Shape sensitivity analysis of an elastic contact problem: convergence of the Nitsche based finite element approximation

Élie Bretin * Julien Chapelat † Charlie Douanla-Lontsi ‡ Thomas Homolle §
Yves Renard ¶

January 13, 2023

Abstract

In a recent work, we introduced a finite element approximation for the shape optimization of an elastic structure in sliding contact with a rigid foundation where the contact condition (Signorini's condition) is approximated by Nitsche's method and the shape gradient is obtained via the adjoint state method. The motivation of this work is to propose an a priori convergence analysis of the numerical approximation of the variables of the shape gradient (displacement and adjoint state) and to show some numerical results in agreement with the theoretical ones. The main difficulty comes from the non-differentiability of the contact condition in the classical sense which requires the notion of conical differentiability.

Keywords : unilateral contact, linearized elasticity, Nitsche's method, finite element method, shape optimization, conical derivative, shape gradient, adjoint state method, sensitivity analysis.

1 Introduction

In many industrial applications, shape optimization has become an essential tool to improve the quality and performance of mechanical structures. In some contexts, complexity arises while the mathematical formulations involve non-linear or non-differentiable terms. In this study, the motivation is based on the shape optimization of an elastic structure in sliding contact via a gradient descent strategy that requires in particular the shape derivative of the optimization criterion. Unfortunately, the introduction of a non-linear frictionless contact condition in the mechanical problem leads to a tricky formulation of the shape gradient. The elastic problem with sliding contact becomes an elliptic variational inequality whose differentiation is difficult to obtain especially since it is not well defined in the classical sense. We refer to [1] for an overview of shape optimization results for contact problems.

A first approach consists in defining a weak notion of the differentiability, the so-called conical differentiability initially introduced by F. Mignot in [2], leading to optimality conditions. We refer to the work of J. Sokolowski and J.-P. Zolesio [3, 4, 5, 6, 7, 8]. A way to get optimality conditions is to consider a sequence of penalized problems (see for instance [9, 10, 11] and for numerical

*Univ Lyon, INSA Lyon, UJM, UCBL, ECL, CNRS UMR 5208, ICJ, F-69621, France. email: elie.bretin@insa-lyon.fr

†Univ Lyon, INSA Lyon, CNRS UMR5259, LaMCoS, F-69621, France. email: julien.chapelat@insa-lyon.fr

‡MFP MICHELIN, Campus RDI Ladoux, France. email: charlie.douanla-lontsi@michelin.com

§MFP MICHELIN, Campus RDI Ladoux, France. email: thomas.homolle@michelin.com

¶Univ Lyon, INSA Lyon, UJM, UCBL, ECL, CNRS UMR 5208, ICJ, CNRS UMR5259, LaMCoS, F-69621, France. email: yves.renard@insa-lyon.fr

applications see [12, 13, 14]). B. Chaudet and J. Deteix prove the conical differentiability of the solution to the contact problem using the penalization method in [15] and the augmented Lagrangian method in [16].

A second approach to deal with the non-differentiability in the classical sense consists in formulating the discrete variational inequality and then differentiating the discrete formulation. We refer to the work of J. Haslinger et al. [17, 18, 19, 20, 21, 22, 23] where the mechanical problem is approximated by the finite element method. In particular, a convergence analysis is performed in [24] according to the discretization parameter. Penalising [17, 25, 26, 10, 27] or regularising [28, 29] the contact condition can make it easier to obtain the shape derivatives, at the cost of an additional approximation.

While friction is considered in the contact conditions, the derivation is even more tricky. The Tresca model for friction is studied in [4] and a conical derivative is reached for specific directions and only in a two-dimensional framework. Some results are given for Coulomb friction in [18].

In the recent work [30], we are interested in the optimization of an elastic structure under contact conditions while trying to minimize criteria that couple compliance terms and additional terms allowing pressure uniformizations. We propose the use of Nitsche-based methods [31] to efficiently discretize the contact terms. The optimization of the elastic structure is also performed using gradient descent strategy where the gradient is estimated via the adjoint state method applied directly on the discrete formulation of the problem. Although the proposed method allows us to obtain convincing structure optimization, no results of convergence analysis about the discretization of the adjoint state problem were given in [30]. The aim of this paper is therefore to analyze and propose a first result in this direction.

First of all, in Section 2, we recall the elastic formulation with the contact problem. We recall then, in Section 3, some results about the conical directional differentiability of the solution to the contact problem and the link with the shape gradient mainly following [16, 32]. In a second step, as in [30], we present in Section 4 the discretization of the adjoint state problem consisting in applying the adjoint state method on the discrete Nitsche version of the direct problem. Unfortunately, we note a lack of consistency of this approach. We then consider alternatively the discrete Nitsche-based approximation of the continuous adjoint state. We then show an a priori convergence result of this numerical discretization under assumptions of convergence rate of the discrete contact area. By slightly modifying the Nitsche-based formulation of the adjoint state, we introduce a method that allows to dispense with this assumption. Finally, numerical experiments will illustrate in Section 5 these convergence results on the discretization of the adjoint state.

2 Formulation of the contact problem

We consider a linearly elastic structure occupying in its reference configuration a domain $\Omega \subset \mathbb{R}^d, d = 2$ or 3 whose shape is to be optimized. An example is depicted in Figure 1. The boundary $\partial\Omega$ of Ω is split into three non-overlapping parts, Γ_N, Γ_D and Γ_C. We consider a Neumann condition on Γ_N, where a force density g_N is prescribed, and a homogeneous Dirichlet condition on Γ_D. Moreover, Γ_D is supposed to be of non-zero Lebesgue measure to ensure the coercivity of the elastic problem. A frictionless contact might occur with a flat and horizontal rigid obstacle on Γ_C. Let n_y be the inward unit vector to the rigid flat obstacle and g be the initial gap at each point $x \in \Gamma_C$, i.e. the distance function to the obstacle (see Figure 1). It is defined by

$$g = n_y \cdot (y - x),$$

where y is the orthogonal projection of x upon the obstacle. We adopt the following decomposition into normal and tangential components for displacement fields and contact stresses on Γ_C:

$$u_n = u \cdot n_y, \qquad u_t = (I - n_y \otimes n_y)u,$$

$$\sigma_n(v) = (\sigma(v)\ n) \cdot n_y, \qquad \sigma_t(v) = (I - n_y \otimes n_y)(\sigma(v)\ n),$$

where n is the outward unit vector to Ω.

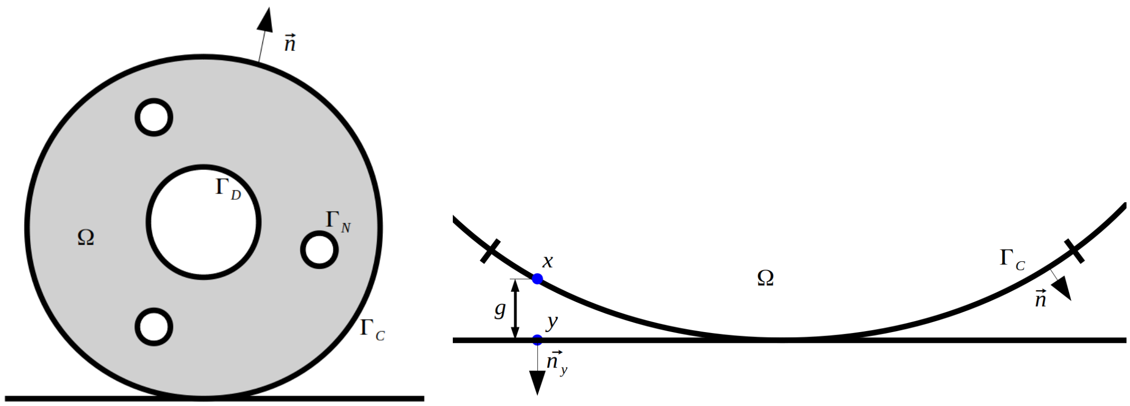

Figure 1: Left: schematic representation of a possible domain Ω. Right: Contact surface representation.

Then, the unilateral contact problem consists in finding $u_\Omega : \Omega \to \mathbb{R}^d$ the displacement of the body according to its reference configuration as the solution to the following problem:

$$\begin{cases} \text{- div } \sigma(u_\Omega) = f & \text{in } \Omega \text{ with } \sigma(u_\Omega) = A\,\varepsilon(u_\Omega), \\ \sigma(u_\Omega)\ n\ = g_N & \text{on } \Gamma_N, \\ u_\Omega\ = 0 & \text{on } \Gamma_D, \\ (u_n - g) \leq 0, \sigma_n(u) \leq 0, (u_n - g)\ \sigma_n(u) = 0 & \text{on } \Gamma_C, \end{cases} \tag{1}$$

where A is the fourth order symmetric tensor of elasticity, $\varepsilon(u) = (\nabla u + \nabla u^T)/2$ is the small deformation tensor and f is an external volumic force density. Assuming the isotropy of the material and denoting μ and λ the constant positive Lamé material parameters, the tensor A reads

$$A\,\varepsilon(u) = 2\mu\varepsilon(u) + \lambda tr(\varepsilon(u))I.$$

Let us introduce the Hilbert space V and the convex cone K of admissible displacements satisfying the non-penetration condition on the contact boundary Γ_C:

$$V := \{v \in H^1(\Omega; \mathbb{R}^d) | v = 0 \text{ on } \Gamma_D\},\ K := \{v \in V | v_n - g \leq 0 \text{ on } \Gamma_C\},$$

where here and in the rest of the paper, $H^s(\Omega)$ denotes the usual Hilbert functional space (see [33], for instance). In order to derive the weak formulation, we introduce two applications $a : V \times V \to \mathbb{R}$ and $\ell : V \to \mathbb{R}$, defined for all $(u, v) \in V \times V$ by

$$a(u, v) = \int_\Omega A\varepsilon(u) : \varepsilon(v)\ \mathrm{d}x, \qquad \ell(v) = \int_\Omega f(x) \cdot v\ \mathrm{d}x + \int_{\Gamma_N} g_N \cdot v\ \mathrm{d}s(x).$$

We deduce from the previous assumptions that $a(\cdot, \cdot)$ is a bilinear, V-elliptic and continuous form on $V \times V$ and $\ell(\cdot)$ is a linear continuous form on V. The weak formulation of Problem (1) in case of frictionless contact reads as a variational inequality (see [34, 35, 36, 37]):

$$\begin{cases} \text{Find } u_\Omega \in K \text{ such that} \\ a(u_\Omega, v - u_\Omega) \geq \ell(v - u_\Omega), \quad \forall v \in K. \end{cases} \tag{2}$$

Under standard assumptions, the existence and uniqueness of the solution to problem (2) is a direct consequence of Stampacchia's theorem (see [38]). Moreover, the solution to (2) is the unique minimizer on K of the functional

$$\inf_{v\in K} \varphi(v) := \inf_{u\in K} \frac{1}{2}a(u,u) - \ell(u).$$

Contact conditions are often approximated in numerical application using the penalty method, which has the advantage of simplicity and robustness at the price of a supplementary approximation. Another classical strategy is the use of Lagrangian or augmented Lagrangian formulations which are fully consistent in contrary to the penalty approach but requires supplementary unknowns (the Lagrange multipliers) and the satisfaction of inf-sup conditions. In this work, we consider a third approach, namely Nitsche's method, which is also fully consistent and avoid the use of supplementary unknowns.

3 Geometric shape optimization

The geometric shape optimization aims at minimizing a criterion $J(\Omega) = J(\Omega, u(\Omega))$. It explicitly depends on the domain Ω, but also implicitly on the solution u_Ω to Problem (2). For each part of the boundary Γ_C, Γ_D and Γ_N, it is supposed that a part is non-optimizable, denoted $\Gamma_C^{no}, \Gamma_D^{no}$ and Γ_N^{no}, the remaining parts Γ_C^o, Γ_D^o and Γ_N^o being optimizable. To preserve the coervicity of the problem, it is supposed that Γ_D^{no} is of non-zero Lebesgue measure. Let $\mathcal{D} \subset \mathbb{R}^d$ be a fixed bounded and smooth domain having $\Gamma_C^{no}, \Gamma_D^{no}$ and Γ_N^{no} as part of its boundary. The shape optimization consists in minimizing the criterion $J(\Omega)$ on the set of admissible domains composed of all smooth (of class $\mathscr{C}^1$) open domains $\Omega \subset \mathcal{D}$ accompanied with a partition Γ_C, Γ_D and Γ_N of its boundary with the constraint $\Gamma_C^{no} \subset \Gamma_C, \Gamma_D^{no} \subset \Gamma_D$, and $\Gamma_N^{no} \subset \Gamma_N$ (see Figure 1). The generic formulation for the target criterion can be expressed as

$$J(\Omega) = \int_\Omega \mathcal{M}(u_\Omega)\, \mathrm{d}x + \int_{\partial\Omega} \mathcal{N}(u_\Omega)\, \mathrm{d}s(x), \tag{3}$$

where the properties of $\mathcal{M}$ and $\mathcal{N}$ will be specified later. In the following, we denote $\Gamma_m = \Gamma_C^o \cup \Gamma_D^o \cup \Gamma_N^o$ the optimizable (moving) boundary.

3.1 Notions of shape derivative

We recall here some results coming mainly from [16, 32]. The differentiation with respect to the domain aims at modifying the reference state of the domain Ω using the boundary method first described by J. Hadamard in [39] and then developed for instance in [40, 41, 42, 4, 43]. Let $\Theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d) \cap \mathscr{C}^1(\mathbb{R}^d)$ be a vector field displacing the reference domain Ω towards different admissible shapes Ω_t. The associated transported domain Ω_t in the direction Θ is defined by

$$\Omega_t = (Id + \Theta)(\Omega),$$

for Θ small enough so that $Id + \Theta$ is a diffeomorphism (see for instance [44]). Then the classical notion of differentiability in Banach spaces can define shape differentiability. We refer to [45] for the different notions of differentiability. We recall the definition of a conical derivative as expressed in [2].

Definition 1. *Let V_1 and V_2 be two Banach spaces. A continuous function $u : V_1 \to V_2$ admits a conical derivative at x if there exists an operator $Q : V_1 \to V_2$ positively homogeneous such that:*

$$\forall h \in V_1, \forall t \geq 0, u(x + th) = u(x) + tQ(h) + o(t).$$

For $u_\Omega \in V$ the solution of a variational formulation posed on Ω, there are two ways to define the derivative of u according to Ω as proposed for instance in [32]: a Lagrangian and an Eulerian one. First we define the Lagrangian derivative or material derivative following the point x during its transportation by the diffeomorphism $I_d + \Theta$.

Definition 2. *Let V be a reflexive Banach set and assume that $u_\Omega(x) \in V$, and $u_{(I_d+\Theta)\Omega}(x + \Theta(x)) \in V$. We call $d_\Omega u[\Theta]$, the directional Lagrangian derivative of $u_\Omega(x)$ in the direction Θ, the linear form in Θ from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ to V satisfying:*

$$u_{(I_d+\Theta)\Omega}(x + \Theta) = u_\Omega(x) + d_\Omega u[\Theta] + o(\Theta).$$

where $o(\Theta)$ is to be understood as

$$\lim_{\Theta \to 0} \frac{\|o(\Theta)\|_V}{\|\Theta\|_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}} = 0.$$

The other definition refers to the Eulerian derivative or shape derivative which is more easy to use but causes additional difficulties to be properly defined. There is actually no difficulty if we define it for a point x belonging to both Ω and $(I_d + \Theta)(\Omega)$. Yet it is much more intricate for points located in the boundary $\partial\Omega$ which do not belong to $(I_d + \Theta)(\Omega)$ or its boundary. We only differentiate the point values of $u(x)$, without carrying the points on the boundary which does not lead to rigorous definitions of functional space for u and its derivative.

Definition 3. *We call $\mathcal{D}_\Omega\, u[\Theta]$, the directional Eulerian derivative of $u_\Omega(x)$, the linear form in Θ that satisfies:*

$$u_{(I_d+\Theta)\Omega}(x) = u_\Omega(x) + \mathcal{D}_\Omega\, u[\Theta] + o(\Theta).$$

Note that while the additional condition $\nabla u_\Omega \cdot \Theta \in V$ holds for $\Theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d) \cap \mathscr{C}^1(\mathbb{R}^d)$, we use the following notation of the directional Eulerian or shape derivative of an element u according to Ω in the direction Θ:

$$\mathcal{D}_\Omega\, u[\Theta] = d_\Omega u[\Theta] - \nabla u_\Omega \cdot \Theta. \tag{4}$$

The relation (4) correctly defines the Eulerian derivative, preventing from the difficulties previously mentioned. Finally, we note that the solution u_Ω is directionally shape differentiable if it admits a directional derivative for any admissible direction Θ. In case the map $\Theta \mapsto \mathcal{D}_\Omega\, u_\Omega[\Theta]$ is positively homogeneous from $\mathscr{C}^1(\mathbb{R}^d)$ to V, u_Ω is conically differentiable. Finally, this map is shape differentiable if it is linear continuous from $\mathscr{C}^1(\mathbb{R}^d)$ to V.

3.2 Shape differentiability

It is known that the projection operator used in the contact condition is not Fréchet-differentiable, the consequence being that u_Ω is not differentiable in a classical sense. However, it has been proved that the solution u_Ω of (2) admits conical material derivative and conical shape derivatives, as for instance [5] for the Signorini's problem.

In view of Zolésio-Hadamard structure theorem, we make the usual choice to limit the geometric deformation fields $\Theta \in \mathscr{C}^1(\mathbb{R}^d)$ along the direction of the normal n (see [16] for instance). The vector n is extended to $\mathscr{C}^1(\mathbb{R}^d)$ as $\partial\Omega$ is assumed to have $\mathscr{C}^1$ regularity. In order to perform a domain transport, the variables must have a certain regularity for usual reasons of differentiability. This is the aim of the following assumption.

Assumption 4. *$f \in H^1(\Omega;\mathbb{R}^d)$ and $g \in H^2(\Omega;\mathbb{R}^d)$.*

We suppose also as in [16, 15] that for $u_\Omega \in H^{(\frac{3}{2}+\nu)}(\Omega)$ with $\nu \in]0,1[$. This implies in particular $\sigma_n(u_\Omega) \in L^2(\Gamma_C)$. The contact boundary Γ_C is split into three parts (with $\sigma_n(u_\Omega)$ a particular representative of its class in $L^2(\Gamma_C)$):

- $\Gamma_{C,a} := \{x \in \Gamma_C | \sigma_n(u_\Omega) < 0, (u_\Omega)_n = g\}$, the active set, or effective contact area,
- $\Gamma_{C,i} := \{x \in \Gamma_C | \sigma_n(u_\Omega) = 0, (u_\Omega)_n < g\}$, the inactive set, or non-contact area,
- $\Gamma_{C,b} := \{x \in \Gamma_C | \sigma_n(u_\Omega) = 0, (u_\Omega)_n = g\}$, the bi-active set, or grazing contact area.

Theorem 5. *Under Assumption 4, the solution u_Ω of (2) is conically shape differentiable with respect to the domain Ω and its conical shape derivative $\mathcal{D}_\Omega\, u[\Theta]$ in the direction Θ satisfies $\mathcal{D}_\Omega\, u[\Theta] \in S(K_0)$ and*

$$a(\mathcal{D}_\Omega\, u[\Theta], \phi - \mathcal{D}_\Omega\, u[\Theta]) \geq \ell'(\phi - \mathcal{D}_\Omega\, u[\Theta])[\Theta] - a'(u_\Omega, \phi - \mathcal{D}_\Omega\, u[\Theta])[\Theta], \qquad \forall \phi \in S(K_0), \qquad (5)$$

where $S(K_0) = \{\phi \in V | \phi_n \leq 0$ a.e. on $\Gamma_{C,a} \cup \Gamma_{C,b}$ and $(a(u_\Omega, \phi) = \ell(\phi))\}$ and where

$$\begin{aligned} a'(u,v)[\Theta] &= \int_{\Gamma_m} (\Theta \cdot n)\ A\varepsilon(u) : \varepsilon(v)\ \mathrm{d}s(x), \\ \ell'(v)[\Theta] &= \int_{\Gamma_m} (\Theta \cdot n)\ f \cdot v\ \mathrm{d}s(x) + \int_{\Gamma_m \cap \Gamma_N} (\Theta \cdot n)\ (\kappa_m g_N \cdot v + \nabla(g_N \cdot v) \cdot n)\ \mathrm{d}s(x). \end{aligned}$$

Here Γ_m is still the optimizable boundary of Ω and κ_m is the mean curvature of $\partial\Omega$.

The proof can be found in [32], Section 5.2. Note that Formulation (5) relies on the set $S(K_0)$ that is not easy to handle. It is however possible to rewrite this formulation as a standard optimization problem under the assumption that there exists no isolated point (see [46]):

Assumption 6. $\Gamma_{C,a} \cup \Gamma_{C,b} = \overline{\mathrm{int}(\Gamma_{C,a} \cup \Gamma_{C,b})}$.

Theorem 7. *Under assumptions 4 and 6, $\mathcal{D}_\Omega\, u[\Theta]$ is solution of (5) if and only if it solves:*

$$\inf_{\phi \in K_{\Gamma_{C,a}}} \frac{1}{2} a(\phi,\phi) - \ell'(\phi)[\Theta] + a'(u_\Omega, \phi)[\Theta],$$

where $V_{\Gamma_{C,a}} := \{\phi \in V | \phi_n = 0$ a.e. on $\Gamma_{C,a}, \phi = 0$ a.e. on $\Gamma_D\}$ and $K_{\Gamma_{C,a}} := \{\phi \in V_{\Gamma_{C,a}} | \phi_n \leq 0$ a.e. on $\Gamma_{C,b}\}$.

The proof can be found in [16] and shows in particular that $S(K_0) = K_{\Gamma_{C,a}}$ in that case. Some additional results can then be obtain in the case $K_{\Gamma_{C,a}} = V_{\Gamma_{C,a}}$, which implies the use of the following assumption:

Assumption 8. *The subset $\Gamma_{C,b}$ is of zero Lebesgue measure in Γ_C.*

The non-differentiability coming from the points in $\Gamma_{C,b}$, the analysis can be simplified when the assumption 8 is considered.

Remark 9. *An element $x \in \Gamma_{C,b}$ is a point where $(u_\Omega)_n = g$ and $\sigma_n(u_\Omega) = 0$ at the same time which means that contact occurs with a vanishing contact pressure. The set $\Gamma_{C,b}$ is often referred as the set of grazing contact. Assumption 8 is verified while the set of grazing contact points is a zero measure set between contact and non contact areas. Interestingly, this corresponds, in fact, to most of the practical situations.*

Theorem 10. *Under assumptions 4 and 6 and if in addition Assumption 8 holds, then* u_Ω *solution of* (2) *is shape differentiable in* $L^2(\Omega)$*. Its shape derivative in the direction* Θ *denoted* $\mathcal{D}_\Omega\, u[\Theta]$ *is defined as the unique solution of*

$$a(\mathcal{D}_\Omega\, u[\Theta], \phi) = \ell'(\phi)[\Theta] - a'(u_\Omega, \phi)[\Theta],\ \forall \phi \in V_{\Gamma_{C,a}}. \tag{6}$$

The proof can be found in [47] section 1.3.3.

3.3 Shape gradient formulation

Still considering the generic formulation for a criterion in (3) given by $J(\Omega) = \int_\Omega \mathcal{M}(u_\Omega)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}(u_\Omega)\,\mathrm{d}s(x)$, we assume that the two functions $\mathcal{M}$ and $\mathcal{N}$ are in $\mathscr{C}^1(\mathbb{R}^d)$ and their derivatives $\mathcal{M}'$ and $\mathcal{N}'$ are Lipschitz-continuous.

Suppose Ω is of class $\mathscr{C}^2$ and Assumption 4 holds, then $J(\Omega)$ is also conically shape differentiable at Ω and its derivative in the direction $\Theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ reads (see [32]):

$$\begin{aligned}\mathcal{D}\,J(\Omega)[\Theta] \ &= \int_\Omega \mathcal{M}'(u_\Omega)\cdot \mathcal{D}_\Omega\, u[\Theta]\,\mathrm{d}x + \int_{\partial\Omega} (\Theta\cdot n)\mathcal{M}(u_\Omega)\,\mathrm{d}s(x) \\ &+ \int_{\partial\Omega} (\mathcal{N}'(u_\Omega)\cdot \mathcal{D}_\Omega\, u[\Theta] + (\Theta\cdot n)\,(\kappa_m\,\mathcal{N}(u_\Omega) + \nabla\mathcal{N}(u_\Omega)\cdot n))\,\mathrm{d}s(x).\end{aligned} \tag{7}$$

From a numerical point of view, this expression of the shape derivative is difficult to use in the sense that it does not allow to define a gradient algorithm. Therefore, in order to isolate a quantity independent of Θ and get rid of the Eulerian derivative, we classically introduce the adjoint state variable $p_\Omega \in V_{\Gamma_{C,a}}$ solution to the following problem:

$$a(v, p_\Omega) = -\int_\Omega \mathcal{M}'(u_\Omega)\cdot v\,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega)\cdot v\,\mathrm{d}s(x),\ \forall v \in V_{\Gamma_{C,a}}. \tag{8}$$

The corresponding strong formulation is the following:

$$\begin{cases} -\mathrm{div}\,(\sigma(p_\Omega)) = -\mathcal{M}'(u_\Omega) & \text{in } \Omega, \\ \sigma(p_\Omega)n \ = -\mathcal{N}'(u_\Omega) & \text{on } \Gamma_{C,b}\cup\Gamma_{C,i}\cup\Gamma_N, \\ p_\Omega \ = 0 & \text{on } \Gamma_D, \\ (p_\Omega)_n \ = 0 & \text{on } \Gamma_{C,a}, \\ \sigma_t(p_\Omega) \ = -(\mathcal{N}'(u_\Omega))_t & \text{on } \Gamma_{C,a}. \end{cases} \tag{9}$$

This allows to rewrite the shape derivative of J in (7) for $v = \mathcal{D}_\Omega\, u$ as

$$\begin{aligned}\mathcal{D}\,J(\Omega)[\Theta] \ &= -a(\mathcal{D}_\Omega\, u[\Theta], p_\Omega) \\ &+ \int_{\partial\Omega} (\Theta\cdot n)\mathcal{M}(u_\Omega)\,\mathrm{d}x + \int_{\partial\Omega} (\Theta\cdot n)\,(\kappa_m\,\mathcal{N}(u_\Omega) + \nabla\mathcal{N}(u_\Omega)\cdot n)\,\mathrm{d}s(x).\end{aligned} \tag{10}$$

Considering Assumption 8 and taking $\phi = p_\Omega$ in (6), it holds

$$\begin{aligned}\mathcal{D}\,J(\Omega)[\Theta] \ &= \int_{\Gamma_m} (\Theta\cdot n)\,(\mathcal{M}(u_\Omega) + A\varepsilon(u_\Omega) : \varepsilon(p_\Omega) - f(x)\cdot p_\Omega)\,\mathrm{d}s(x) \\ &+ \int_{\Gamma_m} (\Theta\cdot n)\,(\kappa_m\,\mathcal{N}(u_\Omega) + \nabla\mathcal{N}(u_\Omega)\cdot n)\,\mathrm{d}s(x) \\ &- \int_{\Gamma_m\cap\Gamma_N} (\Theta\cdot n)\,(\kappa_m\,p_\Omega\cdot g_N + \nabla(p_\Omega\cdot g_N)\cdot n)\,\mathrm{d}s(x).\end{aligned} \tag{11}$$

In particular, this formula now allows us to easily obtain the gradient expression of J from

$$\mathcal{D}\, J(\Omega)[\Theta] = \langle \nabla J(\Omega), \Theta \rangle_{L^2(\Gamma_m)} = \int_{\Gamma_m} \nabla J(\Omega)(x) \cdot \Theta(x) \, \mathrm{d}s(x),$$

which is defined for all $x \in \Gamma_m$ by

$$\begin{aligned}\nabla J(\Omega)(x) = &(\mathcal{M}(u_\Omega(x)) + A\varepsilon(u_\Omega(x)) : \varepsilon(p_\Omega(x)) - f(x) \cdot p_\Omega(x))n(x) \\ &+ (\kappa_m(x)\, \mathcal{N}(u_\Omega(x)) + \nabla\mathcal{N}(u_\Omega(x)) \cdot n(x))n(x) \\ &+ (\kappa_m(x)\, p_\Omega \cdot g_N(x) + \nabla(p_\Omega(x) \cdot g_N(x)) \cdot n(x))\chi_{\Gamma_N}(x)n(x).\end{aligned} \tag{12}$$

Note that since $a(\cdot,\cdot)$ is a continuous and coercive bilinear form whereas $\mathcal{M}$ and $\mathcal{N}$ are supposed to be Lipschitz-continuous, Lax-Milgram theorem ensures the well-posedness of problem (8) which admits a unique solution $p_\Omega \in V_{\Gamma_{C,a}}$.

Remark 11. *If assumption 8 does not hold, is is not possible to obtain the formulation* (11) *since the shape derivative depends nonlinearly on the direction* Θ*. In this case, the functional* J *is not differentiable in the classical sense.*

4 Nitsche-based formulations

In this section, we conduct a convergence analysis of a finite element approximation of the adjoint state equation (8). We introduce Nitsche's method to deal with the boundary condition on Γ_C. We verify its the consistency and finally detail its convergence analysis.

4.1 Nitsche-based formulation for the direct problem

Let $V^h \subset V$ be a family of finite dimensional vector spaces (see [48]) indexed by h coming from a family $\mathcal{T}^h$ of triangulations of the domain Ω ($h = \max_{T \in \mathcal{T}^h} h_T$ where h_T is the diameter of T). The family of triangulations is supposed uniformly regular for simplicity, i.e., there exists $\sigma > 0$ and $\zeta > 0$ such that $\forall T \in \mathcal{T}^h, h_T/\rho_T \le \sigma$ and $h_T > \zeta h$ where ρ_T denotes the radius of the inscribed ball in T. For instance, a standard Lagrange finite element method of degree k reads

$$V^h := \{v^h \in \mathscr{C}^0(\bar{\Omega})^d | v^h|_T \in (P_k(T))^d, \forall T \in \mathcal{T}^h, v^h = 0 \text{ on } \Gamma_D\}. \tag{13}$$

Let γ be a piecewise constant function on the contact interface Γ_C defined for any $x \in \Gamma_C$ lying on the relative interior of $\Gamma_C \cap T$ for a (closed) element T having a non-empty intersection of dimension $d-1$ with Γ_C by

$$\gamma(x) = \frac{\gamma_0}{h_T},$$

where γ_0 is a positive given constant. The Nitsche-based formulation is built on the equivalent reformulation of the contact conditions which has been originally derived from the augmented Lagrangian approach [49] and reads as

$$\sigma_n(u_\Omega) = -[\sigma_n(u_\Omega) - \gamma((u_\Omega)_n - g)]_-,$$

where the negative part is defined by $[x]_- = \frac{1}{2}(|x| - x), \forall x \in \mathbb{R}$. The generalized Nitsche-based approximation $u^h_\Omega \in V^h$ is then the solution of

$$a(u^h_\Omega, v^h) + \mathcal{I}(u^h_\Omega, v^h, n) = \ell(v^h), \quad \forall v^h \in V^h, \tag{14}$$

where the frictionless contact term $\mathcal{I}(u,v,n)$ reads

$$\mathcal{I}(u,v,n) = -\int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(u)\sigma_n(v)\,\mathrm{d}s(x) + \int_{\Gamma_C} \frac{1}{\gamma}\quad [\sigma_n(u) - \gamma(u_n - g)]_- \,(\theta\,\sigma_n(v) - \gamma v_n)\,\mathrm{d}s(x). \tag{15}$$

In the following proposition, we recall some results due to P. Hild, F. Chouly and Y. Renard [31].

Proposition 12. *Suppose that the solution u to Problem (2) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $\nu \in]0, k-1/2[$ ($k = 1, 2$ is the degree of the finite element method given in (13))and $d = 2, 3$. When $\theta \neq -1$, suppose in addition that the parameter γ_0 is sufficiently large. The solution u^h_Ω of Problem (14) satisfies the following error estimates for $C > 0$ a constant independent of h:*

$$\|u^h_\Omega - u_\Omega\|_{1,\Omega} \le Ch^{\frac{1}{2}+\nu}\|u_\Omega\|_{\frac{3}{2}+\nu,\Omega}, \tag{16}$$

$$\|\sigma_n(u^h_\Omega) - \sigma_n(u_\Omega)\|_{0,\Gamma_C} \le Ch^{\nu}\|u_\Omega\|_{\frac{3}{2}+\nu,\Omega}, \tag{17}$$

$$\|[\sigma_n(u^h_\Omega) - \gamma((u_\Omega)^h_n - g)]_- + \sigma_n(u_\Omega)\|_{0,\Gamma_C} \le Ch^{\nu}\|u_\Omega\|_{\frac{3}{2}+\nu,\Omega}, \tag{18}$$

where here and in the rest of this paper, $\|\cdot\|_{s,\omega}$ stands for the $H^s(\omega)-$norm.

Note that these convergence results make an important use of the following classical property whose proof can be found for instance in [31].

Lemma 13. *There exists $C > 0$ independent of the parameter γ_0 and of the mesh size h, such that for all $v^h \in V^h$*

$$\|\gamma^{-\frac{1}{2}}\sigma_n(v^h)\|^2_{0,\Gamma_C} \le \frac{C}{\gamma_0}\|v^h\|^2_{1,\Omega}. \tag{19}$$

4.2 Adjoint state of the Nitsche-based formulation

For the minimization of the discrete criterion $J^h(\Omega) = \int_\Omega \mathcal{M}(u^h_\Omega)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}(u^h_\Omega)\,\mathrm{d}s(x)$, where $u^h_\Omega \in V^h$ solution of (14), a first approach is to derive the adjoint state of the discrete formulation, for instance using a Lagrangian approach. This is presented in [30] and leads to the following formulation:

$$\begin{aligned} \mathcal{D}\,J^h(\Omega)[\Theta] \;=\; & \int_{\Gamma_m} (\Theta\cdot n)\,(\mathcal{M}(u^h_\Omega) + A\varepsilon(u^h_\Omega) : \varepsilon(\tilde{p}^h_\Omega) - f(x)\cdot\tilde{p}^h_\Omega)\,\mathrm{d}s(x) \\ & + \int_{\Gamma_m} (\Theta\cdot n)\,(\kappa_m\,\mathcal{N}(u^h_\Omega) + \nabla\mathcal{N}(u^h_\Omega)\cdot n)\,\mathrm{d}s(x) \\ & - \int_{\Gamma_m\cap\Gamma_N} (\Theta\cdot n)\,(\kappa_m\,\tilde{p}^h_\Omega\cdot g_N + \nabla(\tilde{p}^h_\Omega\cdot g_N)\cdot n)\,\mathrm{d}s(x), \end{aligned} \tag{20}$$

where the discrete adjoint state $\tilde{p}^h_\Omega \in V^h$ is defined by

$$\left\{\begin{array}{l} \text{Find } \tilde{p}^h_\Omega \in V^h \text{ such that } \forall q^h \in V^h \\ a(\tilde{p}^h_\Omega, q^h) - \displaystyle\int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(\tilde{p}^h_\Omega)\sigma_n(q^h)\mathrm{d}s(x) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma}H(-(\sigma_n(u^h_\Omega) - \gamma((u^h_\Omega)_n - g))))(\sigma_n(q^h) - \gamma q^h_n)(\theta\sigma_n(\tilde{p}^h_\Omega) - \gamma(\tilde{p}^h_\Omega)_n)\mathrm{d}s(x) \\ = -\displaystyle\int_\Omega \mathcal{M}'(u^h_\Omega)\cdot q^h\,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u^h_\Omega)\cdot q^h\,\mathrm{d}s(x), \end{array}\right. \tag{21}$$

with $H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$ being the Heaviside function.

Since expressions (20) and (11) are more than similar and that there are some convergence results of u_Ω^h towards u_Ω, a question that naturally arises is to know if a similar convergence result of $\tilde{p}_\Omega^h$ towards p_Ω can be expected. Unfortunately the answer seems to be negative in the general case, due to a consistency issue in the definition of $\tilde{p}^h$ which does not allow to ensure the right boundary conditions on $\Gamma_{C,a}$, at least in the case $\theta \neq 1$. To be convinced of this, it is enough to notice that assuming for simplicity $H(-(\sigma_n(u_\Omega^h) - \gamma((u_\Omega^h)_n - g)))) = \chi_{\Gamma_{C,a}}$, then $\tilde{p}_\Omega^h$ satisfies after application of Green's formula and for simplicity for $\theta = 0$:

$$
\begin{aligned}
0 = &- \int_\Omega (\operatorname{div}(\sigma(\tilde{p}_\Omega^h)) - \mathcal{M}'(u_\Omega^h)) \cdot q^h dx + \int_{\Gamma_N \cup \Gamma_{C,b} \cup \Gamma_{C,i}} (\sigma(\tilde{p}_\Omega^h) n + \mathcal{N}'(u_\Omega^h)) \cdot q^h ds(x) \\
&+ \int_{\Gamma_{C,a}} \left(\sigma(\tilde{p}_\Omega^h) n \cdot q^h + \gamma (\tilde{p}_\Omega^h)_n q_n^h - \sigma_n(q^h)(\tilde{p}_\Omega^h)_n + \mathcal{N}'(u_\Omega^h) \cdot q^h \right) ds(x),
\end{aligned}
$$

which enforces both $(\tilde{p}_\Omega^h)_n = 0$ and $\sigma(\tilde{p}_\Omega^h) n = -\mathcal{N}'(u_\Omega^h)$ asymptotically on $\Gamma_{C,a}$ when h goes to zero. This is symptomatic of the non-self-adjoint nature of Nitsche's method for $\theta \neq 1$.

Remark 14. *Although we cannot demonstrate a convergence result from the discrete adjoint state to its continuous counterpart, at least for $\theta \neq 1$, the use of $\tilde{p}_\Omega^h$ in* (20) *allows to properly define the gradient of the discrete energy J^h which can be use to minimize J^h using a gradient algorithm, as we proposed in [30].*

4.3 Nitsche-based formulation for the adjoint state and consistency

A second approach is the discretization of Problem (9) with a Nitsche-based method. It can be formulated as follows:

$$
\begin{cases}
\text{Find } p_\Omega^h \in V^h \text{ such that } \forall q^h \in V^h \\
a(p_\Omega^h, q^h) - \displaystyle\int_{\Gamma_C} \frac{\theta}{\gamma} (\sigma_n(p_\Omega^h) + (\mathcal{N}'(u_\Omega^h))_n) \sigma_n(q^h) \mathrm{d}s(x) \\
+ \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} H(-(\sigma_n(u_\Omega^h) - \gamma(u_n^h - g))))(\sigma_n(p_\Omega^h) + (\mathcal{N}'(u_\Omega^h))_n - \gamma(p_\Omega^h)_n)(\theta\sigma_n(q^h) - \gamma q_n^h) \mathrm{d}s(x) \\
= - \displaystyle\int_\Omega \mathcal{M}'(u_\Omega^h) \cdot q^h \, \mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega^h) \cdot q^h \, \mathrm{d}s(x),
\end{cases}
\tag{22}
$$

where $\theta \in \mathbb{R}$ and $\gamma > 0$. Note that expressions (22) and (21) are identical in the case $\theta = 1$ (this corresponds to the symmetric version of Nitsche's method) and when $\mathcal{N}'(u_\Omega^h)$ vanishes on Γ_C. The advantage of Formulation (22) over (21) is that a consistency result can be proved for (22).

Lemma 15. *The Nitsche-based adjoint state formulation* (22) *is consistent in the following sense: suppose that the solution p_Ω to* (9) *lies in $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $\nu \geq 0$ and $d = 2, 3$. Then if assumption 8 holds, p_Ω is also solution, $\forall q^h \in V^h$, of*

$$
\begin{aligned}
a(q^h, p_\Omega) &- \int_{\Gamma_C} \frac{\theta}{\gamma} (\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n) \sigma_n(q^h) \mathrm{d}s(x) \\
&+ \int_{\Gamma_C} \frac{1}{\gamma} H(-(\sigma_n(u_\Omega) - \gamma((u_\Omega)_n - g))))((\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n) - \gamma(p_\Omega)_n)(\theta\sigma_n(q^h) - \gamma q_n^h) \mathrm{d}s(x) \\
&= - \int_\Omega \mathcal{M}'(u_\Omega) \cdot q^h \, \mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega) \cdot q^h \, \mathrm{d}s(x).
\end{aligned}
\tag{23}
$$

Proof. Using Green's formula on the adjoint state problem (9), $\forall q^h \in V^h$, it holds

$$a(q^h, p_\Omega) - \int_{\partial\Omega} (\sigma(p_\Omega)n + \mathcal{N}'(u_\Omega)) \cdot q^h \mathrm{d}s(x) = -\int_\Omega \mathcal{M}'(u_\Omega) \cdot q^h \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega) \cdot q^h \,\mathrm{d}s(x). \tag{24}$$

As p_Ω satisfies $\sigma(p_\Omega)n = -\mathcal{N}'(u_\Omega)$ in $\Gamma_{C,i} \cup \Gamma_N$, we have

$$\int_{\Gamma_{C,i}\cup\Gamma_N} (\sigma(p_\Omega)n + \mathcal{N}'(u_\Omega)) \cdot q^h \mathrm{d}s(x) = \int_{\Gamma_{C,i}} (\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n)\sigma_n(q^h)\mathrm{d}s(x) = 0.$$

Recall also that p_Ω satisfies $(p_\Omega)_n = 0$ and $\sigma(p_\Omega)_t = -\mathcal{N}'(u_\Omega)_t$ in $\Gamma_{C,a}$, which gives

$$\int_{\Gamma_{C,a}} (\sigma(p_\Omega)n + \mathcal{N}'(u_\Omega)) \cdot q^h \mathrm{d}s(x) \quad = \int_{\Gamma_{C,a}} q_n^h(\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n - \gamma(p_\Omega)_n)\mathrm{d}s(x),$$

and $\displaystyle\int_{\Gamma_{C,a}} \theta\sigma_n(q^h)(p_\Omega)_n \mathrm{d}s(x) = 0$. These equalities show that the adjoint state field $p_\Omega \in V$ satisfies

$$\begin{aligned} a(q^h, p_\Omega) &- \int_{\Gamma_{C,i}} \frac{\theta}{\gamma}(\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n)\sigma_n(q^h)\mathrm{d}s(x) - \int_{\Gamma_{C,a}} \theta\sigma_n(q^h)(p_\Omega)_n\mathrm{d}s(x) \\ &- \int_{\Gamma_{C,a}} q_n^h(\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n - \gamma(p_\Omega)_n)\mathrm{d}s(x) \\ &= -\int_\Omega \mathcal{M}'(u_\Omega) \cdot q^h \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega) \cdot q^h \,\mathrm{d}s(x), \; \forall q^h \in V^h, \end{aligned} \tag{25}$$

leading then to

$$\begin{aligned} a(q^h, p_\Omega) &- \int_{\Gamma_C} \frac{\theta}{\gamma}(\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n)\sigma_n(q^h)\mathrm{d}s(x) \\ &+ \int_{\Gamma_{C,a}} \frac{1}{\gamma}(\sigma_n(p_\Omega) + (\mathcal{N}'(u_\Omega))_n - \gamma(p_\Omega)_n)(\theta\sigma_n(q^h) - \gamma q_n^h)\mathrm{d}s(x) \\ &= -\int_\Omega \mathcal{M}'(u_\Omega) \cdot q^h \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u_\Omega) \cdot q^h \,\mathrm{d}s(x). \end{aligned}$$

Since $\Gamma_{C,a} := \{x \in \Gamma_C | \sigma_n(u_\Omega) < 0, (u_\Omega)_n = g\}$, then $H(-(\sigma_n(u_\Omega(x)) - \gamma(((u_\Omega)_n(x) - g))) = \chi_{\Gamma_{C,a}} = \begin{cases} 1 \text{ if } x \in \Gamma_{C,a}, \\ 0 \text{ otherwise.} \end{cases}$, which implies that (23) is satisfied. ☐

Remark 16. *In the event that Assumption 8 is not satisfied, one cannot expect a convergence result because* (8) *prescribes a Neumann condition on* $\Gamma_{C,b}$ *which will not necessarily be asymptotically satisfied by the solution to* (22). *We address this problem in Section 4.5 by a slight modification of the equation satisfied by* p_Ω^h.

4.4 Convergence analysis

The aim of this section is to present an a priori convergence result of the Nitsche-based formulation (8) with respect to the mesh parameter h. This result requires a supplementary assumption on the convergence of the effective contact area (i.e. a supplementary condition on the convergence of u^h_Ω towards u_Ω). For the sake of simplicity and clarity of this section and the next one, we will no longer indicate the dependence of the solution with respect to Ω and just use

$$u = u_\Omega, u^h = u^h_\Omega, p = p_\Omega \text{ and } p^h = p^h_\Omega.$$

Moreover, we introduce the two following quantities relative to the contact status:

$$\beta_h = -\sigma_n(u^h) + \gamma(u^h_n - g), \beta = -\sigma_n(u) + \gamma(u_n - g),$$

and recall that $\Gamma_{C,a} := \{x \in \Gamma_C | \beta > 0\}$ and introduce also the discrete effective contact area

$$\Gamma^h_{C,a} := \{x \in \Gamma_C | \beta_h > 0\}.$$

Remark 17. *In practice, β actually depends on h as $\gamma = \gamma_0/h_T$. However, $H(\beta)$ being the characteristic function of $\Gamma_{C,a}$, $H(\beta) = \chi_{\Gamma_{C,a}} = \begin{cases} 1 \text{ for } x \in \Gamma_{C,a} \\ 0 \text{ otherwise} \end{cases}$, it does not depend on h.*

We first introduce the following lemma on the weak convergence of $H(\beta^h)$ that is required for the main convergence result.

Lemma 18. *Suppose that the solution u to Problem (2) belongs to $(H^{\frac{3}{2}+\nu})^d$ with $\nu > 0$ and $d = 2$ or $d = 3$ and that assumptions 8 holds. Then, $|H(\beta) - H(\beta_h)| \xrightarrow{*} 0$ in $L^\infty(\Gamma_C)$, in the sense that $\forall \phi \in L^1(\Gamma_C)$*

$$\lim_{h \to 0} \int_{\Gamma_C} |H(\beta) - H(\beta_h)| \phi \, \mathrm{d}s(x) = 0.$$

Consequently, $H(\beta_h) \xrightarrow{*} \chi_{\Gamma_{C,a}}$ in $L^\infty(\Gamma_C)$.

Proof. Still for $\sigma_n(u)$ a particular element of its class in $L^2(\Gamma_C)$, we introduce the measurable set

$$A_\delta = \{x \in \Gamma_C | \sigma_n(u) \leq -\delta\} \subset \Gamma_{C,a}. \tag{26}$$

It corresponds to the contact area where contact actually occurs for u and where the contact pressure is greater than δ. We also introduce N^h_δ a subset of A_δ where the contact does not occur for u^h defined by

$$N^h_\delta = \{x \in A_\delta | \sigma_n(u^h) - \gamma(u^h_n - g) > 0\}.$$

So on N^h_δ, it holds

$$|[\sigma_n(u^h) - \gamma(u^h_n - g)]_- + \sigma_n(u)| \geq \delta,$$

which implies

$$\int_{N^h_\delta} |[\sigma_n(u^h) - \gamma(u^h_n - g)]_- + \sigma_n(u)|^2 \, \mathrm{d}s(x) \geq \delta^2 |N^h_\delta|,$$

where $|\cdot|$ stands for the Lebesgue measure. Using (18) in Proposition 12, it finally holds

$$|N^h_\delta| \leq \frac{Ch^{2\nu}}{\delta^2}. \tag{27}$$

Now, introducing I_δ the measurable set where no contact occurs for u with a separation greater than δ defined by

$$I_\delta = \{x \in \Gamma_C | u_n \leq g - \delta\},$$

and

$$M^h_\delta = \{x \in I_\delta, \sigma_n(u^h) - \gamma(u^h_n - g) \leq 0\},$$

its subset where contact occurs for u^h, we can write on M^h_δ

$$|-\frac{\sigma_n(u^h)}{\gamma}+(u^h_n-g)-(u_n-g)|\geq \delta.$$

This implies

$$\int_{M^h_\delta}|-\frac{\sigma_n(u^h)}{\gamma}+(u^h_n-u_n)|^2\,\mathrm{d}s(x)\geq \delta^2|M^h_\delta|.$$

Using (16) in Proposition 12, it finally holds

$$|M^h_\delta|\leq \frac{Ch^{1+2\nu}}{\delta^2}. \tag{28}$$

Under Assumption 8, $\forall\delta>0$ and $\forall\phi\in L^1(\Gamma_C)$, we write

$$\int_{\Gamma_C}|H(\beta)-H(\beta_h)|\phi\,\mathrm{d}s(x)=\int_{A_\delta}(1-H(\beta_h))\phi\,\mathrm{d}s(x)+\int_{I_\delta}H(\beta_h)\phi\,\mathrm{d}s(x)$$
$$+\int_{\Gamma_C/(A_\delta\cup I_\delta)}|H(\beta)-H(\beta_h)|\phi\,\mathrm{d}s(x). \tag{29}$$

However,

$$\int_{A_\delta}(1-H(\beta_h))\phi\,\mathrm{d}s(x)=-\int_{N^h_\delta}\phi\,\mathrm{d}s(x),$$

and using (27)

$$\lim_{h\to 0}\int_{N^h_\delta}\phi\,\mathrm{d}s(x)=0.$$

Similarly

$$\int_{I_\delta}H(\beta_h)\phi\,\mathrm{d}s(x)=\int_{M^h_\delta}\phi\,\mathrm{d}s(x),$$

and using (28)

$$\lim_{h\to 0}\int_{M^h_\delta}\phi\,\mathrm{d}s(x)=0.$$

Since the measure $\Gamma_C\setminus(A_\delta\cup I_\delta)$ tends to 0 when δ tends to 0 under assumption 8, we finally obtain

$$\lim_{h\to 0}\int_{\Gamma_C}|H(\beta)-H(\beta_h)|\phi\,\mathrm{d}s(x)=0.$$

□

Let us consider the following assumption on the convergence of the effective contact area.

Assumption 19. *There exist $\omega>0$, $C>0$ independent of h such that $\Gamma^h_{C,a}\cap\Gamma_{C,i}$ is bounded as follows:*

$$|\Gamma^h_{C,a}\cap\Gamma_{C,i}|\leq Ch^\omega.$$

We present now our main convergence result of the discrete Nitsche-based adjoint state formulation (22).

Theorem 20. *Suppose that the solution p to Problem (8) and the solution u to Problem (2) belong to* $(H^{\frac{3}{2}+\nu}(\Omega))^d$ *with* $\nu > 0$ *and* $d = 2$ *or* $d = 3$. *Suppose that the parameter* γ_0 *is sufficiently large and that Assumptions 8 holds, then, it exists* $C > 0$ *independent of h such that the solution* $p^h \in V^h$ *to Problem (22) satisfies*

$$\begin{aligned}
&\|p - p^h\|_{1,\Omega}^2 + \|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h - p) - \gamma(p_n^h - p_n))\|_{0,\Gamma_C}^2 \\
&\quad \le C\left(\int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) + \int_{\Gamma_{C,i}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x)\right) \\
&\quad + C \inf_{q^h \in V^h} \left\{\left(\|\gamma^{-\frac{1}{2}}\sigma_n(q^h - p)\|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}}(q_n^h - p_n)\|_{0,\Gamma_C}^2 + \|q^h - p\|_{1,\Omega}^2\right)\right\} \\
&\quad + C\left(\|u - u^h\|_{1,\Omega}^2 + \|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C}\right).
\end{aligned}$$

Moreover, if Assumption 19 holds for $\omega > 1$, *we deduce that*

$$\lim_{h \to 0} \|p^h - p\|_{1,\Omega}^2 = 0. \tag{30}$$

Proof. Using the coercivity and continuity of $a(\cdot, \cdot)$, we write for any $q^h \in V^h$

$$\begin{aligned}
\alpha\|p - p^h\|_{1,\Omega}^2 &\le a(p - p^h, p - p^h) \\
&= a(p - p^h, p - q^h + q^h - p^h) \\
&\le C\|p - p^h\|_{1,\Omega}\, \|p - q^h\|_{1,\Omega} + a(p - p^h, q^h - p^h) \\
&\le \frac{\alpha}{2}\|p - p^h\|_{1,\Omega}^2 + \frac{C^2}{2\alpha}\|p - q^h\|_{1,\Omega}^2 + a(p, q^h - p^h) - a(p^h, q^h - p^h),
\end{aligned}$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot, \cdot)$, and $C > 0$ a generic constant independent of h in the whole study. We can rewrite the term $a(p, q^h - p^h) - a(p^h, q^h - p^h)$ as p solves (8), p^h solves (22) and using Lemma 15, it yields:

$$\begin{aligned}
&\frac{\alpha}{2}\|p - p^h\|_{1,\Omega}^2 \le \frac{C^2}{2\alpha}\|p - q^h\|_{1,\Omega}^2 - \int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(p^h - p)\sigma_n(q^h - p^h)\,\mathrm{d}s(x) \\
&+ \int_{\Gamma_C} \frac{1}{\gamma} H(\beta_h)(\sigma_n(p^h) - \gamma p_n^h - (\sigma_n(p) - \gamma p_n))(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\,\mathrm{d}s(x) \\
&+ \int_{\Gamma_C} \frac{1}{\gamma}(H(\beta_h) - H(\beta))(\sigma_n(p) - \gamma p_n)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\,\mathrm{d}s(x). \\
&+ \int_{\Gamma_C} \frac{\theta}{\gamma}((1 - H(\beta))\mathcal{N}'(u) - (1 - H(\beta^h))\mathcal{N}'(u^h))_n\sigma_n(q^h - p^h)\,\mathrm{d}s(x) \\
&+ \int_{\Gamma_C} (H(\beta)\mathcal{N}'(u) - H(\beta^h)\mathcal{N}'(u^h))_n(q_n^h - p_n^h)\,\mathrm{d}s(x) \\
&- \int_{\Omega} (\mathcal{M}'(u) - \mathcal{M}'(u^h)).(q^h - p^h)dx - \int_{\partial\Omega} (\mathcal{N}'(u) - \mathcal{N}'(u^h)).(q^h - p^h)ds(x).
\end{aligned} \tag{31}$$

The first integral term in (31) is bounded as follows, using Young's inequality for any $\xi_1 > 0$:

$$
\begin{aligned}
&-\int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(p^h-p)\sigma_n(q^h-p^h)\,\mathrm{d}s(x) = -\int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n((p^h-q^h)+(q^h-p))\sigma_n(q^h-p^h)\,\mathrm{d}s(x)\\
&= \int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(q^h-p^h)\sigma_n(q^h-p^h)\,\mathrm{d}s(x) - \int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(q^h-p)\sigma_n(q^h-p^h)\,\mathrm{d}s(x)\\
&\le \theta\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}^2 + |\theta|\;\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p)\|_{0,\Gamma_C}\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}\\
&\le \theta\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}^2 + \frac{\xi_1\theta^2}{2}\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p)\|_{0,\Gamma_C}^2 + \frac{1}{2\xi_1}\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}^2\\
&\le (\theta+\frac{1}{2\xi_1})\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}^2 + \frac{\xi_1\theta^2}{2}\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p)\|_{0,\Gamma_C}^2,\\
&\le \frac{C_0}{\gamma_0}(\theta+\frac{1}{2\xi_1})\left(\|p-q^h\|_{1,\Omega}^2+\|p-p^h\|_{1,\Omega}^2\right) + \frac{\xi_1\theta^2}{2}\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p)\|_{0,\Gamma_C}^2.
\end{aligned}
\tag{32}
$$

Concerning the second integral term in (31), we derive the following estimate for any $\xi_2 > 0$:

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma}H(\beta_h)(\sigma_n(p^h)-\gamma p_n^h-(\sigma_n(p)-\gamma p_n))(\theta\sigma_n(q^h-p^h)-\gamma(q_n^h-p_n^h))\,\mathrm{d}s(x)\\
&= -\int_{\Gamma_C} \frac{1}{\gamma}H(\beta_h)(\sigma_n(p^h-p)-\gamma(p_n^h-p_n))^2\,\mathrm{d}s(x)\\
&+ \int_{\Gamma_C} \frac{1}{\gamma}H(\beta_h)(\sigma_n(p^h-p)-\gamma(p_n^h-p_n))(\sigma_n(q^h-p)-\gamma(q_n^h-p_n))\,\mathrm{d}s(x)\\
&+ (\theta-1)\int_{\Gamma_C} \frac{1}{\gamma}H(\beta_h)(\sigma_n(p^h-p)-\gamma(p_n^h-p_n))\sigma_n(q^h-p^h)\,\mathrm{d}s(x)\\
&\le (-1+|\theta-1|\frac{\xi_2}{2}+\frac{\xi_2}{2})\|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h-p)-\gamma(p_n^h-p_n))\|_{0,\Gamma_C}^2\\
&+ \frac{1}{2\xi_2}\|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(q^h-p)-\gamma(q_n^h-p_n))\|_{0,\Gamma_C}^2 + \frac{1}{2\xi_2}|\theta-1|\;\|H(\beta_h)\gamma^{-\frac{1}{2}}\sigma_n(q^h-p^h)\|_{0,\Gamma_C}^2,\\
&\le (-1+|\theta-1|\frac{\xi_2}{2}+\frac{\xi_2}{2})\|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h-p)-\gamma(p_n^h-p_n))\|_{0,\Gamma_C}^2\\
&+ \frac{1}{2\xi_2}\left(\|\gamma^{-\frac{1}{2}}\sigma_n(q^h-p)\|_{0,\Gamma_C}^2+\|\gamma^{\frac{1}{2}}(q_n^h-p_n)\|_{0,\Gamma_C}^2\right) + \frac{C_0}{\gamma_0}\frac{1}{2\xi_2}(\|p-q^h\|_{1,\Omega}^2+\|p-p^h\|_{1,\Omega}^2).
\end{aligned}
\tag{33}
$$

The third integral term in (31) is split as follows:

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma}(H(\beta_h)-H(\beta))(\sigma_n(p)-\gamma p_n)(\theta\sigma_n(q^h-p^h)-\gamma(q_n^h-p_n^h))\,\mathrm{d}s(x)\\
&= \int_{\Gamma_{C,a}} \frac{1}{\gamma}(H(\beta_h)-1)\;\sigma_n(p)(\theta\sigma_n(q^h-p^h)-\gamma(q_n^h-p_n^h))\,\mathrm{d}s(x)\\
&+ \int_{\Gamma_{C,i}} \frac{H(\beta_h)}{\gamma}\;(\sigma_n(p)-\gamma p_n)(\theta\sigma_n(q^h-p^h)-\gamma(q_n^h-p_n^h))\,\mathrm{d}s(x).
\end{aligned}
\tag{34}
$$

For the first integral term of the right hand side of (34), on $\Gamma_{C,a}$, we obtain, using the trace inequality and for any $\xi_3 > 0$

$$
\begin{aligned}
&\int_{\Gamma_{C,a}} \frac{1}{\gamma}(H(\beta_h) - 1)\, \sigma_n(p)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\, \mathrm{d}s(x) \\
&\leq \frac{1}{2\xi_3} \int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) + \xi_3 \int_{\Gamma_{C,a}} (\frac{\theta^2}{\gamma^2}\sigma_n^2(q^h - p^h) + (q^h - p^h)^2)\mathrm{d}s(x) \\
&\leq \frac{1}{2\xi_3} \int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) + \xi_3(\theta^2\|\gamma^{-1}\sigma_n(q^h - p^h)\|^2_{0,\Gamma_{C,a}} + C\|q^h - p^h\|^2_{1,\Omega}) \\
&\leq \frac{1}{2\xi_3} \int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) + \xi_3(\frac{\theta^2 C_0 h^T}{\gamma_0^2} + C)(\|q^h - p\|^2_{1,\Omega} + \|p - p^h\|^2_{1,\Omega}).
\end{aligned}
\tag{35}
$$

For the second integral term of the right hand side of (34) on $\Gamma_{C,i}$, we obtain for any $\xi_4 > 0$

$$
\begin{aligned}
&\int_{\Gamma_{C,i}} \frac{H(\beta_h)}{\gamma}\, (\sigma_n(p) - \gamma p_n)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\, \mathrm{d}s(x) \\
&\leq \frac{1}{2\xi_4} \int_{\Gamma_{C,i}} \frac{H(\beta_h)}{\gamma}(\gamma p_n)^2 \mathrm{d}s(x) \\
&+ \frac{\xi_4}{2} \int_{\Gamma_{C,i}} \frac{H(\beta_h)}{\gamma}(\theta\sigma_n(q^h - p + p - p^h) - \gamma(q_n^h - p_n + p_n - p_n^h))^2\, \mathrm{d}s(x) \\
&\leq \frac{1}{2\xi_4} \int_{\Gamma_{C,i}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x) + 2\xi_4\|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h - p) - \gamma(p_n^h - p_n))\|^2_{0,\Gamma_{C,i}}, \\
&+ 2\xi_4|\theta - 1|\ \|H(\beta_h)\gamma^{-\frac{1}{2}}\sigma_n(q^h - p^h)\|^2_{0,\Gamma_{C,i}} + 2\xi_4\|H(\beta_h)(\sigma_n(q^h - p) - \gamma(q_n^h - p_n))\|^2_{0,\Gamma_{C,i}} \\
&\leq \frac{1}{2\xi_4} \int_{\Gamma_{C,i}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x) + 2\xi_4\|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h - p) - \gamma(p_n^h - p_n))\|^2_{0,\Gamma_{C,i}}, \\
&+ 2\xi_4|\theta - 1|\frac{C_0}{\gamma_0}\left(\|q^h - p\|^2_{1,\Omega} + \|p - p^h\|^2_{1,\Omega}\right) \\
&+ 2\xi_4\left(\|\gamma^{-\frac{1}{2}}\sigma_n(q^h - p)\|^2_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}}(q_n^h - p_n)\|^2_{0,\Gamma_C}\right).
\end{aligned}
\tag{36}
$$

The fourth integral term in (31) can be estimated as follows using Lemma 13, the Lipschitz-continuity of $\mathcal{N}'$ and for any $\xi_5 > 0$

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{\theta}{\gamma}((1 - H(\beta))\mathcal{N}'(u) - (1 - H(\beta^h))\mathcal{N}'(u^h))_n\sigma_n(q^h - p^h)\, \mathrm{d}s(x) \\
&= \int_{\Gamma_C} \frac{\theta}{\gamma}((H(\beta^h) - H(\beta))\mathcal{N}'(u) + (1 - H(\beta^h))(\mathcal{N}'(u) - \mathcal{N}'(u^h)))_n\sigma_n(q^h - p^h)\, \mathrm{d}s(x) \\
&\leq |\theta|\left((\|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C} + \|\mathcal{N}'(u) - \mathcal{N}'(u^h)\|_{0,\Gamma_C})\|\gamma^{-1}\sigma_n(q^h - p^h)\|_{0,\Gamma_C}\right) \\
&\leq C\frac{h^{1/2}}{\gamma_0}|\theta|(\|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C} + \|u - u^h\|_{0,\Gamma_C})\|q^h - p^h\|_{1,\Omega} \\
&\leq C\frac{h^{1/2}}{\gamma_0}|\theta|\left(\frac{1}{\xi_5}\|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C}\right. \\
&\quad \left. + \frac{1}{\xi_5}\|u - u^h\|^2_{0,\Gamma_C} + \xi_5\|q^h - p\|^2_{1,\Omega} + \xi_5\|p - p^h\|^2_{1,\Omega}\right),
\end{aligned}
\tag{37}
$$

and similarly for the fifth integral term in (31), we obtain

$$\int_{\Gamma_C} (H(\beta)\mathcal{N}'(u) - H(\beta^h)\mathcal{N}'(u^h))_n(q_n^h - p_n^h)\, \mathrm{d}s(x)$$
$$= \int_{\Gamma_C} ((H(\beta) - H(\beta^h))\mathcal{N}'(u) + H(\beta^h)(\mathcal{N}'(u) - \mathcal{N}'(u^h)))_n(q_n^h - p_n^h)\, \mathrm{d}s(x)$$
$$\le \int_{\Gamma_C} (H(\beta) - H(\beta^h))(\mathcal{N}'(u))_n(q_n^h - p_n^h)\, \mathrm{d}s(x) + \|\mathcal{N}'(u) - \mathcal{N}'(u^h)\|_{0,\Gamma_C}\|q_n^h - p_n^h\|_{0,\Gamma_C},$$
$$\le C\left(\frac{1}{\xi_5}\|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C} + \frac{1}{\xi_5}\|u - u^h\|_{1,\Omega}^2 + \xi_5\|q^h - p\|_{1,\Omega}^2 + \xi_5\|p - p^h\|_{1,\Omega}^2\right), \tag{38}$$

and for the two last integral terms in (31) using additionally the Lipschitz-continuity of $\mathcal{M}'$

$$-\int_{\Omega} (\mathcal{M}'(u) - \mathcal{M}'(u^h)).(q^h - p^h)dx - \int_{\partial\Omega} (\mathcal{N}'(u) - \mathcal{N}'(u^h)).(q^h - p^h)ds(x)$$
$$\le C\left(\frac{1}{2\xi_5}\|u - u^h\|_{1,\Omega}^2 + \xi_5\|q^h - p\|_{1,\Omega}^2 + \xi_5\|p - p^h\|_{1,\Omega}^2\right). \tag{39}$$

Gathering now (32), (33), (35), (36), (37), (38) and (39) we obtain for ξ_2, ξ_3, ξ_4 and ξ_5 sufficiently small and for γ_0 sufficiently large the existence of $C > 0$ such that

$$\|p - p^h\|_{1,\Omega}^2 + \|H(\beta_h)\gamma^{-\frac{1}{2}}(\sigma_n(p^h - p) - \gamma(p_n^h - p_n))\|_{0,\Gamma_C}^2$$
$$\le C\left(\int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) + \int_{\Gamma_{C,i}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x)\right)$$
$$+ C\left(\|\gamma^{-\frac{1}{2}}\sigma_n(q^h - p)\|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}}(q_n^h - p_n)\|_{0,\Gamma_C}^2 + \|q^h - p\|_{1,\Omega}^2\right)$$
$$+ C\left(\|u - u^h\|_{1,\Omega}^2 + \|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C}\right).$$

Finally, the proof of convergence is obtained thanks to the interpolation error exposed in [31] (Theorem 3.8), which shows that choosing q^h the Lagrange interpolate of p leads to

$$\lim_{h\to 0} \|p - q^h\|_{1,\Omega}^2 = 0, \qquad \lim_{h\to 0} \|\gamma^{-\frac{1}{2}}\sigma_n(q^h - p)\|_{0,\Gamma_C}^2 = 0, \qquad \lim_{h\to 0} \|\gamma^{\frac{1}{2}} q_n^h - p_n)\|_{0,\Gamma_C}^2 = 0.$$

Moreover, thanks to Lemma 18, $H(\beta_h) \xrightarrow{\ *\ } \chi_{\Gamma_{C,a}}$ gives $\lim\limits_{h\to 0} \int_{\Gamma_{C,a}} (1 - H(\beta_h))\sigma_n^2(p)\mathrm{d}s(x) = 0$ and $|H(\beta) - H(\beta_h)| \xrightarrow{\ *\ } 0$ ensures $\lim\limits_{h\to 0} \|(H(\beta^h) - H(\beta))(\mathcal{N}'(u))_n\|_{0,\Gamma_C} = 0$. Moreover, the continuity of p_n ensures $|p_n| \le C$ with $C > 0$. With assumption 19, we can bound the first term in (36) as

$$\int_{\Gamma_{C,i}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x) = \int_{\Gamma_{C,a}^h \cap \Gamma_{C,i}} \gamma\, p_n^2\, \mathrm{d}s(x) \le \frac{\gamma_0}{h} C h^{\omega} \le C h^{\omega - 1}.$$

It suffices that $\omega > 1$ so that

$$\lim_{h\to 0} \int_{\Gamma_{C,a}^h \cap \Gamma_{C,i}} \gamma\, p_n^2\, \mathrm{d}s(x) = 0. \tag{40}$$

□

Note that in the numerical tests we provide in section 5.1, the condition $\omega > 1$ is satisfied for the studied range of mesh size.

4.5 Improved convergence result with an extended Neumann zone for the adjoint state

The aim of this section is to give a convergence result without the consideration of assumptions 8 and 19, i.e. without the consideration of zero measure of $\Gamma_{C,b}$ and assumption on the rate of convergence of the effective contact area. This result is obtained with a slight modification of the discrete adjoint state, extending a bit the part of the boundary where the Neumann condition is applied and with the use of quadratic finite elements.

Let us consider $\xi > 0$ a small parameter which is assumed to tend to zero when $h \to 0$, then the consideration of the following modified problem for the adjoint state:

$$\begin{cases} \text{Find } p^h \in V^h \text{ such that } \forall q^h \in V^h \\ a(p^h, q^h) - \displaystyle\int_{\Gamma_C} \frac{\theta}{\gamma}(\sigma_n(p^h) + (\mathcal{N}'(u^h_\Omega))_n)\sigma_n(q^h)\mathrm{d}s(x) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} H(-(\sigma_n(u^h) - \gamma(u^h_n - g)) - \xi))(\sigma_n(p^h) + (\mathcal{N}'(u^h_\Omega))_n - \gamma p^h_n)(\theta\sigma_n(q^h) - \gamma q^h_n)\mathrm{d}s(x) \\ = -\displaystyle\int_\Omega \mathcal{M}'(u^h) \cdot q^h \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{N}'(u^h) \cdot q^h \,\mathrm{d}s(x), \end{cases} \tag{41}$$

allows to state the following result.

Theorem 21. *Suppose that the solution u to Problem (2) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $\nu > 1/2$ and p the solution to Problem (8) belongs to $(H^{\frac{3}{2}+\nu_2}(\Omega))^d$ with $\nu_2 > 0$ and $d = 2$ or $d = 3$. Suppose that the parameter γ_0 is sufficiently large, $k = 2$ (k being the degree of the finite element method) and $\xi \geq Ch^{\nu-1/2}$ with $C > 0$ arbitrary small enough and $\lim_{h\to 0} \xi = 0$. Then, the solution $p^h \in V^h$ to Problem (41) satisfies*

$$\lim_{h\to 0} \|p^h - p\|_{1,\Omega} = 0.$$

Proof. We observe first that the consistency result of Lemma 15 is still valid using the convention $H(0) = 0$ (i.e. replacing $H(-(\sigma_n(u) - \gamma(u_n - g)))$ by $\chi_{\Gamma_{C,a}}$). Then the proof of Theorem 20 can be followed with limited modifications that we focus on. Let us denote

$$\tilde{\beta}_h = -\sigma_n(u^h) + \gamma(u^h_n - g) - \xi.$$

The estimate (31) of the proof of Theorem 20 becomes

$$\begin{aligned} \frac{\alpha}{2}\|p - p^h\|^2_{1,\Omega} &\leq \frac{C^2}{2\alpha}\|p - q^h\|^2_{1,\Omega} - \int_{\Gamma_C} \frac{\theta}{\gamma}\sigma_n(p^h - p)\sigma_n(q^h - p^h)\,\mathrm{d}s(x) \\ &+ \int_{\Gamma_C} \frac{1}{\gamma} H(\tilde{\beta}_h)(\sigma_n(p^h) - \gamma p^h_n - (\sigma_n(p) - \gamma p_n))(\theta\sigma_n(q^h - p^h) - \gamma(q^h_n - p^h_n))\,\mathrm{d}s(x) \\ &+ \int_{\Gamma_C} \frac{1}{\gamma}(H(\tilde{\beta}_h) - H(\beta))(\sigma_n(p) - \gamma p_n)(\theta\sigma_n(q^h - p^h) - \gamma(q^h_n - p^h_n))\,\mathrm{d}s(x). \end{aligned} \tag{42}$$

The first and second integral terms in (42) are estimated thanks to (32) and (33), respectively, replacing β_h by $\tilde{\beta}_h$ and the same convergence to zero is obtained at the end. The third term in (42) is split similarly as in (34) taking into account the fact that $\Gamma_{C,b}$ is no longer supposed to be of zero measure. So it gives

$$\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma}(H(\tilde{\beta}_h) - H(\beta))(\sigma_n(p) - \gamma p_n)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\,\mathrm{d}s(x) \\
&= \int_{\Gamma_{C,a}} \frac{1}{\gamma}(H(\tilde{\beta}_h) - 1)\,\sigma_n(p)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\,\mathrm{d}s(x) \\
&+ \int_{\Gamma_{C,i}\cup\Gamma_{C,b}} \frac{H(\tilde{\beta}_h)}{\gamma}\,(\sigma_n(p) - \gamma p_n)(\theta\sigma_n(q^h - p^h) - \gamma(q_n^h - p_n^h))\,\mathrm{d}s(x).
\end{aligned} \tag{43}$$

The first integral term of the right hand side of (43) is treated as in (35). It remains to verify that

$$\lim_{h\to 0} \frac{1}{2\xi_3} \int_{\Gamma_{C,a}} (1 - H(\tilde{\beta}_h))\sigma_n^2(p)\mathrm{d}s(x) = 0. \tag{44}$$

Let us still denote A_δ the set defined by (26) where the contact actually occurs for u and the contact pressure is greater than δ, and consider

$$\tilde{N}_{\delta,\xi}^h = \{x \in A_\delta | \sigma_n(u^h) - \gamma(u_n^h - g) > -\xi\},$$

the subset where the discrete adjoint state is submitted to a Neumann condition. We obtain on $\tilde{N}_{\delta,\xi}^h$ for $\xi < \delta$,

$$|[\sigma_n(u^h) - \gamma(u_n^h - g)]_- + \sigma_n(u)| \geq \delta - \xi,$$

so that

$$\int_{\tilde{N}_\delta^h} |[\sigma_n(u^h) - \gamma(u_n^h - g)]_- + \sigma_n(u)|^2\,\mathrm{d}s(x) \geq (\delta - \xi)^2 |\tilde{N}_\delta^h|,$$

which leads, using (18) in Proposition 12 to $|\tilde{N}_\delta^h| \leq \frac{Ch^{2\nu}}{(\delta-\xi)^2}$. We have

$$0 \leq \int_{\Gamma_{C,a}} (1 - H(\tilde{\beta}_h))\sigma_n^2(p)\mathrm{d}s(x) \leq \int_{\tilde{N}_\delta^h} \sigma_n^2(p)\mathrm{d}s(x) + \int_{\Gamma_{C,a}\setminus A_\delta} \sigma_n^2(p)\mathrm{d}s(x).$$

For an arbitrary $\delta > 0$, the term $\int_{\tilde{N}_\delta^h} \sigma_n^2(p)\mathrm{d}s(x)$ tends to zero as h tends to zero and the term $\int_{\Gamma_{C,a}\setminus A_\delta} \sigma_n^2(p)\mathrm{d}s(x)$ tends to zero when δ tends to zero. So that we obtain (44).

Now, concerning the second integral term of the right hand side of (43), we follow (36) and it remains only to prove that

$$\lim_{h\to 0} \int_{\Gamma_{C,i}\cup\Gamma_{C,b}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x) = 0. \tag{45}$$

To this aim, denoting $\Gamma_{C,a}^h = \{x \in \Gamma_C | \tilde{\beta} > 0\}$, we obtain

$$\int_{(\Gamma_{C,i}\cup\Gamma_{C,b})\cap\Gamma_{C,a}^h} |[\sigma_n(u^h) - \gamma(u_n - g)]_- + \sigma_n(u)|^2 \mathrm{d}s(x) \geq \xi^2 |(\Gamma_{C,i}\cup\Gamma_{C,b})\cap\Gamma_{C,a}^h|,$$

and still using (18) we deduce $|(\Gamma_{C,i}\cup\Gamma_{C,b})\cap\Gamma_{C,a}^h| \leq C\frac{h^{2\nu}}{\xi^2}$. So that

$$\int_{\Gamma_{C,i}\cup\Gamma_{C,b}} H(\beta_h)\gamma p_n^2 \mathrm{d}s(x) = \int_{(\Gamma_{C,i}\cup\Gamma_{C,b})\cap\Gamma_{C,a}^h} \gamma p_n^2 \mathrm{d}s(x) \leq C\gamma_0 \frac{h^{2\nu-1}}{\xi^2},$$

since p is bounded on Γ_C. Consequently, (45) holds for $\nu > 1/2$ and $\xi > Ch^{\nu-1/2}$ which ends the proof. □

Extending the part of the boundary on which a Neumann condition is considered, makes the discrete adjoint problem tend to the continuous adjoint which satisfies a Neumann condition on $\Gamma_{C,b}$. Of course, this continuous adjoint may not allow to recover the conical shape derivative given by Theorem 5 for all direction Θ. An interesting and open question would be to verify that it allows to obtain a descent direction of the shape optimization problem.

5 Numerical experiments

In this section, we illustrate the convergence analysis with some numerical tests on an elastic hollow cylinder in contact with a plane rigid foundation. We refer to [30] for more details on our optimization strategy. The different tests are performed using GetFEM++ [50] with quadratic Lagrange finite elements on a polar mesh shown in Figure 2.

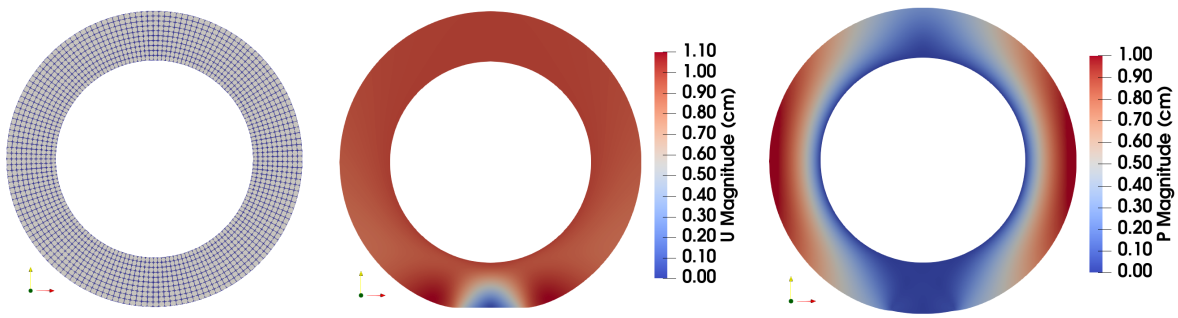

Figure 2: Hollow cylinder in contact with the obstacle. From left to right: structured polar mesh of the hollow cylinder; approximated displacement; approximated adjoint state.

We consider the elastic hollow cylinder presented in Figure 2 with an interior radius $R_i = 20$ cm and an exterior radius $R_e = 30$cm. The contact might occur between the boundary Γ_C on the exterior radius with a horizontal and rigid obstacle located at the bottom of the cylinder. The gravity forces are neglected and $f = 0$. We impose a vertical displacement $u_D = [0, -1\text{cm}]$ on the rigid boundary Γ_D. The optimization criteria are set to $\mathcal{M}(u) = F \cdot u$ where $F = [0, -1N]$ and $\mathcal{N}(u) = 0$ for the sake of simplicity. The result of the finite element computation for both the contact problem and the consistent Nitsche-based adjoint state problem (22) are shown in Figure 2.

5.1 Convergence of the Nitsche-based approximation of the adjoint state problem

We first focus on the convergence rate of the Nitsche-based approximation of the adjoint state problem (22). The reference solution p^h_{ref} of (22) is computed on a very thin mesh ($h = 0.0625$ cm). The slopes plotted in Figure 3 describe the convergence rates associated to the direct problem (14) for the variable u^h and the adjoint state problem (22) for the variable p^h. The relative $H^1(\Omega)-$norm is $\frac{\|p^h_{ref}-p^h\|^2_{1,\Omega}}{\|p^h_{ref}\|^2_{1,\Omega}}$ for $\|v^h\|^2_{1,\Omega} = \displaystyle\int_\Omega (v^h)^2 \mathrm{d}x + \int_\Omega |\nabla v^h|^2 \mathrm{d}x$. The left graph of Figure 3 presents both the convergence rate for the solution u^h to the direct problem (14) and the solution p^h to the adjoint state problem (22). Compared to the theoretical results given in [31] and recalled in Proposition 12 and due to the limitation of regularity of the solution due to the contact transitions (typically $u \in H^\nu(\Omega)$ for $\nu < 5/2$, see for instance [51]), the convergence rate for u^h is in good accordance although a little bit sub-optimal.

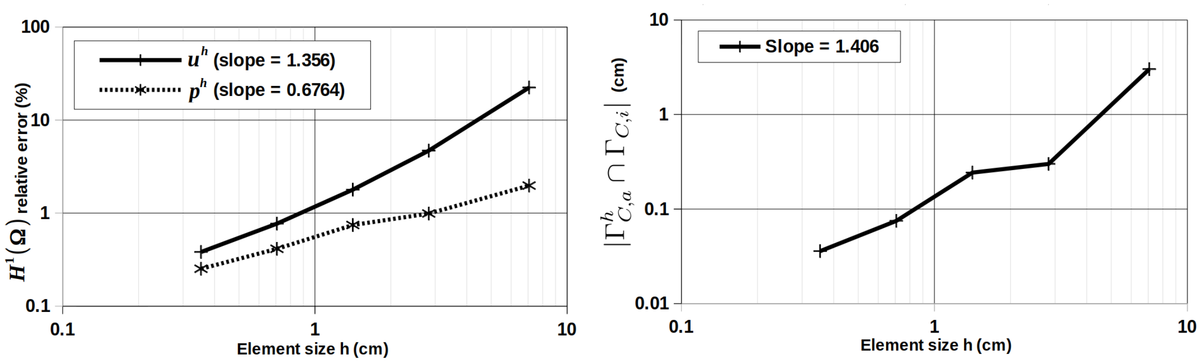

Figure 3: Error curves for $\theta = -1$. Left: relative $H^1(\Omega)$-norm on the displacement and the adjoint state variable. Right: length of $\Gamma^h_{C,a} \cap \Gamma_{C,i}$.

The convergence of p^h towards p solution to (8) is also observed, accordingly to our theoretical results, but with a slower convergence rate compared to u^h. This slower convergence has at least two causes: a Dirichlet-Neumann transition between $\Gamma_{C,a}$ and $\Gamma_{C,i}$, which limits the regularity of p, and the convergence of $\Gamma^h_{C,a}$ towards $\Gamma_{C,a}$ which depends on u^h. The convergence of the effective contact area is illustrated in the right graph of Figure 3. The coefficient ω of Assumption 19 is found approximately equal to 1.406, which is compatible with the requirement of Theorem 20 ($\omega > 1$). One can see on the left part of Figure 4 that the maximum of difference between p and p^h is indeed located on the transition between $\Gamma_{C,a}$ and $\Gamma_{C,i}$.

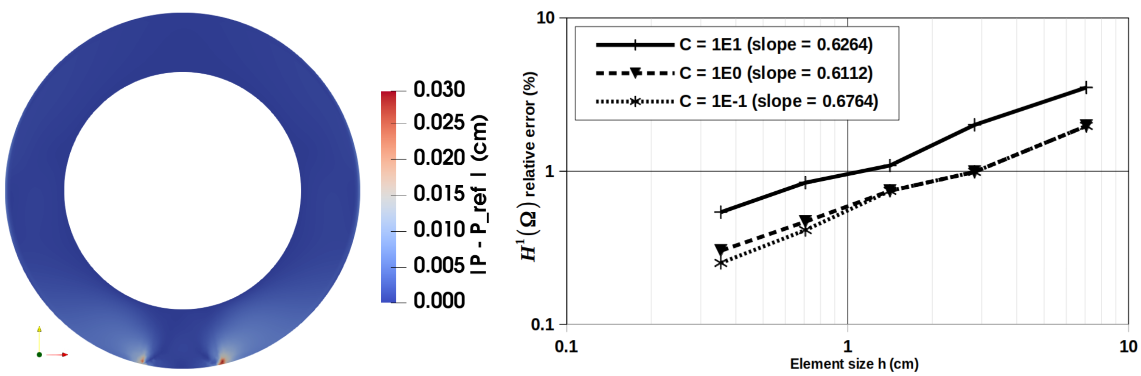

Figure 4: Left: Error map $|p^h - p^h_{ref}|$. Right: Error curves for the adjoint state problem p^h for $\theta = -1$. Relative $H^1(\Omega)$-norm on the displacement and the adjoint state variable for different values of C.

Now, concerning the strategy described in Section 4.5 with an extended Neumann zone on the contact boundary, we present a convergence test in the right graph of Figure 4. We recall that this strategy consists in replacing $H(\beta_h)$ by $H(\beta_h - \xi)$ in the proposed Nitsche-based method. We choose $\xi = C\sqrt{h_T}$ with different values of $C > 0$. Theorem 21 ensures the convergence of p^h for any value of $C > 0$. The strategy is respectively performed for C values of $0.1, 1$ and 10. We can see that this strategy does not deteriorate the order of convergence of p^h and starts to degrade the approximation error for a too high value of the constant ($C = 10$). This strategy can therefore be interesting since it ensures convergence without degradation of the approximation as soon as the constant C is taken with a moderate value.

5.2 Comparison of Nitsche-based adjoint state formulations

We focus now on the convergence rate of the adjoint state of the Nitsche-based formulation (21). Again, the reference solution p^h_{ref} is computed on problem (22) for a very thin mesh ($h = 0.0625$cm). Despite the non-consistence of this formulation, one can see on the slopes presented in Figure (5) that the convergence of $\tilde{p}^h$ solution to problem (21) is still ensured, with a convergence rate slightly deteriorated according to the one for the Nitsche-based approximation of the adjoint state presented in Figure 3.

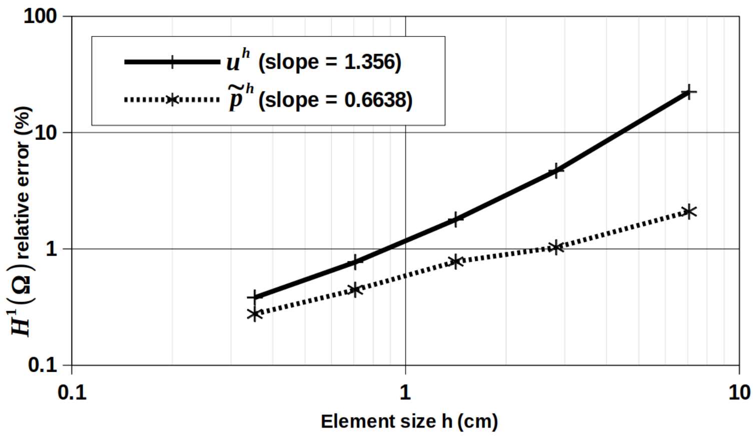

Figure 5: Error curves for the direct problem using Nitsche's method and for the adjoint state problem of the Nitsche-based approximation ($\theta = -1$).

Finally, in Figure 6, we present an example of shape optimization process which is taken from [30] and performed with an additional constraint on the periodicity of the structure. The optimizable boundary is only the interior part, which is submitted to a homogeneous Neumann condition. For the same initial geometry, the shape optimization is performed either with the adjoint state variable approximated by (22) or (21). One can see on Figure 6 that both of the two approximations lead to quasi-identical shapes, meaning that, at least for this example, the two strategies can be indifferently applied.

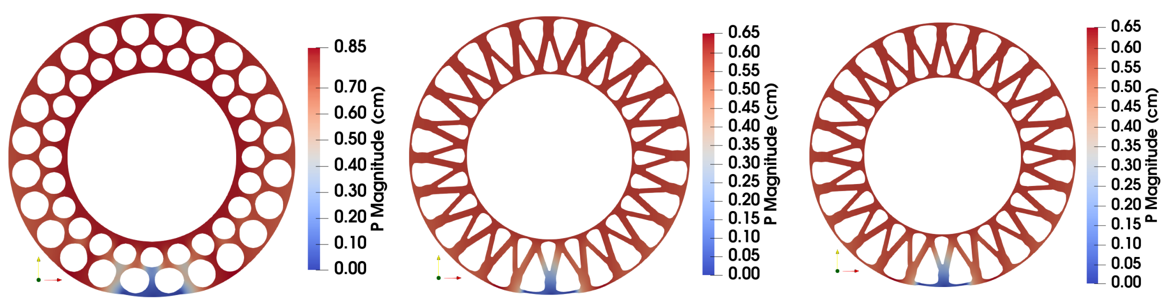

Figure 6: Shape optimization. The adjoint state variable is displayed. From left to right: initial geometry, optimal geometry with adjoint state computed on formulation (22), optimal geometry with adjoint state computed on formulation (21).

6 Conclusion

The context of this work is the shape optimization of an elastic structure under frictionless contact where the contact condition is treated with Nitsche's method and the shape gradients are calculated using the adjoint state method.

In a previous work, we proposed an adjoint state discretization based on the discrete approximation of the optimization criterion. Unfortunately, this approach does not seem to be consistent although in practice it allows to optimize elastic structures. The objective of this work was therefore to propose a more consistent discretization based on the Nitsche approximation of the continuous adjoint state. We have thus developed an a priori convergence analysis of our new approach in the case where the bi-active contact area $\Gamma_{C,b}$ is of zero measure and under assumptions of convergence rate of the contact zones. We also explained how to slightly adapt the discretization method in order to relax these assumptions while keeping a convergence result. Some numerical experiments were also presented to illustrate these convergence results.

An interesting and open question is the obtention of a convergence result of the shape gradient (11) itself. Such a convergence has been studied in the linear framework for instance in [52, 53, 54]. However, the extension to our nonlinear contact problem is a non-immediate adaptation.

Acknowledgements

We are grateful for the French National Association for Research and Technology (ANRT, CIFRE grant number 2019/1027) and the MFP Michelin for their support. We acknowledge the reviewers for their suggestions and comments which allowed to significantly improve the paper.

References

[1] D. Hilding, A. Klarbring, J. Petersson, Optimization of structures in unilateral contact (1999).

[2] F. Mignot, Contrôle dans les inéquations variationelles elliptiques, Journal of Functional Analysis 22 (2) (1976) 130–185.

[3] J. Sokolowski, J.-P. Zolésio, Shape sensitivity analysis of unilateral problems, SIAM Journal on Mathematical Analysis 18 (5) (1987) 1416–1437.

[4] J. Sokolowski, J.-P. Zolésio, Introduction to shape optimization, in: Introduction to Shape Optimization, Springer, 1992, pp. 5–12.

[5] J. Sokolowski, J.-P. Zolesio, Shape sensitivity analysis of contact problem with prescribed friction, Nonlinear Analysis: Theory, Methods & Applications 12 (12) (1988) 1399–1411.

[6] J. Sokolowski, J.-P. Zolesio, Differential stability of solutions to unilateral problems, Free Boundary Problems: Application and Theory 4 (1993) 537–547.

[7] J. Sokolowski, J.-P. Zolesio, Dérivée par rapport au domaine de la solution d'un problème unilatéral, CR Acad. Sc. Paris 301 (4) (1985) 103–106.

[8] J. Sokolowski, J.-P. Zolesio, Shape sensitivity analysis of variational inequalities, in: Introduction to Shape Optimization, Springer, 1992, pp. 163–239.

[9] V. Barbu, Optimal control of variational inequalities, Research Notes in Math. 100 (1984).

[10] A. Amassad, D. Chenais, C. Fabre, Optimal control of an elastic contact problem involving Tresca friction law, Nonlinear Analysis 48 (8) (2002) 1107–1135.

[11] A. Touzaline, Optimal control of a frictional contact problem, Acta Mathematicae Applicatae Sinica, English Series 31 (4) (2015) 991–1000.

[12] I. Páczelt, T. Szabó, Optimal shape design for contact problems, Structural Optimization 7 (1-2) (1994) 66–75.

[13] N. H. Kim, K. K. Choi, J. S. Chen, Structural optimization of finite deformation elastoplasticity using continuum-based shape design sensitivity formulation, Computers & Structures 79 (20-21) (2001) 1959–1976.

[14] B. Desmorat, Structural rigidity optimization with frictionless unilateral contact, International Journal of Solids and Structures 44 (3-4) (2007) 1132–1144.

[15] B. Chaudet-Dumas, J. Deteix, Shape derivatives for the penalty formulation of elastic contact problems with Tresca friction, SIAM Journal on Control and Optimization 58 (6) (2020) 3237–3261.

[16] B. Chaudet-Dumas, J. Deteix, Shape derivatives for an augmented Lagrangian formulation of elastic contact problems, ESAIM: Control, Optimisation and Calculus of Variations 27 (2021) S14.

[17] J. Haslinger, P. Neittaanmäki, T. Tiihonen, Shape optimization in contact problems based on penalization of the state inequality, Aplikace Matematiky 31 (1) (1986) 54–77.

[18] J. Haslinger, Signorini problem with Coulomb's law of friction. shape optimization in contact problems, International Journal for Numerical Methods in Engineering 34 (1) (1992) 223–231.

[19] J. Haslinger, A. Klarbring, Shape optimization in unilateral contact problems using generalized reciprocal energy as objective functional, Nonlinear Analysis: Theory, Methods & Applications 21 (11) (1993) 815–834.

[20] J. Haslinger, J. V. Outrata, R. Pathó, Shape optimization in 2D contact problems with given friction and a solution-dependent coefficient of friction, Set-Valued and Variational Analysis 20 (1) (2012) 31–59.

[21] J. Haslinger, P. Neittaanmäki, On the existence of optimal shapes in contact problems, Numerical Functional Analysis and Optimization 7 (2-3) (1985) 107–124.

[22] J. Haslinger, R. Mäkinen, Introduction to shape optimization: theory, approximation, and computation, SIAM, 2003.

[23] J. Haslinger, P. Neittaanmäki, Shape optimization in contact problems. approximation and numerical realization, ESAIM: Mathematical Modelling and Numerical Analysis 21 (2) (1987) 269–291.

[24] J. Haslinger, P. Neittaanmäki, Finite element approximation for optimal shape, material, and topology design, John Wiley & Sons, 1996.

[25] N. H. Kim, K. K. Choi, J. S. Chen, Shape design sensitivity analysis and optimization of elasto-plasticity with frictional contact, AIAA Journal 38 (9) (2000) 1742–1753.

[26] N. H. Kim, Y. H. Park, K. K. Choi, Optimization of a hyper-elastic structure with multibody contact using continuum-based shape design sensitivity analysis, Structural and Multidisciplinary Optimization 21 (3) (2001) 196–208.

[27] S. Stupkiewicz, J. Lengiewicz, J. Korelc, Sensitivity analysis for frictional contact problems in the augmented Lagrangian formulation, Computer Methods in Applied Mechanics and Engineering 199 (33-36) (2010) 2165–2176.

[28] N. Strömberg, A. Klarbring, Topology optimization of structures with contact constraints by using a smooth formulation and nested approach, Proceeding of the 8th World Congress on Structural and Multidisciplinary Optimization, Lisbon, Portugal (2009).

[29] N. Strömberg, A. Klarbring, Topology optimization of structures in unilateral contact, Structural and Multidisciplinary Optimization 41 (1) (2010) 57–64.

[30] E. Bretin, J. Chapelat, P.-Y. Outtier, Y. Renard, Shape optimization of a linearly elastic rolling structure under unilateral contact using Nitsche's method and cut finite elements, Computational Mechanics 70 (2021) 205–224.

[31] F. Chouly, P. Hild, Y. Renard, Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments, Mathematics of Computation 84 (293) (2013) 1089–1112.

[32] A. Maury, Shape optimization for contact and plasticity problems thanks to the level set method, Ph.D. thesis, Université Pierre et Marie Curie-Paris VI (2016).

[33] R. A. Adams, J. J. Fournier, Sobolev spaces, Elsevier, 2003.

[34] G. Fichera, Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno, atti acc, Naz. Lincei, Memoria presentata il (1964).

[35] G. Duvaut, J.-L. Lions, Les inéquations en mécanique et en physique, Dunod, 1972.

[36] N. Kikuchi, J. T. Oden, Contact problems in elasticity: a study of variational inequalities and finite element methods, SIAM, 1988.

[37] J. Haslinger, I. Hlavácek, J. Necas, Handbook of Numerical Analysis (eds. PG Ciarlet and JL Lions), vol. iv (1996).

[38] I. Ekeland, R. Temam, Convex analysis and variational problems, SIAM, 1999.

[39] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées, Vol. 33, Imprimerie nationale, 1908.

[40] F. Murat, J. Simon, Étude de problèmes d'optimal design, in: IFIP Technical Conference on Optimization Techniques, Springer, 1975, pp. 54–62.

[41] O. Pironneau, Optimal shape design for elliptic systems, in: System Modeling and Optimization, Springer, 1982, pp. 42–66.

[42] J. Simon, Differentiation with respect to the domain in boundary value problems, Numerical Functional Analysis and Optimization 2 (7-8) (1980) 649–687.

[43] A. Henrot, M. Pierre, Shape variation and optimization, Vol. 28 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2018.

[44] A. Henrot, M. Pierre, Variation et optimisation de formes: une analyse géométrique, Vol. 48, Springer Science & Business Media, 2006.

[45] M. C. Delfour, J.-P. Zolésio, Shapes and geometries: metrics, analysis, differential calculus, and optimization, SIAM, 2011.

[46] M. Hintermüller, A. Laurain, Optimal shape design subject to elliptic variational inequalities, SIAM Journal on Control and Optimization 49 (3) (2011) 1015–1047.

[47] A. Maury, G. Allaire, F. Jouve, Shape optimisation with the level set method for contact problems in linearised elasticity, The SMAI Journal of Computational Mathematics 3 (2017) 249–292.

[48] P. G. Ciarlet, J.-L. Lions, Handbook of Numerical Analysis: VOL II: Finite Element Methods.(Part 1)., North-Holland, 1991.

[49] P. Alart, A. Curnier, A mixed formulation for frictional contact problems prone to Newton like solution methods, Computer Methods in Applied Mechanics and Engineering 92 (3) (1991) 353–375.

[50] Y. Renard, K. Poulios, GetFEM: Automated FE modeling of multiphysics problems based on a generic weak form language, Transactions on Mathematical Software 47:1 (2020).

[51] M. Moussaoui, K. Khodja, Régularité des solutions d'un problème mêlé Dirichlet–Signorini dans un domaine polygonal plan, Communications in partial differential equations 17 (5-6) (1992) 805–826.

[52] R. Hiptmair, A. Paganini, S. Sargheini, Comparison of approximate shape gradients, BIT Numerical Mathematics 55 (2) (2015) 459–485.

[53] S. Zhu, Z. Gao, Convergence analysis of mixed finite element approximations to shape gradients in the stokes equation, Computer Methods in Applied Mechanics and Engineering 343 (2019) 127–150.

[54] W. Gong, S. Zhu, On discrete shape gradients of boundary type for pde-constrained shape optimization, SIAM Journal on Numerical Analysis 59 (3) (2021) 1510–1541.