\end{array} \right. \tag{35}$$

which is precisely a problem with penalized contact and friction conditions; γ is now the penalty parameter (see Section 3.5). Therefore, it is possible to solve the problem with penalization (35), then to perform, on a poorly satisfied contact condition or friction condition criterion one or more iterations of the Uzawa algorithm to obtain a better solution.

4.4 Nitsche's method

We saw in Section 3.3 that the Nitsche method could be interpreted as a stabilized Lagrangian method in which a multiplier condensation operation was performed. Although the stabilized Lagrangian method makes it possible to perform this condensation quite simply, in special cases of multiplier spaces (see (Chouly 2023 ; Gustafsson *et al.* 2017, 2019)), it should be noted that such strategies also exist on the augmented Lagrangian for constant multipliers or bi-orthogonal bases in (Popp *et al.* 2012, 2013 ; Seitz 2019), the condensation being however limited to the tangent problem.

The Nitsche method will be first presented, deriving it from an energy functional to find its symmetric variant $\theta = 1$. We will then see that in the context of contact between several elastic solids (or self-contact), it is possible to write an unbiased variant of this method.

4.4.1 Differentiation from a potential

For the frictionless problem, it is possible to obtain a Nitsche method starting from the potential (15) obtained in small strains (see (Chouly *et al.* 2017 ; Mlika *et al.* 2017)) which is transposed here to the framework of large strains:

$$\mathcal{J}_{\mathrm{N}}(u^h) = \mathcal{J}(u^h) - \int_{\Gamma_C^1} \frac{1}{2\gamma}\hat{\sigma}_n(u)^2 \; d\Gamma \; + \int_{\Gamma_C^1} \frac{1}{2\gamma}(\hat{\sigma}_n(u) + \gamma g(u))_-^2 \; d\Gamma$$

where $\gamma > 0$ is again the Nitsche parameter. The optimality system of this potential is written as:

$$\begin{cases} \mathcal{D}\mathcal{J}(u^h)[\delta u^h] \\ -\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}\hat{\sigma}_n(u^h)\mathcal{D}\hat{\sigma}_n(u^h)[\delta u^h] \; d\Gamma \\ -\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}(\hat{\sigma}_n(u^h) + \gamma g(u^h))_-\mathcal{D}(\hat{\sigma}_n(u^h) + \gamma g(u^h))[\delta u^h] \; d\Gamma \; = 0, \\ \forall\, \delta u^h \in V_0^h \end{cases}$$

The term $\mathcal{D}\hat{\sigma}_n[\delta u]$ appears here in which can be seen the derivative of the normal:

$$\mathcal{D}\hat{\sigma}_n[\delta u] = \mathcal{D}((\hat{\sigma}N)\cdot n)[\delta u] = (\mathcal{D}\hat{\sigma}[\delta u]N)\cdot n + (\hat{\sigma}N)\cdot \mathcal{D}n[\delta u]$$

This term may obviously seem problematic since it involves the tangent rigidity $\mathcal{D}\hat{\sigma}[\delta u]$ in the weak form, which implies, rather unusually, the computation of the derivative of this tangent rigidity to get the expression of the tangent problem.

Still for the frictionless problem, the different variants of the Nitsche method are then obtained quite naturally for $\theta \in \mathbb{R}$:

$$\begin{cases} \mathcal{D}\mathcal{J}(u^h)[\delta u^h] \\ -\displaystyle\int_{\Gamma_C^1} \frac{\theta}{\gamma}\hat{\sigma}_n(u^h)\mathcal{D}\hat{\sigma}_n(u^h)[\delta u^h] \; d\Gamma \\ -\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma}(\hat{\sigma}_n(u^h) + \gamma g(u^h))_-\mathcal{D}(\theta\hat{\sigma}_n(u^h) + \gamma g(u^h))[\delta u^h] \; d\Gamma \; = 0, \\ \forall\, \delta u^h \in V_0^h \end{cases}$$

In this context, the variant $\theta = 0$ seems especially interesting because it makes the term $\mathcal{D}\hat{\sigma}[\delta u]$ vanish, of course always at the cost of a non-symmetric tangent problem.

To obtain the Nitsche method in problems with Coulomb friction, it seems preferable to start from the simplified augmented Lagrangian Formulation (32). Actually, replacing the multiplier λ by $\hat{\sigma}N$ directly yields the variant $\theta = 0$, then by generalizing for any $\theta \in \mathbb{R}$, leads then to obtaining (see also (Mlika *et al.* 2017)):

$$\begin{cases} \mathcal{D}\mathcal{J}(u^h)[\delta u^h] - \displaystyle\int_{\Gamma_c^1} \frac{\theta}{\gamma}(\hat{\sigma}^h N) \cdot (\mathcal{D}\hat{\sigma}^h[\delta u^h]N)\ d\Gamma \\ -\displaystyle\int_{\Gamma_c^1} \frac{1}{\gamma}\left((\hat{\sigma}_n^h + \gamma g^h)_- n - P_{B(\mathscr{F}(\hat{\sigma}_n^h+\gamma g^h)_-)}(\hat{\sigma}_t^h - \gamma d_t^h)\right) \\ \cdot\Big(\theta \mathcal{D}\hat{\sigma^h}[\delta u^h]N - \gamma(\delta u^{1,h}(X) - \delta u^{2,h}(Y))\Big)\ d\Gamma\ = 0, \qquad \forall\, \delta u^h \in V_0^h \end{cases}$$

As in the case of the augmented Lagrangian, this formulation avoids normal differentiation terms but at the cost of losing the symmetry of the tangent problem, even when there is no friction ($\mathscr{F} = 0$).

It should also be noted that other variants of the Nitsche method can be obtained through slightly different principles. It can be seen for example in (Burman *et al.* 2017) in small strains, and also within the framework of large strains in (Seitz 2019 ; Seitz *et al.* 2019), where a family is introduced with an additional parameter covering the method presented here.

4.4.2 Unbiased variant

An approximation method is said to be unbiased if no difference is made between master surface and slave surface (see for example (Sauer and De Lorenzis 2015)). In this case, we denote by $\Gamma_C = \Gamma_C^1 \cup \Gamma_C^2$ the total potential contact surface. We will always denote by X the current point of Γ_C and $Y \in \Gamma_C$ the point found by way of the projection strategy or raytracing, which is obviously no longer necessarily on the master surface Γ_C^2. It is then easy to see that an unbiased formulation is simply obtained by adding factors 1/2 and extending the integrals over any Γ_C (see (Mlika *et al.* 2017)):

$$\begin{cases} \mathcal{D}\mathcal{J}(u^h)[\delta u^h] - \dfrac{1}{2}\displaystyle\int_{\Gamma_C} \frac{\theta}{\gamma}(\hat{\sigma}^h N) \cdot (\mathcal{D}\hat{\sigma}^h[\delta u^h]N)\ d\Gamma \\ -\dfrac{1}{2}\displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\left((\hat{\sigma}_n^h + \gamma g^h)_- n - P_{B(\mathscr{F}(\hat{\sigma}_n^h+\gamma g^h)_-)}(\hat{\sigma}_t^h - \gamma d_t^h)\right) \\ \cdot\Big(\theta \mathcal{D}\hat{\sigma^h}[\delta u^h]N - \gamma(\delta u^{1,h}(X) - \delta u^{2,h}(Y))\Big)\ d\Gamma\ = 0, \qquad \forall\, \delta u^h \in V_0^h \end{cases}$$

The advantage of this formulation, in addition to providing additional symmetry, lies mainly when searching for self-contact. The lack of separation in the contact surface into master part and slave part implies that no *a priori* at the actual point where the contact will occur is necessary. There is of course an additional cost related to the need to integrate over the entire contact surface.

4.5 About the value of the parameter γ

The parameter γ which appears as a penalization parameter in the penalized formulation in (35), as an augmentation parameter in the augmented Lagrangian formulation of (32) and as a Nitsche parameter in the formulation in (35) plays nonetheless a rather similar role in all three approaches. Although the numerical solution is relatively unaffected by the value of the parameter γ in the augmented Lagrangian and Nitsche formulations, a minimum value must be respected with regard to the Nitsche method to preserve the problem coercivity. For the penalty method, this will mean that a good compromise will have to be found between a large value of γ that will ensure a good approximation of the contact and friction conditions and a moderate value that does not affect the convergence of the Newton method. Despite that the concerns are different in the three approaches, the optimal values of the parameter γ are yet similar. Actually, *a priori* error analyses within the framework of small strains, both for the Nitsche method (Chouly *et al.* 2015) and for the first-order convergence of the penalty method (Chouly and Hild 2013*b*), bring forward a dependency on $\gamma = \gamma_0/h$ where h is the mesh size. In addition, the study by (Renard 2013) and the dimension analysis carried out in (Poulios and Renard 2015) in the context of the augmented Lagrangian lead to the conclusion that γ_0 has the dimension of an elastic modulus. It is therefore quite natural to choose for γ:

$$\gamma = K/h \text{ where } K = \frac{1}{3}\frac{E}{1-2\nu} = \lambda + \frac{2}{3}\mu$$

denotes the *bulk modulus*, which is equal to Young's modulus E when the Poisson coefficient ν is equal to $1/3$ (λ and μ denote here the Lamé coefficients).

Naturally, and mainly for the Nitsche method for large strains, this value may not be sufficient to ensure coercivity. Indeed, when the deformation is very important, the value of γ_0 should rather be connected to the maximum value of the tangent moduli of elasticity. One way to proceed consists then, as proposed in (Seitz *et al.* 2019), in adapting the value of γ_0 to the maximum eigenvalues of the elementary tangent stiffness matrices.

4.6 Numerical tests

We reproduce here numerical tests that were performed using GetFEM, a public domain software library (Renard and Poulios 2020) both in (Poulios and Renard 2015) following an augmented Lagrangian strategy and in (Mlika *et al.* 2017) with the unbiased Nitsche method previously presented. These results obtained with both methods are very similar. They are further corroborated by (Seitz 2019) where a similar study is presented.

The concern for the numerical integration on the contact boundary can be included in the practical aspects for the implementation of these methods, whether for the augmented Lagrangian or the Nitsche method. This is common to all mortar-type methods, where an integration of quantities between two incompatible meshes (those of the slave surface and the master surface) is necessary. Given that integrals involve piecewise polynomials, optimal integration *a priori* requires sub-slicing the interface that complies with both meshes. When performed, this is usually achieved by sub-slicing the faces of the elements of the slave surface in accordance with a projection of the elements of the master surface in a deformed configuration as in (Puso and Laursen 2004). This sub-slicing is not only quite complex but must *a priori* be performed at each iteration of the Newton method. In addition, when curved elements (isoparametric or isogeometric) are employed, or simply for elements with a degree greater than one, the intersections of the element faces become excessively complex and then need to be simplified.

In a linear case with transmission conditions, a comparison is achieved in (Lacour *et al.* 1997). It also proposed non-symmetric integration method, which was then adapted in (Mlika 2018) for the Nitsche method. Although it is clear that the sub-sub-slicing of slave faces for integration provides additional accuracy, it is not clear whether the gain obtained is large enough to justify a step that remains computationally expansive. Therefore, in (Farah *et al.* 2015), a comparison is made between an integration without slicing, with full slicing and with partial slicing. The partial slicing strategy consists of slicing only those elements that satisfy a discontinuity of the master surface. In the tests performed, partial slicing offers approximately the same accuracy as total slicing. In (Mlika 2018), tests were carried out with total slicing and no slicing at all. In most of these tests performed with linear elements, total slicing does not lead to substantial gain in accuracy. However, a greater gain is reported for the use of quadratic elements. The tests presented in the following were carried out without slicing.

4.6.1 Elastic half-ring

This test was introduced in (Fischer and Wriggers 2005). An elastic ring composed of two layers of neo-hookean hyperelastic materials has the following strain energy:

$$W(C) = \frac{\mu}{2}(i_1(C) - 3) + \frac{\lambda}{4}(i_3(C) - 1) - (\frac{\mu}{2} - \frac{\lambda}{4}) \ln(i_3(C))$$

where λ, μ are the Lamé coefficients of the material, $C = F^T F$ is still the Cauchy-Green tensor, and $i_1(C) = \text{trace}\ (C)$ and $i_3(C) = \det\ (C)$ are the first and third invariants of C. The outer ring has a Young's modulus of 10^3 MPa and the inner ring of 10^5 MPa. The Poisson coefficient is 0.3 for both materials. This significant difference in stiffness creates difficulties for the penalty method (see (Fischer and Wriggers 2005)). This half ring is pushed against an elastic block with a Young modulus of 300 MPa and a Poisson coefficient of 0.3. The dimensions are shown on the drawing at the top of Figure 5 which also specifies the geometry of the bodies in contact. The ends of the half ring are fixed and moved vertically 70 mm downwards in 140 steps of 0.5 mm each.

Figure 5 shows four strain states at different loading steps. Graphs a) correspond to frictionless contact while graphs b) correspond to Coulomb friction contact and $\mathscr{F} = 0.5$. The graphs presented correspond to the use of the variant $\theta = 0$ of the unbiased Nitsche method for a Nitsche parameter $\gamma = \gamma_0/h$ with h the mesh size, γ_0 equal to the Young modulus of the material (10^3 MPa for the half ring and 300 MPa for the block) and the use of the raytracing pairing strategy. In order to be able to compare the results obtained to those of the existing literature, the vertical displacement of the midpoint of the ring is plotted on Figure 6 in both cases, with or without friction. The results obtained with Nitsche are in very good agreement with those produced in (Poulios and Renard 2015) with an augmented Lagrangian formulation.

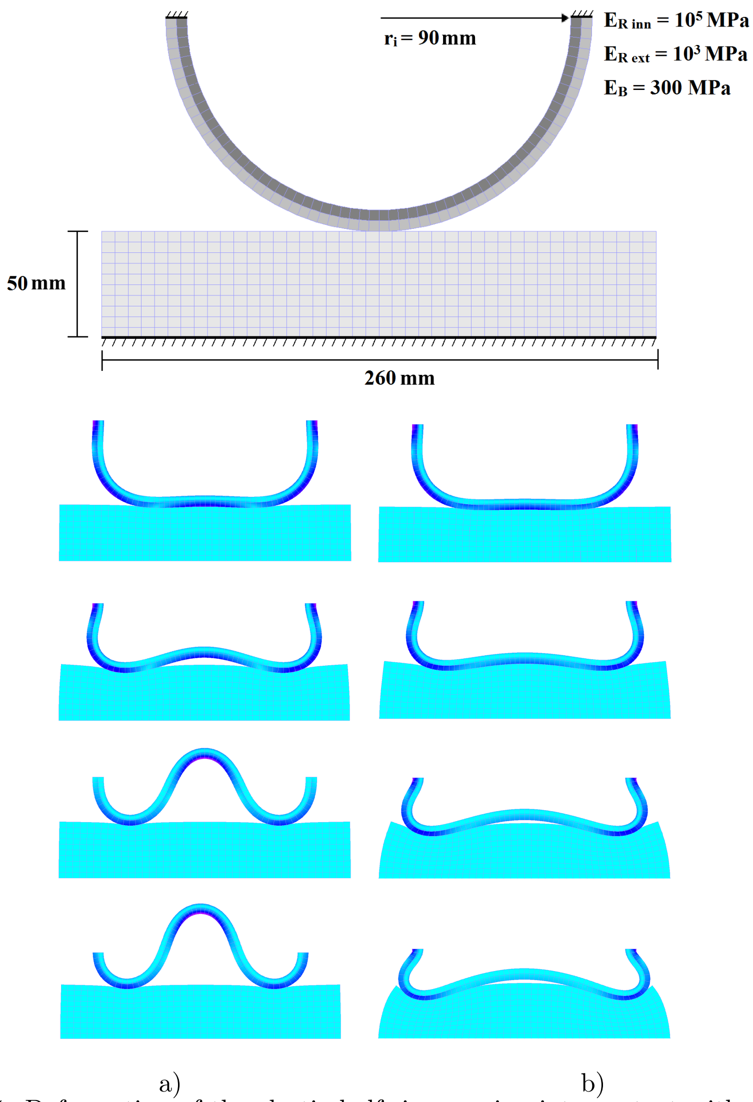

Figure 5: Deformation of the elastic half-ring coming into contact without (a) and with friction for $\mathscr{F}$=0.5 (b) for quadratic quadrilateral finite elements, at loading steps corresponding to 25, 45, 60 and 70 mm of penetration

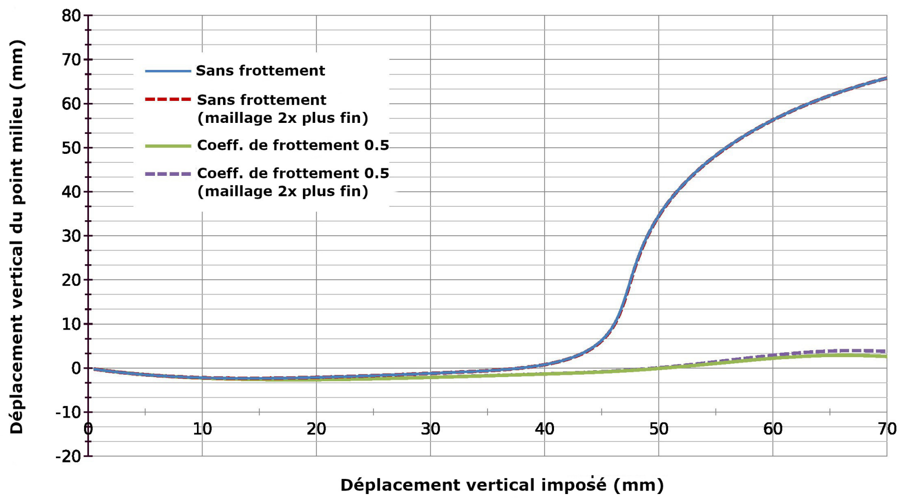

Figure 6: Vertical displacement of the half-ring midpoint for different mesh sizes

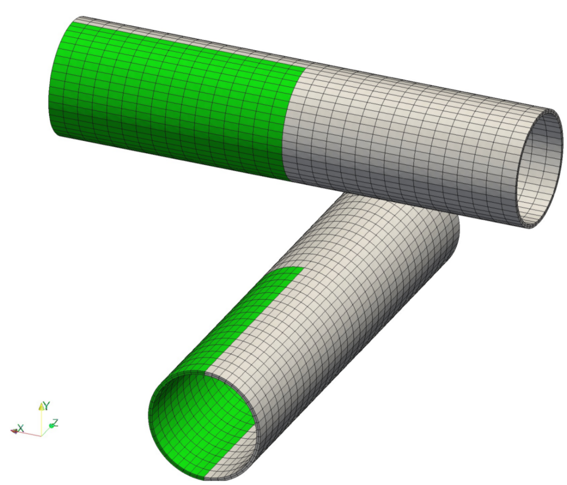

Figure 7: Geometry and mesh of hollow tubes in their undeformed configurations

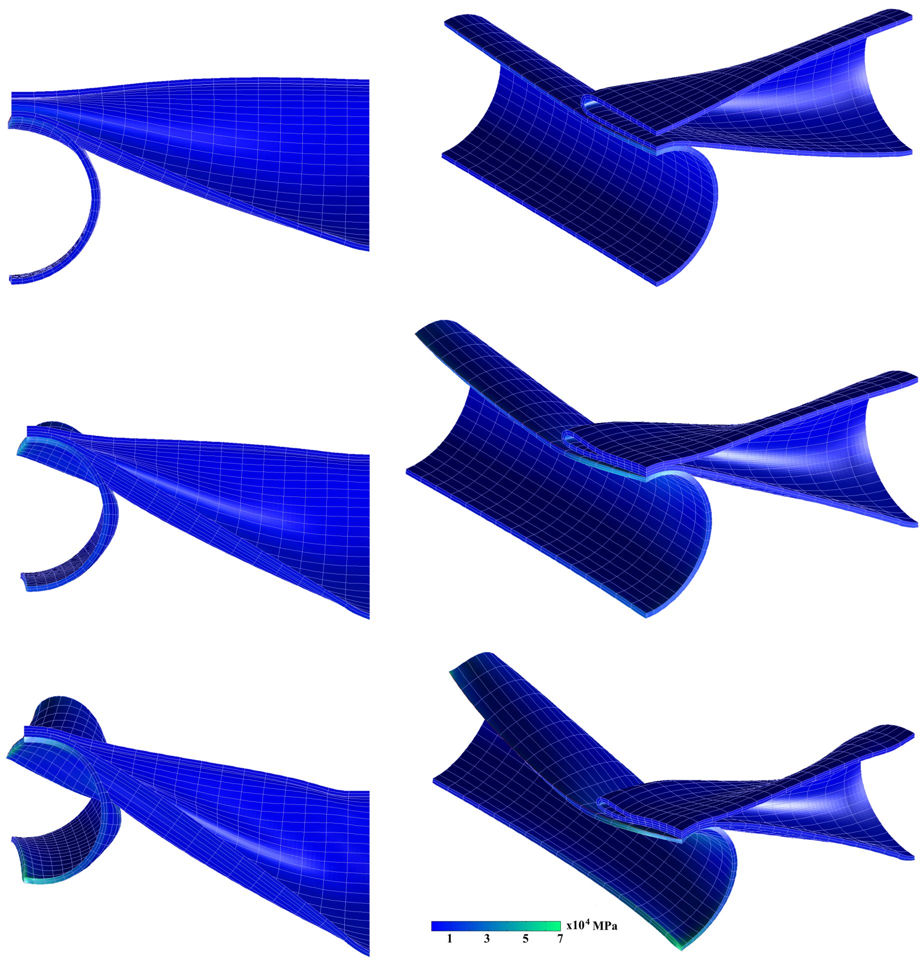

Figure 8: Von-Mises strain and stress of the two crossed tubes. Frictionless example for a penetration of 20 mm, 30 mm and 40 mm respectively

4.6.2 Crossing hollow tubes with self-contact

In this example, we study the contact between two crossing hollow tubes leaning on each other. The tubes are 100 mm long, with an outer diameter of 24 mm and wall thickness of 0.8 mm. The material is considered to be neo-hookean with a Young modulus $E_1 = 10^5$ MPa for the bottom tube and $E_2 = 10^4$ MPa for the top tube. The Poisson coefficient is 0.3 for both tubes. Quadratic hexahedral elements are used with two elements in the thickness.

The meshes that were used are visible in Figure 7 in which are represented in green the parts where the computation actually takes place; the symmetry of the problem is used to reduce the computation area. The ends of the tubes are subject to imposed displacement conditions, zero for the bottom tube and downward for the top tube with a total imposed displacement of 40 mm divided into 80 equal loading steps. This brings the two tubes into contact and induces strains large enough to observe self-contact inside the top tube, as shown in Figure 8. The variant $\theta = 0$ of the unbiased Nitsche method is used with all the inner and outer surfaces of the tubes as contact surface. The Nitsche parameter employed is still $\gamma = \gamma_0/h$ with γ_0 the Young modulus of the solid on which we integrate and h the mesh size.

In Figure 8, it can be seen that the self-contact condition is correctly taken into account despite the very large deformations that are involved. Here too, the results are in very good agreement with those obtained for the augmented Lagrangian on the same test case in (Poulios and Renard 2015).

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