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Hybrid discretization of the Signorini problem with Coulomb friction. Theoretical aspects and comparison of some numerical solvers

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Abstract

The purpose of this work is to present in a general framework the hybrid discretization of unilateral contact and friction conditions in elastostatics. A projection formulation is developed and used. An existence and uniqueness results for the solutions to the discretized problem is given in the general framework. Several numerical methods to solve the discretized problem are presented (Newton, SOR, fixed points, Uzawa) and compared in terms of the number of iterations and the robustness with respect to the value of the friction coefficient.

Keywords: Unilateral contact; Coulomb friction; Signorini problem; Bipotential; Fixed point; Newton method; SOR method; Uzawa method

0. Introduction

This work deals with the hybrid discretization of the contact and friction problem of a linearly elastic structure lying on a rigid foundation, the so-called Signorini problem with Coulomb friction (also called Coulomb problem) introduced by Duvaut and Lions [13].

Since the normal stress on the contact boundary is required to compute the friction threshold, a hybrid formulation seems the natural way to discretize the Coulomb problem. Haslinger in [14] and later Nečas

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et al. in [15] describe a hybrid formulation where a multiplier represents the normal stress. They give existence and uniqueness results for a small friction coefficient.

In this paper, the general hybrid formulation is presented in the case of dual variables introduced for both the contact stress and the friction stress. It is proven in a general framework that an inf–sup condition is sufficient to ensure existence of solutions to the discretized problem for any friction coefficient and uniqueness for a sufficiently small friction coefficient.

Two discretizations are detailed. The first one is almost conformal in displacement in the sense that the non-penetration is prescribed at each finite element contact node. The second one is a hybrid formulation where the normal stress is non-positive at each finite element contact node.

Five different numerical algorithms are presented: a fixed point on the contact boundary stress related to an Uzawa algorithm on the Tresca problem (i.e., the problem with prescribed friction threshold), a fixed point defined using the De Saxcé bipotential, a fixed point on the friction threshold, a SOR like algorithm and a Newton method. All these algorithms are compared in terms of the number of iterations, the robustness with respect to the friction coefficient and the refinement of the mesh.

1. Problem set up

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain which represents the reference configuration of a linearly elastic body submitted to a Neumann condition on Γ_N, a Dirichlet condition on Γ_D and a unilateral contact with Coulomb friction condition on Γ_C between the body and a flat rigid foundation, where Γ_N, Γ_D and Γ_C are non-overlapping open parts of $\partial\Omega$, the boundary of Ω (see Fig. 1). The displacement $u(t, x)$ of the body satisfies the following equations:

$$-\operatorname{div}\sigma(u) = f, \quad \text{in } \Omega, \tag{1}$$

$$\sigma(u) = \mathcal{A}\varepsilon(u), \quad \text{in } \Omega, \tag{2}$$

$$\sigma(u)\mathbf{n} = g, \quad \text{on } \Gamma_N, \tag{3}$$

$$u = 0, \quad \text{on } \Gamma_D, \tag{4}$$

where $\sigma(u)$ is the stress tensor, $\varepsilon(u)$ is the linearized strain tensor, $\mathcal{A}$ is the elasticity tensor which satisfies usual conditions of symmetry and coercivity, $\mathbf{n}$ is the outward unit normal to Ω on $\partial\Omega$, g and f are given force densities.

On Γ_C, it is usual to decompose the displacement and the stress in normal and tangential components as follows:

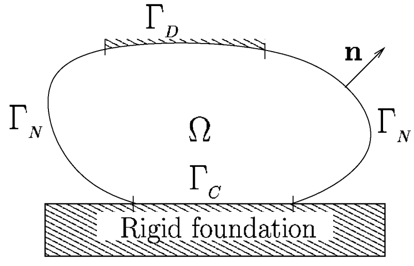

Fig. 1. Linearly elastic body Ω in frictional contact with a rigid foundation.

$$u_N = u.\mathbf{n}, \qquad u_T = u - u_N\mathbf{n},$$
$$\sigma_N(u) = \big(\sigma(u)\mathbf{n}\big).\mathbf{n}, \qquad \sigma_T(u) = \sigma(u)\mathbf{n} - \sigma_N(u)\mathbf{n}.$$

To give a clear sense to this decomposition, we assume Γ_C to have the $\mathcal{C}^1$ regularity. Assuming also that there is no initial gap between the solid and the rigid foundation, the unilateral contact condition is expressed by the following complementary condition:

$$u_N \leqslant 0, \qquad \sigma_N(u) \leqslant 0, \qquad u_N\sigma_N(u) = 0.$$

Denoting $\mathcal{F}$ the friction coefficient, the Coulomb friction condition reads as

$$\text{if } u_T = 0 \quad \text{then } \big|\sigma_T(u)\big| \leqslant -\sigma_N(u)\mathcal{F},$$
$$\text{if } u_T \neq 0 \quad \text{then } \sigma_T(u) = \sigma_N(u)\mathcal{F}\frac{u_T}{|u_T|}.$$

It is possible to express equivalently the contact and friction conditions considering the two following multivalued functions:

$$J_N(\xi) = \begin{cases} \{0\}, & \text{if } \xi < 0, \\ [0, +\infty[, & \text{if } \xi = 0, \\ \emptyset, & \text{if } \xi > 0, \end{cases}$$
$$\mathrm{Dir}_T(v) = \begin{cases} \left\{\frac{v_T}{|v_T|}\right\}, & \forall v \in \mathbb{R}^d, \text{ with } v_T \neq 0, \\ \{w \in \mathbb{R}^d;\ |w| \leqslant 1,\ w_N = 0\}, & \text{if } v_T = 0. \end{cases}$$

J_N and Dir_T are maximal monotone maps representing sub-gradients of the indicator function of interval $]-\infty, 0]$ and the function $v \mapsto |v_T|$ respectively. For a one-dimensional boundary ($d = 2$) Dir_T is the multivalued sign function (see Fig. 2).

With these maps, unilateral contact and friction conditions can be rewritten as:

$$-\sigma_N(u) \in J_N(u_N), \tag{5}$$
$$-\sigma_T(u) \in -\mathcal{F}\sigma_N(u)\,\mathrm{Dir}_T(u_T). \tag{6}$$

The latter expressions are the pointwise corresponding relations to the weak relations in the next section. See, for example, [21,18,23,16] for more details on contact and friction laws in terms of sub or generalized gradients.

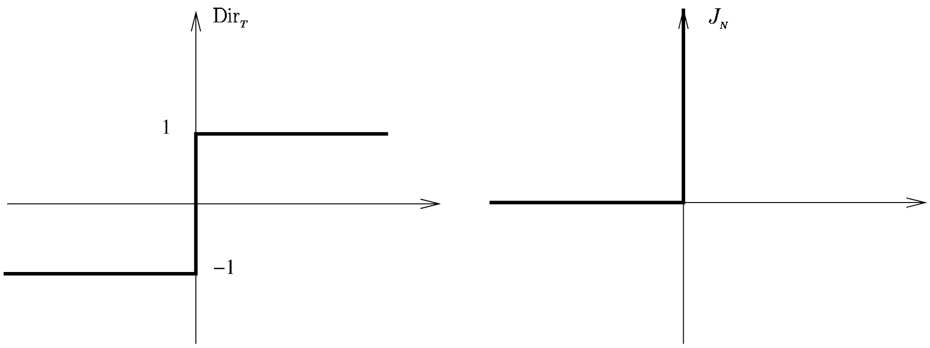

Fig. 2. Multivalued maps J_N and Dir_T for a one-dimensional boundary.

2. Weak formulation of the equations

2.1. Classical weak formulation

In this section, we start with the well-known weak formulation of Duvaut and Lions to finally give systems (11) and (12) which are inclusion formulations of the problem. These formulations, especially the last one, are very helpful to understand the general way to discretize the problem.

Following Duvaut and Lions [13], we introduce the following quantities and sets:

$$V = \big\{v \in H^1\big(\Omega; \mathbb{R}^d\big),\ v = 0 \text{ on } \Gamma_D\big\},$$
$$X_N = \{v_N|_{\Gamma_C}:\ v \in V\}, \qquad X_T = \{v_T|_{\Gamma_C}:\ v \in V\}, \qquad X = X_N \times X_T,$$

and their dual topological spaces V', X_N', X_T' and X',

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x,$$
$$l(v) = \int_\Omega f.v\,\mathrm{d}x + \int_{\Gamma_N} g.v\,\mathrm{d}\Gamma,$$
$$K_0 = \{v \in V:\ v_N \leqslant 0 \text{ on } \Gamma_C\},$$
$$j(\lambda_N, v_T) = -\big\langle \mathcal{F}\lambda_N, |v_T|\big\rangle_{X_N', X_N}.$$

We assume standard hypotheses:

$$a(\cdot\,,\cdot) \text{ bilinear symmetric continuous coercive form on } V \times V, \text{ i.e.,} \tag{7}$$
$$\exists \alpha > 0,\ \exists C_M > 0, \quad a(u,u) \geqslant \alpha \|u\|_V^2 \quad \text{and} \quad a(u,v) \leqslant C_M \|u\|_V \|v\|_V,$$
$$l(\cdot) \text{ linear continuous form on } V, \text{ i.e., } \exists C_L > 0,\ l(u) \leqslant C_L \|u\|_V, \tag{8}$$
$$\mathcal{F} \text{ Lipschitz-continuous non-negative function on } \Gamma_C. \tag{9}$$

Problem (1)–(6) is then formally equivalent to the following weak inequality formulation:

$$\begin{cases} \text{Find } u \in K_0 \text{ satisfying} \\ a(u, v-u) + j(\sigma_N(u), v_T) - j(\sigma_N(u), u_T) \geqslant l(v-u), \quad \forall v \in K_0. \end{cases} \tag{10}$$

The major difficulty with (10) is that this is not a variational inequality because this problem cannot be reduced to an optimization one. This is probably why no uniqueness result has been proved until now for the continuous problem, even for a small (but non-zero) friction coefficient. Some existence result has been proved, for instance in [22] for a sufficiently small friction coefficient.

Introducing $\lambda_N \in X_N'$, $\lambda_T \in X_T'$ two multipliers representing the stresses on the contact boundary, the equilibrium of the elastic body can be written as follows

$$a(u,v) = l(v) + \langle \lambda_N, v_N \rangle_{X_N', X_N} + \langle \lambda_T, v_T \rangle_{X_T', X_T}, \quad \forall v \in V.$$

The weak formulation of the contact condition is

$$u_N \leqslant 0, \quad \langle \lambda_N, v_N \rangle_{X_N', X_N} \geqslant 0, \quad \forall v \in K_0, \quad \langle \lambda_N, u_N \rangle_{X_N', X_N} = 0.$$

Defining the cone of admissible normal displacements as

$$K_N = \{v_N \in X_N:\ v_N \leqslant 0\},$$

we can define its polar cone

$$K_N^* = \left\{ f_N \in X_N' \colon \langle f_N, v_N \rangle_{X_N', X_N} \leqslant 0, \ \forall v_N \in K_N \right\},$$

and one gets

$$\lambda_N \in \Lambda_N,$$

where $\Lambda_N = -K_N^*$ is the set of admissible normal stress on Γ_C. We can also define the normal cone to K_N

$$N_{K_N}(v_N) = \left\{ \mu_N \in X_N' \colon \langle \mu_N, w_N - v_N \rangle_{X_N', X_N} \leqslant 0, \ \forall w_N \in K_N \right\},$$

and express the contact condition as

$$\lambda_N + N_{K_N}(u_N) \ni 0,$$

i.e., $-\lambda_N$ remains in the normal cone to K_N at the point u_N. This inclusion is in fact a weak formulation of the pointwise inclusion (5). Using a Green formula on (10) one obtains

$$\langle \lambda_T, v_T - u_T \rangle_{X_T', X_T} - \left\langle \mathcal{F}\lambda_N, |v_T| - |u_T| \right\rangle_{X_N', X_N} \geqslant 0, \quad \forall v_T \in X_T,$$

which is equivalent to

$$\lambda_T + \partial_2 j(\lambda_N, u_T) \ni 0,$$

due to the convexity in u_T of $j(\cdot, \cdot)$. Details of such correspondences can be found in [16]. This inclusion is a weak formulation of (6). So, problem (10) can be rewritten as the following direct inclusion formulation

$$\begin{cases} \text{Find } u \in V, \ \lambda_N \in X_N', \ \text{and } \lambda_T \in X_T' \text{ satisfying} \\ a(u, v) = l(v) + \langle \lambda_N, v_N \rangle_{X_N', X_N} + \langle \lambda_T, v_T \rangle_{X_T', X_T}, \forall v \in V, \\ \lambda_N + N_{K_N}(u_N) \ni 0, \qquad \text{in } X_N', \\ \lambda_T + \partial_2 j(\lambda_N, u_T) \ni 0, \quad \text{in } X_T'. \end{cases} \tag{11}$$

The two inclusions can be inverted. For the contact condition, inverting N_{K_N} is easy because it is a normal cone to K_N, and K_N is also a cone, thus $(N_{K_N})^{-1}(\lambda_N) = N_{K_N^*}(\lambda_N) = -N_{\Lambda_N}(-\lambda_N)$. So the contact condition is inverted in

$$u_N + N_{\Lambda_N}(\lambda_N) \ni 0.$$

For the friction condition, inverting $\partial_2 j(\lambda_N, u_T)$ is possible by computing the Fenchel conjugate of $j(\cdot, \cdot)$ relative to the second variable because of the relation $(\partial f)^{-1} = \partial(f^*)$ (for more details see [7,19]). One has

$$j^*(\lambda_N, \lambda_T) = I_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T),$$

where $I_{\Lambda_T(\mathcal{F}\lambda_N)}$ is the indicator function of $\Lambda_T(\mathcal{F}\lambda_N)$ with

$$\Lambda_T(g) = \left\{ \lambda_T \in X_T' \colon -\langle \lambda_T, w_T \rangle_{X_T', X_T} + \left\langle g, |w_T| \right\rangle_{X_N', X_N} \leqslant 0, \ \forall w_T \in X_T \right\}.$$

So, the friction condition can be expressed as follows

$$u_T + N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) \ni 0,$$

due to $\partial_{\lambda_T} I_{\Lambda_T(\mathcal{F}\lambda_N)} = N_{\Lambda_T(\mathcal{F}\lambda_N)}$. Finally, problem (10) can be rewritten as the following hybrid inclusion formulation

$$\begin{cases} \text{Find } u \in V,\ \lambda_N \in X_N',\ \text{ and } \lambda_T \in X_T' \text{ satisfying} \\ a(u,v) = l(v) + \langle \lambda_N, v_N \rangle_{X_N',X_N} + \langle \lambda_T, v_T \rangle_{X_T',X_T}, \quad \forall v \in V, \\ -u_N \in N_{\Lambda_N}(\lambda_N), \\ -u_T \in N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T). \end{cases} \tag{12}$$

The two inclusions can also be transformed into variational inequalities as follows:

$$\begin{cases} \text{Find } u \in V, \lambda_N \in X_N',\ \text{ and } \lambda_T \in X_T' \text{ satisfying} \\ a(u,v) = l(v) + \langle \lambda_N, v_N \rangle_{X_N',X_N} + \langle \lambda_T, v_T \rangle_{X_T',X_T}, \quad \forall v \in V, \\ \lambda_N \in \Lambda_N, \quad \langle \mu_N - \lambda_N, u_N \rangle_{X_N',X_N} \geqslant 0, \qquad \forall \mu_N \in \Lambda_N, \\ \lambda_T \in \Lambda_T(\mathcal{F}\lambda_N), \quad \langle \mu_T - \lambda_T, u_T \rangle_{X_T',X_T} \geqslant 0, \quad \forall \mu_T \in \Lambda_T(\mathcal{F}\lambda_N). \end{cases} \tag{13}$$

2.2. Weak inclusion formulation using De Saxcé bipotential method

In a discrete framework, De Saxcé [11] gives a new formulation of the contact and friction conditions allowing to write them using a unique inclusion. The definition of a *bipotential*, given by De Saxcé, is a convex, lower semi-continuous function of each of its variables $b(\zeta, x): H' \times H \to \overline{\mathbb{R}}$ (H is an Hilbert space) satisfying the following generalized Fenchel inequality

$$b(\xi, y) \geqslant \langle \mu, v \rangle_{H',H}, \quad \forall \mu \in H',\ \forall v \in H. \tag{14}$$

In [19], a slightly more restrictive definition is introduced. It is asked for the bipotential to satisfy the two following relations:

$$\inf_{y \in H} \big(b(\zeta, y) - \langle \zeta, y \rangle_{H',H} \big) \in \{0, +\infty\}, \quad \forall \zeta \in H', \tag{15}$$

$$\inf_{\xi \in H'} \big(b(\xi, x) - \langle \mu, x \rangle_{H',H} \big) \in \{0, +\infty\}, \quad \forall x \in H. \tag{16}$$

Of course, (15) or (16) implies (14). The value $+\infty$ cannot be avoided since the bipotential can contain some indicator functions. These conditions are naturally satisfied by the bipotential representing the Coulomb friction law.

Now, a pair (ζ, x) is said to be *extremal* if it satisfies the following relation

$$b(\zeta, x) = \langle \zeta, x \rangle_{H',H}. \tag{17}$$

Subtracting (17) from (14), this means that

$$b(\zeta, y) - b(\zeta, x) \geqslant \langle \zeta, y - x \rangle_{H',H}, \quad \forall y \in H,$$

which is equivalent to

$$-\zeta \in \partial_x b(\zeta, x). \tag{18}$$

A similar reasoning leads to

$$-x \in \partial_\zeta b(\zeta, x). \tag{19}$$

Moreover, due to (15), inclusion (18) is clearly equivalent to (17) and due to (16) inclusion (19) is also equivalent to (17). Thus (18) and (19) are equivalent one to each other (inequality (14) is not sufficient to conclude to this equivalence, this is the reason why (15) and (16)) has been introduced.

De Saxcé defined the so-called bipotential of the Coulomb friction law which can be written in a continuous version as

$$b(-\lambda, u) = \big\langle -\lambda_N, \mathcal{F}|u_T|\big\rangle_{X'_N, X_N} + I_{\Lambda_\mathcal{F}}(-\lambda) + I_{\Lambda_N}(u_N), \tag{20}$$

where $\Lambda_\mathcal{F}$ is the weak friction cone given by

$$\begin{aligned}\Lambda_\mathcal{F} &= \big\{(\lambda_N, \lambda_T) \in X'_N \times X'_T\colon -\langle \lambda_T, v_T\rangle + \big\langle \mathcal{F}\lambda_N, |v_T|\big\rangle \leqslant 0,\ \forall v_T \in X'_T\big\} \\ &= \big\{(\lambda_N, \lambda_T) \in X'_N \times X'_T\colon \lambda_N \in \Lambda_N,\ \lambda_T \in \Lambda_T(\mathcal{F}\lambda_N)\big\}.\end{aligned}$$

The inclusion $\lambda \in \partial_u b(-\lambda, u)$ gives exactly the inclusions of problem (12). Thus, if $b(-\lambda, u)$ is a bipotential it will be equivalent to $-u \in \partial_\lambda b(-\lambda, u)$ which gives

$$-\big(u_N - \mathcal{F}|u_T|, u_T\big) \in N_{\Lambda_\mathcal{F}}(\lambda_N, \lambda_T). \tag{21}$$

Lemma 1. *$b(-\lambda, u)$ defined by* (20) *is a bipotential.*

The proof of this lemma is immediate. More details can be found in [19].

Using inclusion (21), the expression of the Signorini problem with Coulomb friction (12) is equivalent to

$$\begin{cases} \text{Find } u \in V,\ \lambda_N \in X'_N, \text{ and } \lambda_T \in X'_T \text{ satisfying} \\ a(u, v) = l(v) + \langle \lambda_N, v_N\rangle_{X'_N, X_N} + \langle \lambda_T, v_T\rangle_{X'_T, X_T}, \quad \forall v \in V, \\ -\binom{u_N - \mathcal{F}|u_T|}{u_T} \in N_{\Lambda_\mathcal{F}}(\lambda_N, \lambda_T) \\ \quad \Longleftrightarrow\ (\lambda_N, \lambda_T) \in \Lambda_\mathcal{F}, \\ \qquad \langle \mu_N - \lambda_N, u_N - \mathcal{F}|u_T|\rangle_{X'_N, X_N} + \langle \mu_T - \lambda_T, u_T\rangle_{X'_T, X_T} \geqslant 0, \quad \forall (\mu_N, \mu_T) \in \Lambda_\mathcal{F}. \end{cases} \tag{22}$$

3. Hybrid finite element discretization

Let $V^h \subset V$ be a family of finite dimensional vector subspaces indexed by h coming from a regular finite element discretization of the domain Ω (h represents the radius of the largest element). Let us define

$$\begin{aligned} &X_N^h = \big\{v_N^h|_{\Gamma_C}\colon v^h \in V^h\big\}, \qquad X_T^h = \big\{v_T^h|_{\Gamma_C}\colon v^h \in V^h\big\}, \\ &X^h = \big\{v^h|_{\Gamma_C}\colon v^h \in V^h\big\} = X_N^h \times X_T^h. \end{aligned}$$

Let us denote also $X_N'^h \subset X'_N \cap L^2(\Gamma_C)$ and $X_T'^h \subset X'_T \cap L^2(\Gamma_C; \mathbb{R}^{d-1})$ the finite element discretizations of X'_N and X'_T respectively, such that the following discrete Babuška–Brézzi inf–sup conditions hold (see [2])

$$\inf_{\lambda_N^h \in X_N'^h} \sup_{v^h \in V^h} \frac{\langle \lambda_N^h, v_N^h\rangle}{\|v^h\|_V \|\lambda_N^h\|_{X_N'^h}} \geqslant \gamma > 0, \tag{23}$$

$$\inf_{\lambda_T^h \in X_T'^h} \sup_{v^h \in V^h} \frac{\langle \lambda_T^h, v_T^h \rangle}{\|v^h\|_V \|\lambda_T^h\|_{X_T'^h}} \geqslant \gamma > 0, \tag{24}$$

with γ independent of h.

Remark 1. For a regular family of triangulations, it is possible to build an extension operator from X^h to V^h with a norm independent of h (see [6]). The consequence is that it is sufficient to have an inf–sup condition between $X_N'^h$ and X_N^h (respectively $X_T'^h$ and X_T^h). Examples of finite element satisfying the inf–sup condition can be found in [8]. The choice $X_N'^h = X_N^h$ and $X_T'^h = X_T^h$ (via the identification between $L^2(\Gamma_C)$ and its dual space) corresponds to a direct discretization of (10) and always ensures the inf–sup conditions. A P_2 Lagrange element for u and a P_1 Lagrange element for the multipliers also satisfy the Babuška–Brezzi conditions. This is generally not the case for a P_1 Lagrange element for u and a P_0 Lagrange element for the multipliers.

Now, with a particular choice of $\Lambda_N^h \subset X_N'^h$ and $\Lambda_T^h(\mathcal{F}\lambda_N^h) \subset X_T'^h$ closed convex approximations of Λ_N and $\Lambda_T(\mathcal{F}\lambda_N^h)$ respectively (the conditions $\Lambda_N^h \subset \Lambda_N$ and $\Lambda_T^h(\mathcal{F}\lambda_N^h) \subset \Lambda_T(\mathcal{F}\lambda_N^h)$ are generally not satisfied) the finite element discretization of problem (13) reads

$$\begin{cases} \text{Find } u^h \in V^h, \lambda_N^h \in X_N'^h \text{ and } \lambda_T^h \in X_T'^h \text{ satisfying} \\ a(u^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda_N^h u_N^h \,\mathrm{d}\Gamma + \int_{\Gamma_C} \lambda_T^h . u_T^h \,\mathrm{d}\Gamma, \quad \forall v^h \in V^h, \\ \lambda_N^h \in \Lambda_N^h, \quad \int_{\Gamma_C} (\mu_N^h - \lambda_N^h) u_N^h \,\mathrm{d}\Gamma \geqslant 0, \quad \forall \mu_N^h \in \Lambda_N^h, \\ \quad \iff \lambda_N^h = P_{\Lambda_N^h}(\lambda_N^h - r u_N^h), \\ \lambda_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \quad \int_{\Gamma_C} (\mu_T^h - \lambda_T^h). u_T^h \,\mathrm{d}\Gamma \geqslant 0, \quad \forall \mu_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \\ \quad \iff \lambda_T^h = P_{\Lambda_T^h(\mathcal{F}\lambda_N)}(\lambda_T^h - r u_T^h), \end{cases} \tag{25}$$

where the two signs $\iff$ indicate that the inequalities can be replaced by projections. The maps $P_{\Lambda_N^h}$ and $P_{\Lambda_T^h(\mathcal{F}\lambda_N)}$ stand for the L^2 projections onto convexes Λ_N^h and $\Lambda_T^h(\mathcal{F}\lambda_N)$ respectively, and $r > 0$ is an arbitrary augmentation parameter. We refer to [19] for more details on projection formulations of unilateral contact and friction conditions.

Introducing now the following matrix notations

$$u^h(x) = \sum_{i=1}^{k_1} u^i \varphi_i, \qquad \lambda_N^h(x) = \sum_{i=1}^{k_2} \lambda_N^i \psi_i, \qquad \lambda_T^h(x) = \sum_{i=1}^{k_3} \lambda_T^i \xi_i, \tag{26}$$

$$U = (u_i)_{i=1,\dots,k_1}, \qquad L_N = \left(\lambda_N^i\right)_{i=1,\dots,k_2}, \qquad L_T = \left(\lambda_T^i\right)_{i=1,\dots,k_3}, \tag{27}$$

$$(B_N)_{ij} = \int_{\Gamma_C} \psi_i \mathbf{n}.\varphi_j \,\mathrm{d}\Gamma, \qquad (B_T)_{ij} = \int_{\Gamma_C} \xi_i.\varphi_j \,\mathrm{d}\Gamma, \qquad (K)_{ij} = a(\varphi_i, \varphi_j), \tag{28}$$

where φ_i, ψ_i and ξ_i are the shape functions of the finite element methods used, the contact condition

$$\lambda_N^h \in \Lambda_N^h, \quad \left\langle \mu_N^h - \lambda_N^h, u_N^h \right\rangle_{X_N', X_N} \geqslant 0, \quad \forall \mu_N^h \in \Lambda_N^h,$$

can be expressed in a matrix formulation

$$(M_N - L_N)^{\mathrm{T}} B_N U \geqslant 0, \quad \forall M_N \in \overline{\Lambda}_N^h, \tag{29}$$

where

$$\overline{\Lambda}_N^h = \left\{ L_N \in \mathbb{R}^{k_2}\colon \sum_{i=1}^{k_2} \lambda_N^i \psi_i \in \Lambda_N^h \right\}$$

is the corresponding convex of admissible L_N. This is equivalent to $B_N U$ in the normal cone to $\overline{\Lambda}_N^h$ in L_N or equivalently

$$L_N = P_{\overline{\Lambda}_N^h}(L_N - rB_N U),$$

for any $r > 0$ and where $P_{\overline{\Lambda}_N^h}$ stands now for the projection onto $\overline{\Lambda}_N^h$ with respect to the Euclidean scalar product. With the same treatment for the tangential stress, one can express the matrix formulation of problem (25) as follows

$$\begin{cases} \text{Find } U \in \mathbb{R}^{k_1}, L_N \in \mathbb{R}^{k_2} \text{ and } L_T \in \mathbb{R}^{k_3} \text{ satisfying} \\ KU = F + B_N^{\mathrm{T}} L_N + B_T^{\mathrm{T}} L_T, \\ L_N = P_{\overline{\Lambda}_N^h}(L_N - rB_N U), \\ L_T = P_{\overline{\Lambda}_T^h(\mathcal{F}L_N)}(L_T - rB_T U). \end{cases} \tag{30}$$

One can also work with modified multipliers, which in some discretizations correspond to equivalent forces on the contact boundaries: inequality (29) can be rewritten as

$$\left(B_N^{\mathrm{T}} M_N - B_N^{\mathrm{T}} L_N\right)^{\mathrm{T}} U \geqslant 0, \quad \forall M_N \in \overline{\Lambda}_N^h.$$

Then, denoting $\widetilde{L}_N = B_N^{\mathrm{T}} L_N$, $\widetilde{L}_T = B_T^{\mathrm{T}} L_T$, $\widetilde{\Lambda}_N^h = B_N^{\mathrm{T}} \overline{\Lambda}_N^h$, and $\widetilde{\Lambda}_T^h(\mathcal{F}\widetilde{L}_N) = B_T^{\mathrm{T}} \overline{\Lambda}_T^h(\mathcal{F}L_N)$, one obtains the following matrix formulation

$$\begin{cases} \text{Find } U \in \mathbb{R}^{k_1}, \widetilde{L}_N \in \mathbb{R}^{k_1} \text{ and } \widetilde{L}_T \in \mathbb{R}^{k_1} \text{ satisfying} \\ KU = F + \widetilde{L}_N + \widetilde{L}_T, \\ \widetilde{L}_N = P_{\widetilde{\Lambda}_N^h}(\widetilde{L}_N - rU), \\ \widetilde{L}_T = P_{\widetilde{\Lambda}_T^h(\mathcal{F}\widetilde{L}_N)}(\widetilde{L}_T - rU). \end{cases} \tag{31}$$

In fact, $\widetilde{L}_N$ and $\widetilde{L}_T$ are in the range of B_N^{T} and B_T^{T}, respectively and thus remain in a vector subspace of dimension k_2 and k_3, respectively.

The choice between (30) and (31) will depend on which of the convex sets $\overline{\Lambda}_N^h$, $\overline{\Lambda}_T^h(\mathcal{F}L_N)$ or $\widetilde{\Lambda}_N^h$, $\widetilde{\Lambda}_T^h(\mathcal{F}\widetilde{L}_N)$ have the simplest expression. The advantage of these two formulations is that contact and frictions conditions are expressed without constraints and with Lipschitz continuous expressions.

The hybrid discretization of Signorini problems is also discussed in [3–5,16,17].

Remark 2. There is of course a strict equivalence between problem (25) and the two formulations (30) and (31) for an arbitrary $r > 0$. In [19] an analysis of projection formulations has been done and the link between the projection formulations and augmented Lagrangian for the Tresca problem has been also discussed.

Remark 3. In [19], the formulation with projections with respect to the $H^{1/2}$ inner product has been studied and we proved that for the Tresca problem there is no degradation of the contraction constant of the corresponding fixed point (see bellow the definition of T_h^1 and T_h^2). With L^2 projections, the contraction constant tends to 1 when h goes to 0. If one wants to use the $H^{1/2}$ projections, one has to replace B_N and B_T in formulation (30) with the matrix coming from the $H^{1/2}$ inner product. Formulation (31) is not changed (except perhaps the definition of $\widetilde{\Lambda}_N^h$ and $\widetilde{\Lambda}_T^h(\mathcal{F}\widetilde{L}_N)$).

3.1. Discretization of the De Saxcé formulation

It is possible to define $\Lambda_{\mathcal{F}}^h$ as

$$\Lambda_{\mathcal{F}}^h = \left\{\left(\lambda_N^h, \lambda_T^h\right) \in X_N'^h \times X_T'^h\colon\ \lambda_N^h \in \Lambda_N^h,\ \lambda_T^h \in \Lambda_T^h\left(\mathcal{F}\lambda_N^h\right)\right\}. \tag{32}$$

In the following, we will consider this definition, although $\Lambda_{\mathcal{F}}^h$ could be defined independently.

The discretization of problem (22) reads

$$\begin{cases} \text{Find } u^h \in V^h, \lambda_N^h \in X'^h_N, \text{ and } \lambda_T^h \in X'^h_T \text{ satisfying} \\ a(u^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda_N^h u_N^h \,\mathrm{d}\Gamma + \int_{\Gamma_C} \lambda_T^h . u_T^h \,\mathrm{d}\Gamma, \quad \forall v^h \in V^h, \\ -\binom{u_N^h - \mathcal{F}|u_T^h|}{u_T^h} \in N_{\Lambda_{\mathcal{F}}^h}(\lambda_N^h, \lambda_T^h) \iff \binom{\lambda_N^h}{\lambda_T^h} = P_{\Lambda_{\mathcal{F}}^h}\binom{\lambda_N^h - r(u_N^h - \mathcal{F}|u_T^h|)}{\lambda_T^h - r u_T^h}. \end{cases} \tag{33}$$

Here the normal cone has to be understood in a $L^2(\Gamma_C, \mathbb{R}^d)$ sense:

$$N_{\Lambda_{\mathcal{F}}^h}\left(\lambda^h\right) = N_{\Lambda_{\mathcal{F}}^h}\left(\lambda_N^h, \lambda_T^h\right) = \left\{w \in L^2\left(\Gamma_C, \mathbb{R}^d\right)\colon \int_{\Gamma_C} w\left(\mu^h - \lambda^h\right) \mathrm{d}\Gamma \leqslant 0;\ \forall \mu^h \in \Lambda_{\mathcal{F}}^h\right\}.$$

With $\Lambda_{\mathcal{F}}^h$ defined by (32), one can verify that (25) and (33) are equivalent. The matrix formulation of this latter problem is

$$\begin{cases} \text{Find } U \in \mathbb{R}^{k_1}, L_N \in \mathbb{R}^{k_2} \text{ and } L_T \in \mathbb{R}^{k_3} \text{ satisfying} \\ KU = F + B_N^{\mathrm{T}} L_N + B_T^{\mathrm{T}} L_T, \\ \binom{L_N}{L_T} = P_{\overline{\Lambda}_{\mathcal{F}}^h}\binom{L_N - rB_N U + r\mathcal{F}S_T(U)}{L_T - rB_T U}, \end{cases} \tag{34}$$

where $S_T(U)$ is the vector defined by

$$\left(S_T(U)\right)_i = \int_{\Gamma_C} \psi_i |u_T| \,\mathrm{d}\Gamma.$$

3.2. Fixed point formulations and existence and uniqueness of solution to the discrete problems

Problems (25) and (33) leads to fixed points formulations. Let us define two maps T_1^h, T_2^h as follows

$$T_1^h : X'^h \to X'^h$$

$$\begin{pmatrix} \lambda_N^h \\ \lambda_T^h \end{pmatrix} \mapsto \begin{pmatrix} P_{\Lambda_N^h}(\lambda_N^h - r u_N^h) \\ P_{\Lambda_T^h(\mathcal{F}\lambda_N^h)}(\lambda_T^h - r u_T^h) \end{pmatrix},$$

$$T_2^h : X'^h \to X'^h$$

$$\lambda^h \mapsto P_{\Lambda_{\mathcal{F}}^h}\left(\lambda^h - r\begin{pmatrix} u_N^h - \mathcal{F}|u_T^h| \\ u_T^h \end{pmatrix}\right),$$

where u^h is solution to

$$a(u^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda_N^h u_N^h \,\mathrm{d}\Gamma + \int_{\Gamma_C} \lambda_T^h . u_T^h \,\mathrm{d}\Gamma, \quad \forall v^h \in V^h.$$

The fixed points of these two maps are solutions of the discrete Coulomb problem and are independent of the augmentation parameter $r > 0$.

Theorem 1. *Under hypotheses* (7)–(9), (23), (24) *and for $r > 0$ sufficiently small, maps T_1^h and T_2^h have at least one fixed point. Thus, problems* (25), (30), (31), (33) *and* (34) *have at least one solution for any $\mathcal{F}$ and any $r > 0$.*

Proof. The proof is done for T_1^h, the proof for T_2^h is similar.

Let us establish that for a sufficiently small $r > 0$ and sufficiently large λ^h

$$\|T_1^h(\lambda^h)\|_{L^2(\Gamma_C)} \leqslant \|\lambda^h\|_{L^2(\Gamma_C)}, \quad \text{where } \lambda^h = (\lambda_N^h, \lambda_T^h).$$

One has

$$\begin{aligned}\|T_1^h(\lambda^h)\|^2_{L^2(\Gamma_C)} &= \|P_{\Lambda_N^h}(\lambda_N^h - r u_N^h)\|^2_{L^2(\Gamma_C)} + \|P_{\Lambda_T^h(\mathcal{F}\lambda_N^h)}(\lambda_T^h - r u_T^h)\|^2_{L^2(\Gamma_C)} \\ &\leqslant \|\lambda^h - r u^h\|^2_{L^2(\Gamma_C)} \leqslant \|\lambda^h\|^2_{L^2(\Gamma_C)} - 2r \int_{\Gamma_C} \lambda^h . u^h \,\mathrm{d}\Gamma + r^2 \|u^h\|^2_{L^2(\Gamma_C)}.\end{aligned}$$

But

$$\int_{\Gamma_C} \lambda^h u^h \,\mathrm{d}\Gamma = a(u^h, u^h) - l(u^h) \geqslant \alpha \|u^h\|_V^2 - C_L \|u^h\|_V, \tag{35}$$

and

$$\|u^h\|_{L^2(\Gamma_C)} \leqslant \beta \|u^h\|_V, \qquad \|u^h\|_{L^2(\Gamma_C)} \leqslant \frac{\beta}{\alpha}\big(C_L + \beta \|\lambda^h\|_{L^2(\Gamma_C)}\big), \tag{36}$$

and from the inf–sup conditions (23), (24)

$$\|\lambda^h\|_{L^2(\Gamma_C)} \leqslant \frac{1}{\eta^h \gamma}\big(C_M \|u^h\|_V + C_L\big) \quad \text{where } \eta^h \text{ is such that } \|\lambda^h\|_{X'} \geqslant \eta^h \|\lambda^h\|_{L^2(\Gamma_C)},$$

where C_M and C_L are defined by (7) and (8). Finally,

$$\begin{aligned}\|T_1^h(\lambda^h)\|^2_{L^2(\Gamma_C)} &\leqslant \|\lambda^h\|^2_{L^2(\Gamma_C)} - 2r\alpha \|u^h\|_V^2 + 2r C_L \|u^h\|_V + r^2 \|u^h\|^2_{L^2(\Gamma_C)} \\ &\leqslant \|\lambda^h\|^2_{L^2(\Gamma_C)} - 2r\alpha \left(\frac{\eta^h \gamma}{C_M} \|\lambda^h\|_{L^2(\Gamma_C)} - \frac{C_L}{C_M}\right)^2 \\ &\quad + 2r \frac{C_L}{\alpha}\big(C_L + \beta \|\lambda^h\|_{L^2(\Gamma_C)}\big) + r^2 \frac{\beta^2}{\alpha^2}\big(C_L + \beta \|\lambda^h\|_{L^2(\Gamma_C)}\big)^2.\end{aligned}$$

Thus, there exists C^h such that, for $\|\lambda^h\|_{L^2(\Gamma_C)} > C^h$, the term in factor of r is always strictly negative and there will be a r_0 such that

$$\|T_1^h(\lambda^h)\|_{L^2(\Gamma_C)} < \|\lambda^h\|_{L^2(\Gamma_C)},$$

for $\|\lambda^h\|_{L^2(\Gamma_C)} > C^h$ and $0 < r < 2r_0$.

Now, using the triangular inequality, there exist k_1 and k_2 such that

$$\|T_1^h(\lambda^h)\|_{L^2(\Gamma_C)} \leqslant \|\lambda^h\|_{L^2(\Gamma_C)} + r\|u^h\|_{L^2(\Gamma_C)} \leqslant k_1\|\lambda^h\|_{L^2(\Gamma_C)} + k_2,$$

and thus

$$\|T_1^h(\lambda^h)\|_{L^2(\Gamma_C)} \leqslant C^h k_1 + k_2, \quad \text{when } \|\lambda^h\|_{L^2(\Gamma_C)} \leqslant C^h.$$

This means that $T_1^h(\lambda^h)$ is continuous map from the ball of radius $C^h k_1 + k_2$ into itself and then one can conclude with Brouwer's fixed point theorem. □

Theorem 2. *Under hypotheses* (7)–(9), (23), (24) *and for $r > 0$ sufficiently small and $\|\mathcal{F}\|_\infty$ sufficiently small, the mappings T_1^h and T_2^h are strict contractions. Thus, problems* (25), (30), (31), (33), (34) *have a unique solution for $\|\mathcal{F}\|_\infty$ sufficiently small and any $r > 0$.*

Proof. The proof is done for T_2^h, the proof for T_1^h is similar.

Let us denote $\delta T_2^h(\lambda^h) = T_2^h(\lambda_1^h) - T_2^h(\lambda_2^h)$, $\delta\lambda^h = \lambda_1^h - \lambda_2^h = \delta\lambda^h$ and $\delta u^h = u_1^h - u_2^h$. Then

$$\begin{aligned}
\|\delta T_2^h(\lambda^h)\|^2_{L^2(\Gamma_C)} &= \left\| P_{\Lambda^h_{\mathcal{F}}}\left(\lambda_1^h - r\begin{pmatrix} u_{1N}^h - \mathcal{F}|u_{1T}^h| \\ u_{1T}^h \end{pmatrix}\right) - P_{\Lambda^h_{\mathcal{F}}}\left(\lambda_2^h - r\begin{pmatrix} u_{2N}^h - \mathcal{F}|u_{2T}^h| \\ u_{2T}^h \end{pmatrix}\right)\right\|^2_{L^2(\Gamma_C)} \\
&\leqslant \left\| \delta\lambda^h - r\begin{pmatrix} \delta u_N^h - \mathcal{F}\delta|u_T^h| \\ \delta u_T^h \end{pmatrix}\right\|^2_{L^2(\Gamma_C)} \\
&= \|(\delta\lambda^h - r\delta u^h) + r\delta v^h\|^2_{L^2(\Gamma_C)} \quad \text{with } v^h = \begin{pmatrix} \mathcal{F}|u_T^h| \\ 0 \end{pmatrix} \\
&\leqslant \left(\|\delta\lambda^h - r\delta u^h\|_{L^2(\Gamma_C)} + r\|\delta v^h\|_{L^2(\Gamma_C)}\right)^2.
\end{aligned}$$

But

$$\|\delta\lambda^h - r\delta u^h\|^2_{L^2(\Gamma_C)} \leqslant \|\delta\lambda^h\|^2_{L^2(\Gamma_C)} - 2r\int_{\Gamma_C} \delta\lambda^h.\delta u^h \,\mathrm{d}\Gamma + r^2\|\delta u^h\|^2_{L^2(\Gamma_C)},$$

and

$$\int_{\Gamma_C} \delta\lambda^h.\delta u^h \,\mathrm{d}\Gamma \geqslant \alpha\|\delta u^h\|^2_V,$$

moreover

$$\|\delta u^h\|_{L^2(\Gamma_C)} \leqslant \beta\|\delta u^h\|_V \quad \text{and} \quad \|\delta v^h\|_{L^2(\Gamma_C)} \leqslant \|\mathcal{F}\|_\infty\|\delta u^h\|_{L^2(\Gamma_C)}.$$

Thus, with $\xi = \frac{\|\delta u^h\|_V}{\|\delta\lambda^h\|_{L^2(\Gamma_C)}} \geqslant \frac{\eta^h\gamma}{C_M}$, and choosing r sufficiently small such that $(1 - 2r\alpha\xi^2 + r^2\beta^2\xi^2) < 1$, one has

$$\begin{aligned}
\|\delta T_2^h(\lambda^h)\|^2_{L^2(\Gamma_C)} &\leqslant \|\delta\lambda^h\|^2_{L^2(\Gamma_C)}\left((1 - 2r\alpha\xi^2 + r^2\beta^2\xi^2)^{1/2} + r\|\mathcal{F}\|_\infty\beta\xi\right)^2 \\
&\leqslant \|\delta\lambda^h\|^2_{L^2(\Gamma_C)}\left(1 - 2r\alpha\xi^2 + r^2\beta^2\xi^2 + 2r\|\mathcal{F}\|_\infty\beta\xi + r^2\|\mathcal{F}\|^2_\infty\beta^2\xi^2\right).
\end{aligned}$$

Thus, the contraction constant is less than one for r sufficiently small when

$$\|\mathcal{F}\|_\infty \leqslant \frac{\alpha\eta^h\gamma}{C_M\beta},$$

and T_2^h is a contraction for $r < 2r_0$ where

$$r_0 = \frac{\alpha\gamma\eta^h - C_M\beta\|\mathcal{F}\|_\infty}{(1+\|\mathcal{F}\|_\infty)^2\beta^2\eta^h\gamma}.$$

This ensures existence and uniqueness of the solution. □

Remark 4. The constant η^h, in the proofs of the two previous theorems, represents the equivalence constant between the $L^2(\Gamma_C)$ norm and the X' norm. For regular discretizations, this constant is of order $\sqrt{h}$ (see [10] for instance). This means that the bound for $\|\mathcal{F}\|_\infty$ which ensures the uniqueness goes to zero when h goes to zero. This is coherent with the fact that no uniqueness result has been proven for the continuous problem, even for a sufficiently small friction coefficient. As a consequence, it seems not to be possible to give error estimate in a global framework.

4. Example of discretizations

In order to perform numerical tests and comparisons between several approaches, an exhaustive description of the discretization will be given in two cases (of course, many other discretizations are possible):

- an “almost conformal” discretization of the displacement where the same Lagrange finite element method is used for both the displacement and forces on the contact boundary;
- an “almost conformal” discretization of the friction and contact forces with different Lagrange finite element methods for the displacement and the forces on the contact boundary.

For the sake of simplicity, the friction coefficient $\mathcal{F}$ will be assumed to be a constant.

Let us denote a_i, $i = 1, \dots, N_c$, the set of all the finite element nodes and $I_C = \{i\colon a_i \in \Gamma_C\}$ the indices of nodes on Γ_C. We still use notations defined in (26)–(28). For a Lagrange element, it is possible to define $N_i \in \mathbb{R}^{k_1}$ for $i \in I_C$ such that the normal displacement in a finite element node on Γ_C can be written

$$u_N^h(a_i) = N_i^{\mathrm{T}} U.$$

Similarly, we consider at each node a_i an orthonormal basis t_i^α, $\alpha = 1, \dots, d-1$, of the tangent plane to Γ_C. Denoting t_i the corresponding $d \times d-1$ matrices (in which the t_i^α are stored columnwise), it is possible to define T_i some $k_1 \times (d-1)$ matrices for $i \in I_C$ such that

$$u_T^h(a_i) = u_T^i = t_i T_i^{\mathrm{T}} U.$$

4.1. Almost conformal in u_N^h discretization

This case corresponds approximately to a direct discretization of problem (10) (i.e., a standard Galerkin procedure applied to this problem), because since

$$\forall F \in X' \text{ one can find } \widetilde{F} \in X^h \quad \text{such that} \quad \langle F, v^h \rangle = \int_{\Gamma} \widetilde{F}.v^h, \quad \forall v^h \in X^h.$$

A direct discretization is equivalent to the choice $X_N^h = X_N'^h$ and $X_T^h = X_T'^h$. A conformal discretization in u_N^h is obtained when $K_N^h \subset K_N$. A natural choice for K_N^h would be

$$\{u_N^h \in X_N^h\colon\ u_N^h(x) \leqslant 0\}.$$

The drawback with this choice is that for finite element method of degree greater or equal to two, the condition $u_N^i \leqslant 0$ is not easy to express neither on the coefficients of the polynomials nor on the nodal values (see [17]). This is why most of the time, a non-conformal discretization is chosen, where the non-penetration condition is assumed on the finite element nodes as follows:

$$K_N^h = \{u_N^h \in X_N^h\colon\ u_N(a_i) \leqslant 0 \text{ for } i \in I_C\}.$$

In the matrix formulation this corresponds to the condition $U.N_i \leqslant 0$ for $i \in I_C$. The corresponding set of admissible normal stresses is defined by

$$\Lambda_N^h = \left\{\lambda_N^h \in X_N^h\colon\ \int_{\Gamma_C} \lambda_N^h(x) u_N^h(x)\, \mathrm{d}\Gamma \geqslant 0,\ \forall u_N^h \in K_N^h\right\}.$$

Still denoting ψ_i the shape functions of the finite element space X_N^h

$$\psi_i \in X_N^h; \quad \psi_i(a_j) = \delta_{ij}, \quad \forall i, j \in I_C$$

this is equivalent to

$$\Lambda_N^h = \left\{\lambda_N^h \in X_N^h\colon\ \int_{\Gamma_C} \lambda_N^h(x)\psi_i\, \mathrm{d}\Gamma \leqslant 0,\ \forall i \in I_C\right\}.$$

This means that using matrix formulation (31), $\widetilde{\Lambda}_N^h$ is defined by

$$\widetilde{\Lambda}_N^h = \left\{\widetilde{L}_N = \sum_{i \in I_C} \tilde{\lambda}_N^i N_i\colon\ \tilde{\lambda}_N^i \leqslant 0,\ \forall i \in I_C\right\},$$

with the relation $\tilde{\lambda}_N^i = \int_{\Gamma_C} \lambda_N^h(x)\psi_i\, \mathrm{d}\Gamma$. Since $\widetilde{\Lambda}_N^h$ is very simple in this case, we will use matrix formulation (31) instead of formulation (30).

Concerning the tangential stress, a natural way is to consider the set

$$\left\{\lambda_T^h \in X_T^h\colon\ -\int_{\Gamma_C} \lambda_T^h(x).w_T(x)\, \mathrm{d}\Gamma + \int_{\Gamma_C} \mathcal{F}\lambda_N^h(x)\big|w_T(x)\big|\, \mathrm{d}\Gamma \leqslant 0,\ \forall w_T \in X_T^h\right\},$$

but, due to the non-linearity of the term $|w_T(x)|$, this set is not easy to express. A classical way to proceed is to interpolate this term on the Lagrange basis, that is to do the approximation of $|w_T(x)|$ by $\sum_{i \in I_C} |w_T(a_i)|\psi_i(x)$.

Denoting ξ_i^α the shape functions of X_T^h

$$\xi_i^\alpha(a_j) \in X_T^h, \quad \xi_i^\alpha(a_j) = t_i^\alpha \delta_{ij}, \quad \forall i, j \in I_C,\ \alpha = 1, \ldots, d-1.$$

$\varLambda_T^h$ will be defined as

$$\varLambda_T^h\big(\mathcal{F}\lambda_N^h\big) = \left\{\lambda_T^h \in X_T^h\colon -\int\limits_{\varGamma_C} \lambda_T^h.w_T^h \,\mathrm{d}\varGamma + \sum_{i\in I_C} \int\limits_{\varGamma_C} \mathcal{F}\lambda_N^h \big|w_T^h(a_i)\big| \psi_i \,\mathrm{d}\varGamma \leqslant 0,\ \forall w_T^h \in X_T^h\right\}.$$

This is equivalent to

$$\varLambda_T^h\big(\mathcal{F}\lambda_N^h\big) = \left\{\lambda_T^h \in X_T^h\colon \left|\left(\int\limits_{\varGamma_C} \lambda_T^h.\xi_i^\alpha \,\mathrm{d}\varGamma\right)_\alpha\right| \leqslant -\mathcal{F}\tilde{\lambda}_N^i,\ \forall i \in I_C\right\}.$$

This is compatible with the fact that $\tilde{\lambda}_N^i = \int_{\varGamma_C} \lambda_N^h(x)\psi_i(x)\,\mathrm{d}\varGamma \leqslant 0$.

With the matrix formulation (31), $\widetilde{\varLambda}_T^h(\mathcal{F}\widetilde{L}_N)$ is defined by

$$\widetilde{\varLambda}_T^h\big(\mathcal{F}\widetilde{L}_N\big) = \left\{\widetilde{L}_T = \sum_{i\in I_C} T_i \tilde{\lambda}_T^i\colon \big|\tilde{\lambda}_T^i\big| \leqslant -\mathcal{F}\tilde{\lambda}_N^i,\ \forall i \in I_C\right\},$$

where $(\tilde{\lambda}_T^i)_\alpha = \int_{\varGamma_C} \lambda_T^h.\xi_i^\alpha \,\mathrm{d}\varGamma$. The discrete problem can be written with nodal contact and friction conditions as

$$\begin{cases} \text{Find } U \in \mathbb{R}^{k_1},\ \widetilde{L}_N = \sum_{i\in I_C} \tilde{\lambda}_N^i N_i \text{ and } \widetilde{L}_T = \sum_{i\in I_C} T_i \tilde{\lambda}_T^i \text{ satisfying} \\ KU = F + \widetilde{L}_N + \widetilde{L}_T, \\ -\tilde{\lambda}_N^i \in J_N(U.N_i),\ \forall i \in I_C \iff \tilde{\lambda}_N^i = -(rU.N_i - \tilde{\lambda}_N^i)_+, \\ -\tilde{\lambda}_T^i \in -\mathcal{F}\tilde{\lambda}_N^i \operatorname{Dir}_T(u_T^i),\ \forall i \in I_C \iff \tilde{\lambda}_T^i = P_{\overline{B}(0,-\mathcal{F}\tilde{\lambda}_N^i)}(\tilde{\lambda}_T^i - ru_T^i), \end{cases} \tag{37}$$

where $P_{\overline{B}(0,\delta)}$ is the projection over the ball of center 0 and radius δ in $\mathbb{R}^{d-1}$, $(x)_+$ is the non-negative part of $x \in \mathbb{R}$, and $r > 0$ is an arbitrary augmentation parameter.

4.2. Almost conformal in stress hybrid discretization

Here, we assume that the stress on the contact boundary is discretized with a scalar Lagrange finite element (in particular, this implies $k_3 = (d-1)k_2$).

Matrix formulation (30) will be easily exploitable from a numerical viewpoint if the set $\varLambda_N^h$ is simple to express. The simpler approximation of $\varLambda_N$ is

$$\varLambda_N^h = \left\{\lambda_N^h = \sum_{i=1}^{k_2} \lambda_N^i \psi_i(x)\colon \lambda_N^i \leqslant 0\right\}.$$

For the same reason as in the latter section, this is not a conformal approximation of $\varLambda_N$ (i.e., $\varLambda_N^h \subset \varLambda_N$) except for P_1 elements. In the matrix formulation (30) this corresponds to

$$\overline{\varLambda}_N^h = \big\{L_N \in \mathbb{R}^{k_2}\colon (L_N)_i \leqslant 0,\ i = 1, \ldots, k_2\big\}.$$

In a same way, $\overline{\varLambda}_T^h(\mathcal{F}L_N)$ can be defined as

$$\overline{\varLambda}_T^h(\mathcal{F}L_N) = \big\{L_T \in \mathbb{R}^{(d-1)k_2}\colon \big|L_T^i\big| \leqslant -\mathcal{F}(L_N)_i,\ 1 \leqslant i \leqslant k_2\big\},$$

where L_T^i is the vector with $d-1$ components $((L_T)_{(d-1)i}, \ldots, (L_T)_{(d-1)i+d-2})$. The matrix formulation is the following:

$$\begin{cases} \text{Find } U \in \mathbb{R}^{k_1}, L_N \in \mathbb{R}^{k_2} \text{ and } L_T \in \mathbb{R}^{(d-1)k_2} \text{ satisfying} \\ KU = F + B_N^{\mathrm{T}} L_N + B_T^{\mathrm{T}} L_T, \\ (L_N)_i = -(r(B_N U)_i - (L_N)_i)_+, \qquad \forall i = 1, \ldots, k_2, \\ L_T^i = P_{\overline{B}(0, -\mathcal{F}(L_N)_i)}(L_T^i - r(B_T U)^i), \quad \forall i = 1, \ldots, k_2, \end{cases} \tag{38}$$

where $(B_T U)^i$ is the vector $((B_T U)_{(d-1)i}, \ldots, (B_T U)_{(d-1)i+d-2})$.

A classical example of this kind of hybrid discretization is to use a P_K finite element method (piecewise polynomials of degree K) for the displacement and a P_{K-1} method for the multipliers. The inf–sup conditions are satisfied for $K > 1$. For $K = 1$ the inf–sup conditions are generally not satisfied for $d = 2$ and never for $d = 3$, but, it is possible to stabilize the finite element method with bubble functions as in [3] or to use a coarser mesh for the multipliers as in [15].

Remark 5. Formulation (37) can be written in a similar form as formulation (38) defining the matrices $B_N^{\mathrm{T}} = (N_1 N_2 \ldots N_{k_2})$ and $B_T^{\mathrm{T}} = (T_1 T_2 \ldots T_{k_2})$.

5. Numerical study

In this section, two test cases are considered: a disc for the two-dimensional case and a torus for the three-dimensional case. The bodies are submitted to their own weight which has been overvalued in order to have a significant deformation. They are in frictional contact with a flat rigid foundation. The efficiency of different solvers for the discrete problem are compared.

Case (a): a linearly isotropic elastic disc of radius 20 cm with Lamé coefficient $\lambda = 115$ GP; $\mu = 77$ GP (see Fig. 3). The mesh is unstructured with from 16 triangles (82 d.o.f. for u, 18 d.o.f. for λ) to 2760 triangles (11 306 d.o.f. for u, 266 d.o.f. for λ). The finite element method is a P_2 isoparametric one.

Case(b): a linearly isotropic elastic torus of largest radius 20 cm with the same above characteristics (see Fig. 4). The mesh is structured with form 8 hexahedrons (288 d.o.f. for u, 72 d.o.f. for λ) to 512 hexahedrons (13 824 d.o.f. for u, 987 d.o.f. for λ). The finite element method is a Q_2 isoparametric one.

For all numerical tests, the stopping criterion of the methods is reached when the relative residue is smaller than 10^{-9}.

5.1. Fixed point methods

Two fixed point methods are investigated here: the first one is a fixed point on the contact and friction stresses and the second one is a fixed point on the friction threshold. Some theoretical aspects about these methods can be found in [19].

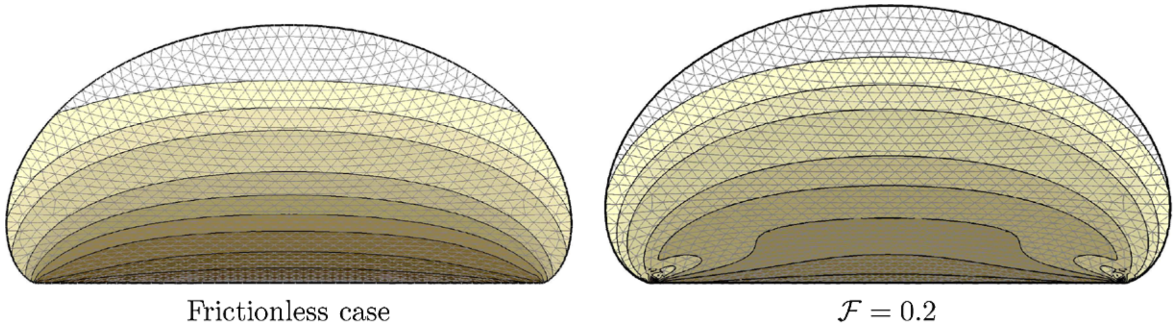

Fig. 3. Case (a), the Von Mises criterion on the deformed disc meshed with P_2 isoparametric FEM in frictional contact with a rigid foundation (with Getfem [25]).

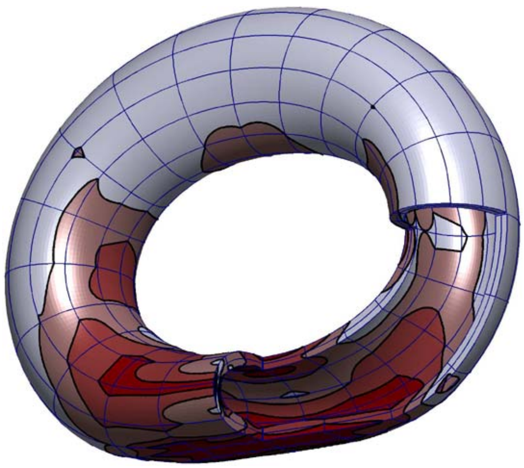

Fig. 4. Case (b), the Von Mises criterion on the deformed torus meshed with one layer of regular hexahedric cells and a Q_2 isoparametric FEM (with Getfem [25]).

5.1.1. *Fixed point on the contact stresses (FPS)*

One of the most straightforward approaches is to use the fixed point T_1^h or the De Saxcé variant T_2^h defined in Section 3.2. In the case of the discretization defined in Section 4.2, the algorithm can be expressed as follows for the fixed point T_1^h:

$$\left|\begin{array}{l}
(0)\ L_N^0, L_T^0 \text{ arbitrary,} \\
(1) \text{ compute } U^k \text{ solution to} \\
\quad KU^k = F + B_N^{\mathrm{T}} L_N^k + B_T^{\mathrm{T}} L_T^k, \\
(2) \text{ compute } L_N^{k+1} \text{ and } L_T^{k+1} \text{ as} \\
\quad (L_N^{k+1})_i = -(r(B_N U^k)_i - (L_N^k)_i)_+, \qquad \forall i = 1, \ldots, k_2, \\
\quad L_T^{i,k+1} = P_{\overline{B}(0, -\mathcal{F}(L_N^k)_i)}(L_T^{i,k} - r(B_T U)^{i,k}), \quad \forall i = 1, \ldots, k_2. \\
(3) \text{ Go to (1) until stop criterion is reached.}
\end{array}\right. \tag{39}$$

Remark 6. For Tresca problem (i.e., Coulomb problem with fixed threshold $-\mathcal{F}(\lambda_N^k)_i = s, \forall i = 1, \dots, k_2$), algorithm (39) coincides with an Uzawa one.

Remark 7. The practical difficulty for both fixed points T_1^h and T_2^h is to choose the value of the augmentation parameter r. The proof of Theorem 2 clearly shows that the contraction property depends on r. Following this proof an estimate of the optimal r is given by

$$r_{\text{opt}} = \frac{1/\lambda_{\max} - \|\mathcal{F}\|_\infty/\lambda_{\min}}{(1 + \|\mathcal{F}\|_\infty)^2}, \tag{40}$$

where $\lambda_{\max}$, $\lambda_{\min}$ are the extremal eigenvalues of $BK^{-1}B^{\mathrm{T}}$ and $B = \binom{B_N}{B_T}$.

Figs. 5 and 6 show the evolution of the number of iterations with respect to the friction coefficient $\mathcal{F}$ and the augmentation parameter r. The linear system at each iteration is solved with a preconditioned conjugate gradient method.

Surprisingly, in the two-dimensional case, the optimal parameter r does not seem to depend on $\mathcal{F}$, whereas the number of iterations increases when $\mathcal{F}$ increases. This does not seem to corroborate the estimate given by (40), which gives very small values of optimal r for $\mathcal{F} > 0$.

The situation is quite different on the three-dimensional case. The friction coefficient has a great influence on the optimal value of the augmentation parameter.

Numerical tests corresponding to Figs. 7 and 8 are done with a fixed friction coefficient ($\mathcal{F} = 0.2$) and different mesh sizes for the disc and the torus.

As it can be seen, the optimal value of the augmentation parameter r strongly depends on the mesh size.

Both the 2D and 3D experimental results show the remarkable property that the number of iterations increases abruptly for an augmentation parameter r slightly greater than the numerical optimal value. We do not have any interpretation of this phenomenon.

5.1.2. Fixed point on the friction threshold (FPT)

This fixed point is a well-known approach to solve the Coulomb problem (see, for instance, [12]). It consists in a sequence of Tresca problem. Each iteration requires the solution of a non-linear problem. The formulation is

$$\left|\begin{array}{l} (0)\ s^0 \geqslant 0 \text{ arbitrary,} \\ (1) \text{ find } U^k, L_N^k \text{ and } L_T^k \text{ solution to the non-linear (Tresca) problem} \\ \quad \begin{cases} KU^k = F + B_N^{\mathrm{T}} L_N^k + B_T^{\mathrm{T}} L_T^k, \\ -(L_N^k)_i \in J_N((B_N U^k)_i), \qquad \forall i = 1, \dots, k_2, \\ -L_T^{i,k} \in s^k \operatorname{Dir}_T((B_T U)^{i,k}), \quad \forall i = 1, \dots, k_2, \end{cases} \\ (2)\ s^{k+1} = -\mathcal{F}(L_N^k)_i. \text{ Go to (1) until stop criterion is reached.} \end{array}\right. \tag{41}$$

The term $(B_T U)^{i,k}$ is a notation for the vector with $d-1$ components $((B_T U^k)_{(d-1)i}, \dots, (B_T U^k)_{(d-1)i+d-2})$.

On Figs. 9 and 10 experimental results for cases (a) and (b) are presented with different values of the mesh size and friction coefficient.

For reasonable values of the friction coefficient, say $\mathcal{F}$ between 0 and 1.5, the number of iterations increases with $\mathcal{F}$.

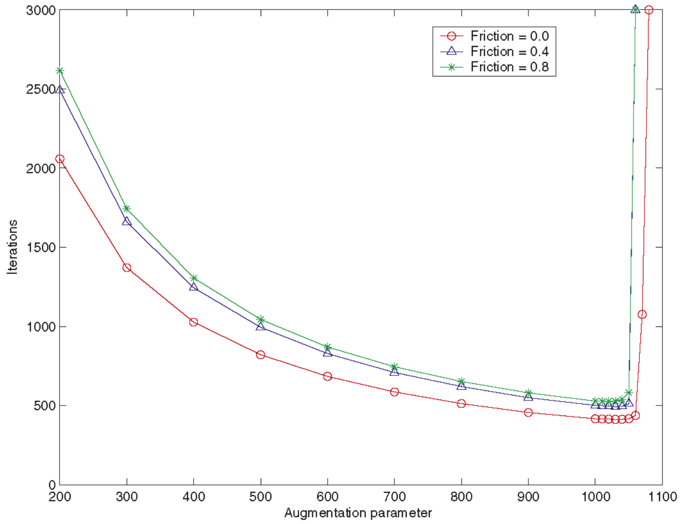

Fig. 5. FPS influence of the augmentation parameter r for the disc with different values of the friction coefficient.

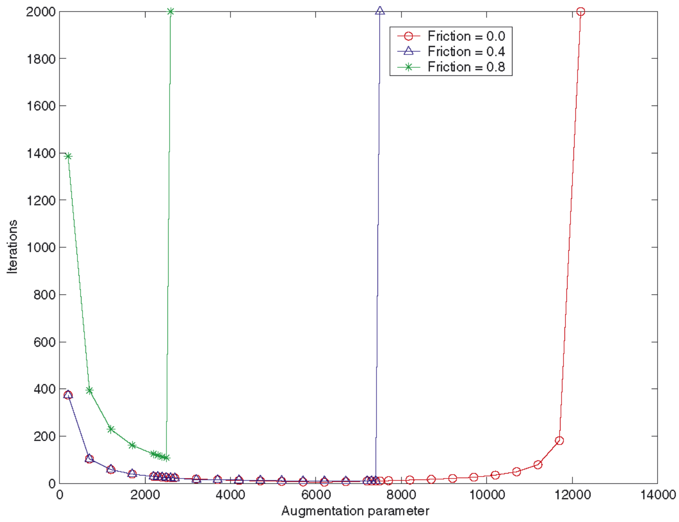

Fig. 6. FPS influence of the augmentation parameter r for the torus with different values of the friction coefficient.

For coarse meshes and high values of $\mathcal{F}$, the algorithm converges in very small amount of iterations. This might be related to the small number of nodes in contact and the fact that they are stuck on ($u_T = 0$). This phenomenon does not persist for fine meshes.

5.2. Successive over relaxed method (SOR)

In the context of friction problems, this method has been proposed by different authors like Lebon in [20] and Raous et al. in [24] in the two-dimensional case. Here, the method is presented for both two- and three-dimensional cases.

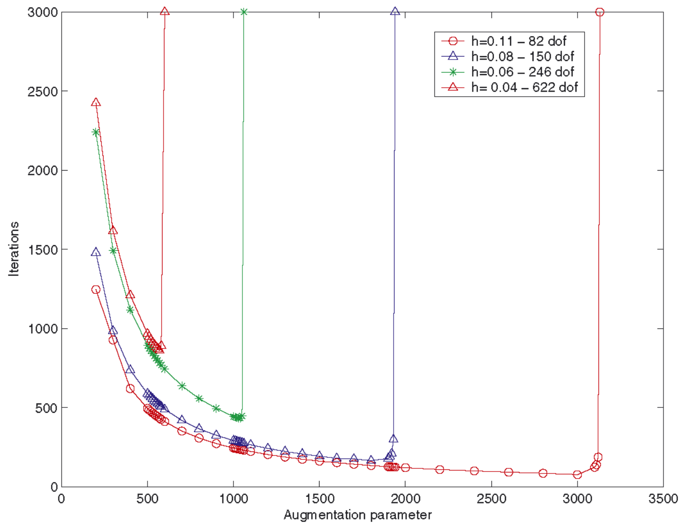

Fig. 7. FPS influence of the augmentation parameter r for the disc with different mesh sizes.

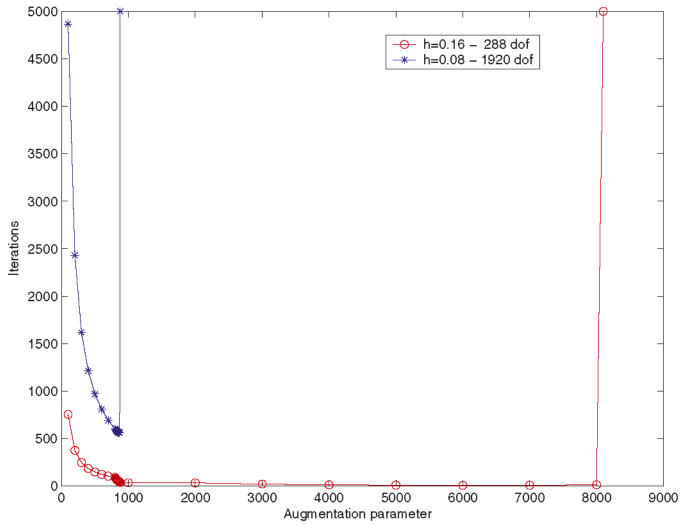

Fig. 8. FPS influence of the augmentation parameter r for the torus with different mesh sizes.

The formulation (37) of Section 4.1, can be equivalently rewritten as

$$\begin{cases} KU = F + \widetilde{L}_N + \widetilde{L}_T, \\ -\widetilde{L}_N.N_i \in J_N(U.N_i), & \forall i = 1, \dots, k_2, \\ -T_i^{\mathrm{T}}\widetilde{L}_T \in -\mathcal{F}\widetilde{L}_N.N_i\,\mathrm{Dir}_T(T_i^{\mathrm{T}}U), & \forall i = 1, \dots, k_2. \end{cases} \tag{42}$$

The resolution of (42) with the SOR method is the following:

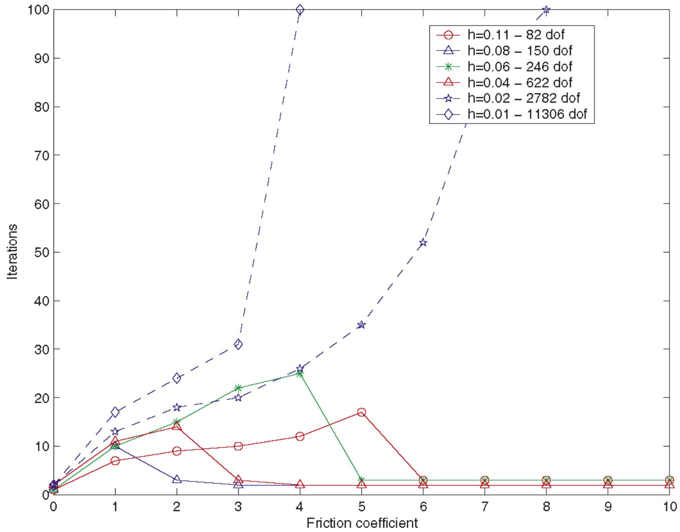

Fig. 9. FPT influence of the friction coefficient for the disc with different mesh sizes.

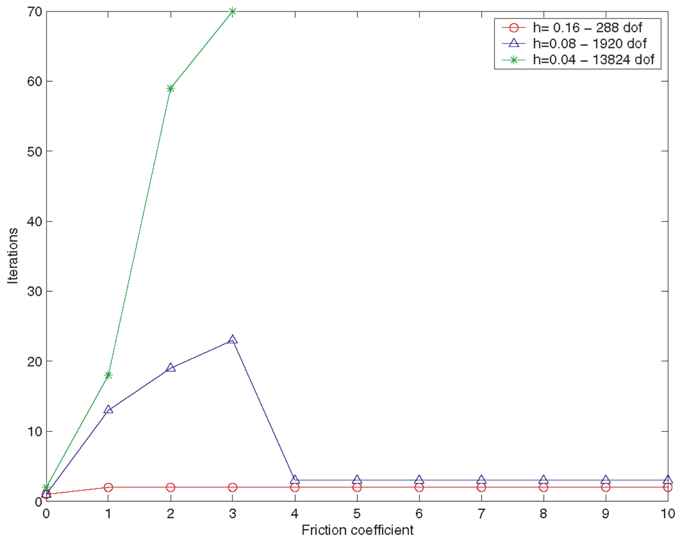

Fig. 10. FPT influence of the friction coefficient for the torus with different mesh sizes.

- for nodes which are not on Γ_C there are two strategies:
  - nodal strategy (i.e., apply a SOR iteration on each d.o.f.)

$$U_i^{k+1} = (1-\omega)U_i^k + \frac{\omega}{K_{ii}}\left(F_i - \sum_{j<i} K_{ij}U_j^{k+1} - \sum_{j>i} K_{ij}U_j^k\right),$$

  - global strategy (i.e., SOR iteration on matrix for interior d.o.f.)

$$\left(B^{\mathrm{T}}U\right)^{k+1} = \left(B^{\mathrm{T}}U\right)^k + \omega\left(B^{\mathrm{T}}KB\right)^{-1}\left(B^{\mathrm{T}}F - B^{\mathrm{T}}KU^k\right),$$

  where ω is the relaxation parameter and B the matrix selecting the interior d.o.f.,

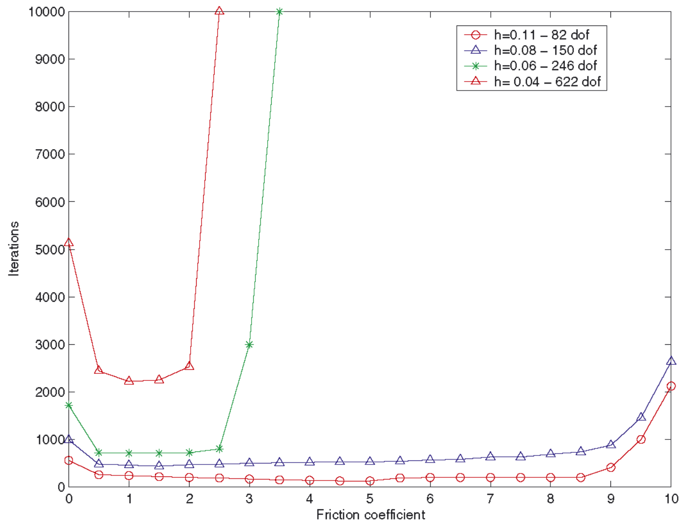

Fig. 11. SOR influence of the mesh size for the disc (with $\omega = 1.5$ and nodal strategy).

- for nodes which are on Γ_C:
  - the normal components are updated with

$$U^{k+1}.N_i = (1-\omega)U^k.N_i + \frac{\omega}{N_i^T K N_i}\big[F.N_i - \big(K\big(U^k - \big(U^k.N_i\big)N_i\big)\big).N_i\big]_-,$$

  - the tangential components are updated with

$$U^{k+1}T_i^{\mathrm{T}} = (1-\omega)U^k T_i^{\mathrm{T}} + \omega X,$$

where X is such that

$$Y \in \bar{A}X + \beta\,\mathrm{Dir}_T(X),$$

and

$$\begin{cases} Y = (F - K(U^k - T_i X))T_i^{\mathrm{T}}, \\ \bar{A} = T_i K T_i^{\mathrm{T}}, \\ \beta = -\mathcal{F}\widetilde{L}_N.N_i. \end{cases}$$

So, if $\|\frac{Y}{\beta}\| \leqslant 1$ then $X = 0$ is a solution.
Else $Y = \bar{A}X + \beta\frac{X}{\|X\|}$. Setting $X = \alpha v$ with $\|v\| = 1$, one has

$$Y = (\alpha\bar{A} + \beta\mathrm{Id})v.$$

Thus

$$\|v\| = \big\|(\alpha\bar{A} + \beta\mathrm{Id})^{-1}Y\big\| = 1. \tag{43}$$

Table 1
The number of iterations for the formulations with global and nodal strategies (with $\omega = 1.8$)

Formulation	Global	Nodal
Case (a)	180	230
Case (b)	31 200	59 800

This means that for tangent components, one has to find α solution of (43). The value of X will be deduced from α with $v = (\alpha \bar{A} + \beta \mathrm{Id})^{-1} Y$.

As it can be seen on Fig. 11, the number of iterations is very high. However, each iteration is very simple to compute.

The number of iteration strictly increases for fine meshes, but it is a natural behavior of SOR, even for linear problems.

There are no theoretical results on the optimum relaxation coefficient for this problem. We experimentally observed that the optimal values of ω are usually between 1.5 and 1.9. This corroborates experimental results of Raous et al. in [24].

Experiments show that the SOR with global strategy has a better behavior than the nodal one. Table 1 represents the number of iterations with the two strategies for the cases (a) and (b). Of course, the cost for an iteration is higher for the global strategy.

5.3. *Semi-smooth Newton method (SSN)*

Semi-smooth Newton method has been proposed by Alart and Curnier in [1] for Coulomb problem. Some development can also be found in [9].

From formulation (38), solving the Coulomb problem is equivalent to find the zero of the function $\mathcal{H}(Z)$ defined by

$$\mathcal{H}(Z) = \begin{pmatrix} KU - F - B_N^{\mathrm{T}} L_N - B_T^{\mathrm{T}} L_T \\ \mathcal{H}_N \\ \mathcal{H}_T \end{pmatrix}, \tag{44}$$

where

$$Z = (U, L_N, L_T)^{\mathrm{T}},$$
$$(\mathcal{H}_N)_i = \frac{1}{r}\big(-(L_N)_i - \big(r(B_N U)_i - (L_N)_i\big)_+\big), \quad \forall i = 1, \ldots, k_2,$$

and

$$\mathcal{H}_T^i = \frac{1}{r}\big(-L_T^i + P_{\bar{B}(0,-\mathcal{F}(L_N)_i)}\big(L_T^i - r(B_T U)^i\big)\big), \quad \forall i = 1, \ldots, k_2.$$

The function $\mathcal{H}(Z)$ is Lipschitz continuous and piecewise $\mathcal{C}^1$.

Algorithm of the semi-smooth Newton method.

Step 1: Z^0 be given.

Step 2: Find a direction d such that

$$\mathcal{H}\big(Z^k\big)+\mathcal{H}'\big(Z^k;d\big)=0, \tag{45}$$

where $\mathcal{H}'(Z^k;d)$ is the directional derivative of $\mathcal{H}$ on Z^k in the direction d.
Step 3: Line search in the direction d to find a convenient α with $Z^{k+1}=Z^k+\alpha d$.
Step 4: If $\|\mathcal{H}(Z^{k+1})\|$ small enough stop. Else, replace k by $k+1$ and return to step 2.

The line search we tested is a very simple one:

$$\left|\begin{array}{l} (0)\ \alpha=1, \\ (1)\ Z^{k+1}=Z^k+\alpha d \\ \quad \text{if } |\mathcal{H}(Z^{k+1})\| < \|\mathcal{H}(Z^k)\| \text{ or if } \alpha \text{ is too small (less than } 1/16 \text{ for instance) then stop,} \\ (2)\ \alpha \ \leftarrow\ \alpha/2. \text{ Go to (1).} \end{array}\right.$$

In Eq. (45), $\mathcal{H}'(Z^k;d)$ is replaced by $\mathcal{H}'(Z^k)d$, the gradient of $\mathcal{H}(Z^k)$ if Z^k is a point of differentiability of $\mathcal{H}$.

The non-differentiability points of $\mathcal{H}$ correspond to very particular situations. The solution to (38) is one of them if and only if

$$\exists i,\ 1\leqslant i\leqslant k_2 \quad \text{such that} \quad \text{either } (L_N)_i=(B_NU)_i=0 \text{ or } (B_TU)^i=0 \text{ and } \big|L_T^i\big|=-\mathcal{F}(L_N)_i.$$

Because this situation is very rare, we consider that $\mathcal{H}$ is differentiable everywhere: Eq. (45) is replaced by

$$\mathcal{H}\big(Z^k\big)+\mathcal{H}'\big(Z^k\big)d=0.$$

Even so, if the algorithm encounter a point of non-differentiability, a gradient on a zone of differentiability around this point is chosen.

The number of iterations does not increase with the friction coefficient in the two-dimensional case, and for the three-dimensional case, there is some fluctuations but the influence is not so important (see Figs. 12 and 13). Of course, the number of iterations increases when h decreases, however, it remains quite small.

The same experiment is done for different values of the mesh size h for the cases (a) and (b). The increase of mesh size h affects on the number of iterations (see Figs. 14 and 15). One can see on these figures that the influence of the augmentation parameter is very less important than in the case of fixed points of Section 5.1.1. The choice of this augmentation parameter does not constitute a difficulty for this method.

5.4. Comparison between different formulations

5.4.1. Partial symmetrization for the semi-smooth Newton method

The expression of $\mathcal{H}(Z)$ given by (44) can be modified to have a more symmetric derivative. This is done using the following definition:

$$\overline{\mathcal{H}}(Z)=\begin{pmatrix} KU-F-B_N^{\mathrm{T}}\overline{L}_N-B_T^{\mathrm{T}}\overline{L}_T \\ \mathcal{H}_N \\ \mathcal{H}_T \end{pmatrix}, \tag{46}$$

where

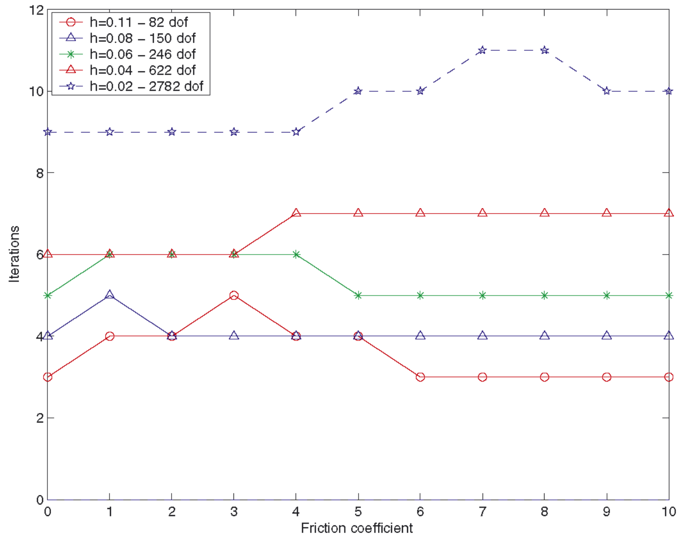

Fig. 12. SSN influence of the friction coefficient for the disc with different mesh sizes.

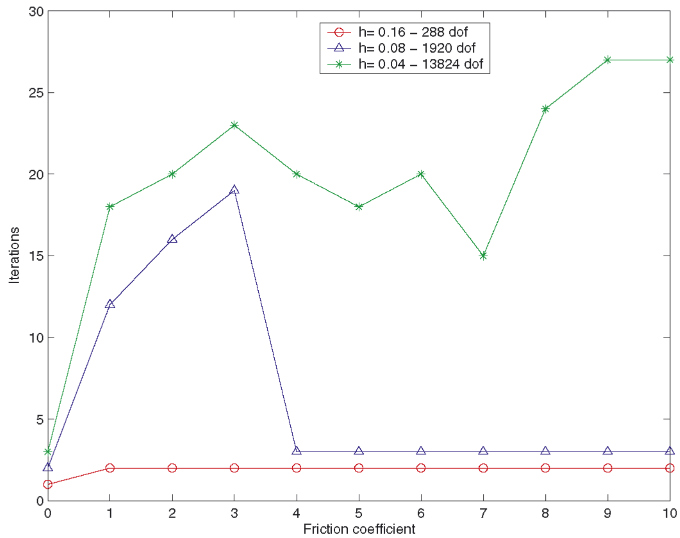

Fig. 13. SSN influence of the friction coefficient for the torus with different mesh sizes.

$$
\begin{aligned}
&Z = (U, L_N, L_T)^{\mathrm{T}}, \\
&\big(\overline{L}_N\big)_i = -\big(r(B_N U)_i - (L_N)_i\big)_+, && \forall i = 1, \ldots, k_2, \\
&\overline{L}_T^i = P_{\overline{B}(0, -\mathcal{F}(L_N)_i)}\big(L_T^i - r(B_T U)^i\big), && \forall i = 1, \ldots, k_2, \\
&(\mathcal{H}_N)_i = \frac{1}{r}\big(-(L_N)_i + \big(\overline{L}_N\big)_i\big), && \forall i = 1, \ldots, k_2
\end{aligned}
$$

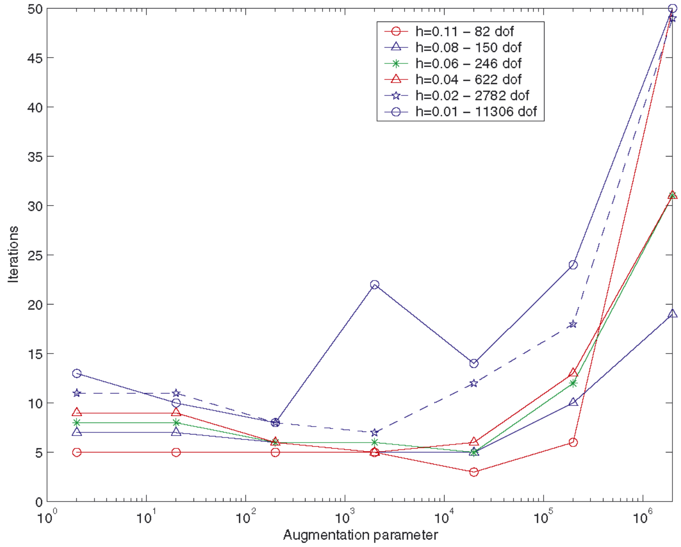

Fig. 14. SSN influence of the augmentation parameter r for the disc with different mesh sizes.

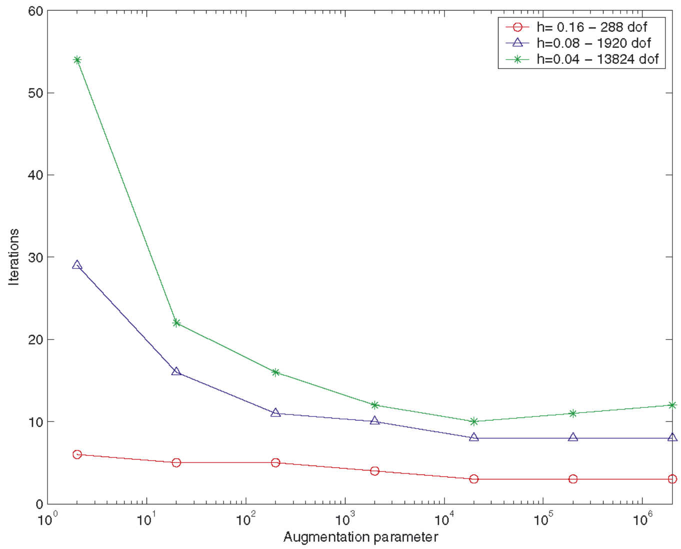

Fig. 15. SSN influence of the augmentation parameter r for the torus with different mesh sizes.

and

$$\mathcal{H}_T^i = \frac{1}{r}\left(-L_T^i + \overline{L}_T^i\right), \quad \forall i = 1, \ldots, k_2.$$

For a Tresca problem, $\overline{\mathcal{H}}(Z)$ has a symmetric derivative because it is the Hessian of an augmented Lagrangian. For a Coulomb problem a non-symmetric part is present which comes from the Coulomb friction condition.

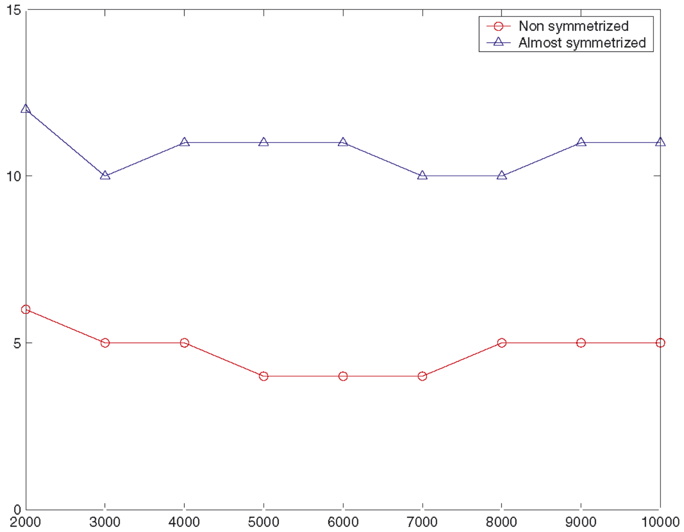

Fig. 16. SSN comparison between the almost symmetrized problem and the non-symmetrized one for the disc.

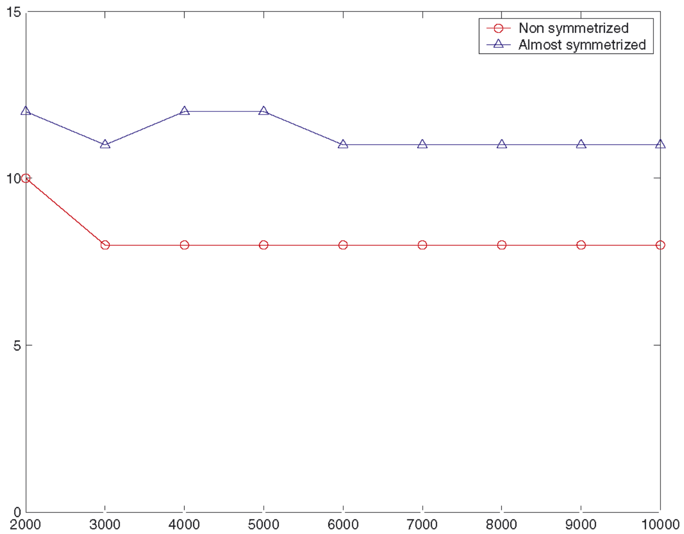

Fig. 17. SSN comparison between the almost symmetrized problem and the non-symmetrized one for the torus.

The comparison is done in the cases (a) and (b) using the semi-smooth Newton method. Figs. 16 and 17 represent the evolution of the number of iterations in function of augmentation parameter r. Apparently, the symmetrization does not seem to be an advantage for the convergence of the semi-smooth Newton method.

5.4.2. *Comparison between De Saxcé and standard formulation*

We now compare the two formulations for the fixed point on the contact stresses: T_1^h and T_2^h. Figs. 18 and 19 show that the two formulations give approximately the same number of iterations.

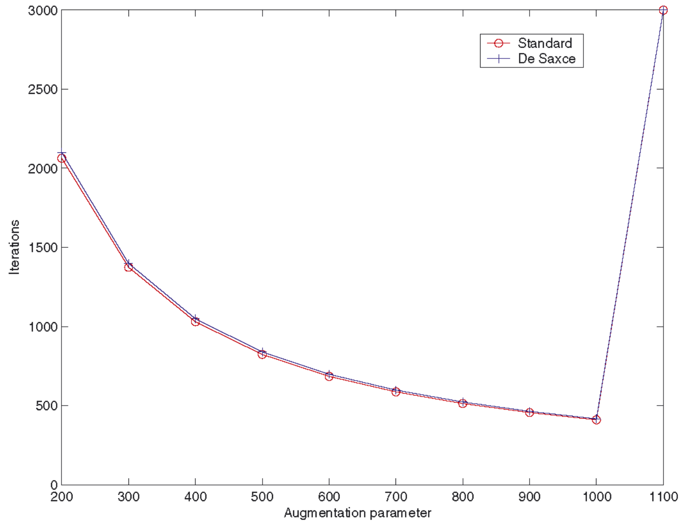

Fig. 18. FPS comparison of the De Saxcé formulation with the standard one for the disc.

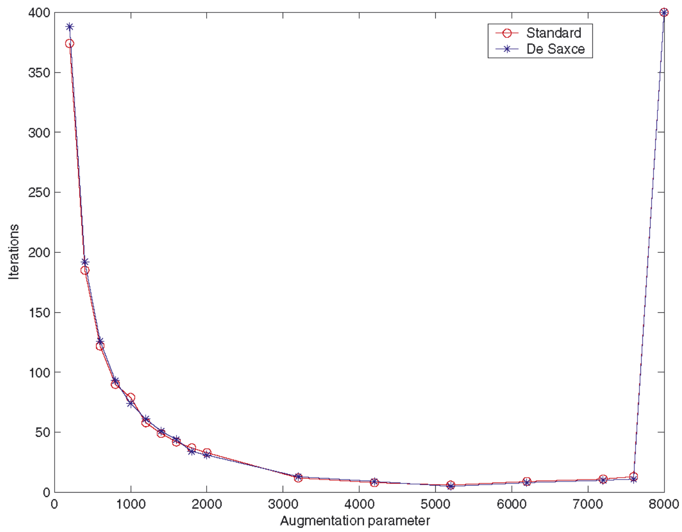

Fig. 19. FPS comparison of the De Saxcé formulation with the standard one for the torus.

5.4.3. Comparison between almost conformal in stress or almost conformal in displacement formulations

All experiments in previous sections are done with the almost conformal in displacement formulation (Section 4.1). The solvers were also tested with the almost conformal in contact stress hybrid formulation (Section 4.2), however, no significant differences of the behavior of the solvers between the two kinds of formulation were found.

6. Conclusion

We have presented in this paper a general framework for the hybrid discretization of contact and friction conditions in elastostatics and we proved an existence and a uniqueness result for the discretized problem in this general framework.

In Section 5, different methods to solve the discrete problem were analyzed from a numerical viewpoint. We did not give the comparison in terms of CPU time because this CPU time depends too much on the implementation details of the method (particularly, the choice of a linear solver and a preconditioner).

The fixed points on the contact and friction stresses T_1^h and T_2^h (Section 5.1.1) correspond to an Uzawa algorithm when the friction threshold is given (Tresca problem). These methods are of order one, the number of iterations increases a lot when the mesh size becomes small and the optimal augmentation parameter is not easy to find. Each iteration requires the solution of a linear symmetric coercive system.

The fixed point on the friction threshold (Section 5.1.2) is a frequently used method. It converges in a few iterations at least for reasonable friction coefficients. Each iteration needs to solve a Tresca problem, which is a non-linear problem. The Tresca problem can be solved with optimization techniques such as conjugate gradient or interior points methods.

The SOR method (Section 5.2) is the simplest method to implement. An iteration of the (nodal) method does not need to solve a linear system and thus is very fast. It is well adapted for small bidimensional problems.

The semi-smooth Newton method on the augmented problem (Section 5.3) is a very efficient method. It appears not to be sensitive to the choice of the augmentation parameter and the number of iterations remains small even for a large value of the friction coefficient. Each iteration requires the solution of linear non-symmetric system involving the tangent matrix.

The conclusion of the numerical study is that the semi-smooth Newton method seems to be the more robust method to solve contact and friction problems for deformable bodies.

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