Haff's Law for Anomalous Granular Gases

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The Boltzmann Equation for Granular Gases

A granular gas is a rarefied gas composed of "macroscopic" particles (pollen, high altitude atmosphere, planetary rings), interacting *via* energy dissipative binary collisions. In the statistical physics point of view, such a gas can be described by a kinetic equation: the **inelastic Boltzmann** equation (also known as the **granular gases** equation). In absence of exterior forces, it is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where $f = f(t, x, v)$; $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ is the density of particles at a point (t, x, v) and $Q(f, g)$ is a dissipative binary collision operator of hard spheres type, localised in time and space, and given in a weak form for smooth function ψ by

$$\langle Q(f,g), \psi\rangle := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |u| f_* \, g \, \left(\psi' + \psi'_* - \psi - \psi_*\right) b_a\left(u\cdot\omega, \mathcal{E}(f)\right) d\omega \, dv \, dv_*, \tag{2}$$

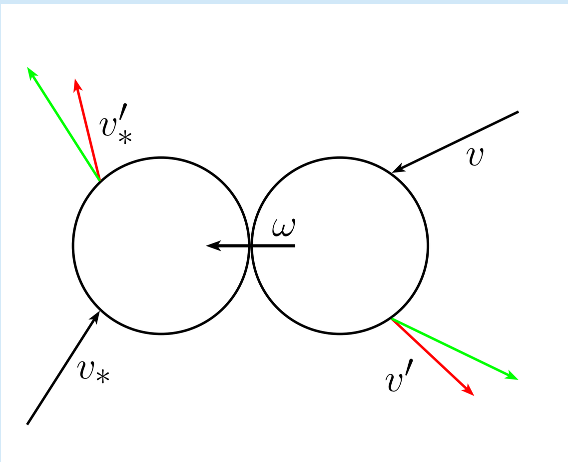
where $\mathcal{E}(f)$ denote the kinetic energy of the gas, namely

$$\mathcal{E}(f)(t,x) := \int_{\mathbb{R}^d} |v|^2 f(t,x,v) dv.$$

The macroscopic energy dissipation is given by a mass and momentum conservative microscopic collision mechanics which also involve the dissipation of energy. It can be described by the following process: given two particles of pre-collisional velocities v and v_*, their respective post-collisional velocities, denoted by v' and v'_*, are given by

$$\begin{cases} v' = v - \dfrac{1+e}{2}(u\cdot\omega)\,\omega \\ v'_* = v_* + \dfrac{1+e}{2}(u\cdot\omega)\,\omega, \end{cases}$$

where $u := v - v_*$ (**relative speed**), $\omega := \frac{u}{|u|}$ (**impact direction**) and $e \in [0,1]$ is the dissipation parameter (**restitution coefficient**) measuring the inelasticity of the collision.



Inelastic collisions (red) and elastic collisions (green)

The Anomalous Gas Model

In the following, the major assumption of space homogeneity for the gas will be made. The x-dependency will thus be dropped and the equation (1) will read

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{3}$$

for $f = f(t, v)$. A gas is said to be **anomalous** if its cooling occurs in finite time, that is if $\mathcal{E}(f)(t)$ goes to 0 when $t \to T_c$ with $T_c < \infty$. It can be modelled by assuming that the particles' collision rate increases with energy dissipation, which may be written as

$$b_a(\cdot, \mathcal{E}) = b(\cdot)\,\mathcal{E}^{-a},$$

where b is a non-negative function of mass 1 on the unity sphere and a a non-negative parameter, to be specified later. Such a problem is well posed in L^1_3, mass and momentum conservative, and has a **cooling time** T_c depending on a. Using the weak form (2) of Q and flux-divergence formula, together with Hölder and Jensen inequality, one has the macroscopic dissipation of energy:

$$\frac{d}{dt}\mathcal{E}(f)(t) \leq -C\mathcal{E}(f)(t)^{-a+3/2}, \; \forall t < T_c, \tag{4}$$

where C is an explicit constant. Then, the cooling time is such that

$$\begin{cases} T_c = +\infty \text{ if } 0 \leq a < 1/2, \\ T_c < +\infty \text{ if } a > 1/2, \end{cases}$$

and the gas is truly anomalous if $a > 1/2$.

Haff's Law

Knowing the time of cooling, we now want to determine the behaviour of the energy. This has been first studied in the Eighties by P.K. Haff, who showed that, if $a = 0$, then

$$\mathcal{E}(f)(t) \sim \frac{1}{1+t^2}.$$

This result is since known as **Haff's Law** and we shall generalise it to our anomalous model.

In order to study this equation and prevent the blow-up, a rescaling of f is needed, in the form $f(t,v) = V(t)^d g\left(T(t), V(t)v\right)$, where f is a solution to the homogeneous equation (3) and g is such that

$0 \leq a < 1/2$	$a > 1/2$
$V(t) = (1+t)^{-\alpha}$, $T = \log(V)$,	$V'(t) = \mathcal{E}(f)(t)^{-a}$, $T = \log(V)$, $T(T_c) = V(T_c) = +\infty$,
$\partial_t g + \nabla_v \cdot (vg) = \mathcal{E}(g)^{-a} Q(g,g)$,	$\partial_t g + \nabla_v \cdot (vg) = Q(g,g)$.

Theorem 1 (Cooling process). *There exist positive constants μ_a, m_i M_i, $i = 1..2$ and T_c, depending only on a and f_{in} such that*

(i) if $0 \leq a < 1/2$ (and then $\alpha := \frac{1}{2a-1} < 0 \to$ ***cooling in infinite time****),*

$$\frac{m_1}{(1+\mu_a t)^{-2\alpha}} \leq \mathcal{E}(f_t) \leq \frac{M_1}{(1+\mu_a t)^{-2\alpha}}, \; \forall t > 0;$$

(ii) if $a > 1/2$ (and then $\alpha > 0 \to$ ***cooling in finite time****),*

$$m_2\left(1 - \frac{t}{T_c}\right)^{2\alpha} \leq \mathcal{E}(f_t) \leq M_2\left(1 - \frac{t}{T_c}\right)^{2\alpha}, \; \forall t < T_c.$$

Elements of proof. The first point is needed in order to prove the second and

(i) Cooling in infinite time: maximum principles on the second and third order moments using estimates on the granular operator due to Bobylev, Gamba and Panferov (*J.S.P.*, 2004).

(ii) Cooling in finite time: use of the cooling process for the rescaled distribution g thanks to the first part of the theorem, and estimates about the time derivative of the energy.

Theorem 2 (Existence of self-similar profiles). *If $0 \leq a < 1/2$, there exists a profile $0 \leq G \in L^1_3$, called* ***self-similar profile*** *of the equation (3), with mass 1 and zero momentum:*

$$\nabla_v \cdot (vG) - \mathcal{E}(G)^{-a} Q_{e_0}(G,G) = 0.$$

Elements of proof. Stability estimates, propagation of L^p norms in self-similar variables and application of Schauder theorem to the evolution semi-group of (3) thanks to the Cauchy theory.

The Spectral Scheme for Granular Gas

If the distribution function f is compactly supported, one can show using another expression of the collision mechanics (the so-called σ-parametrisation) that the collision operator does not spread the support of f. It allows us to deal with solutions of (3) supported in $[-V,V]^d$ and then periodize them over $\mathbb{R}^d$, in order to study numerically their truncated Fourier sums $f_N = \sum_{k=-N}^{N} \hat{f}_k e^{i\frac{\pi}{V}k\cdot v}$ for $N \in \mathbb{N}$. By a direct computation, one can show show that $(f_k)_k$ verify the following system of ordinary differential equations:

$$\frac{\partial \hat{f}_k}{\partial t} = \mathcal{E}(f_N)^{-a} \sum_{\substack{l+m=k \\ l,m=-N}}^{N} \left(\hat{B}(l,m) - \hat{B}(m,m)\right) \hat{f}_l \, \hat{f}_m, \; \forall k \in [-N,N]^d,$$

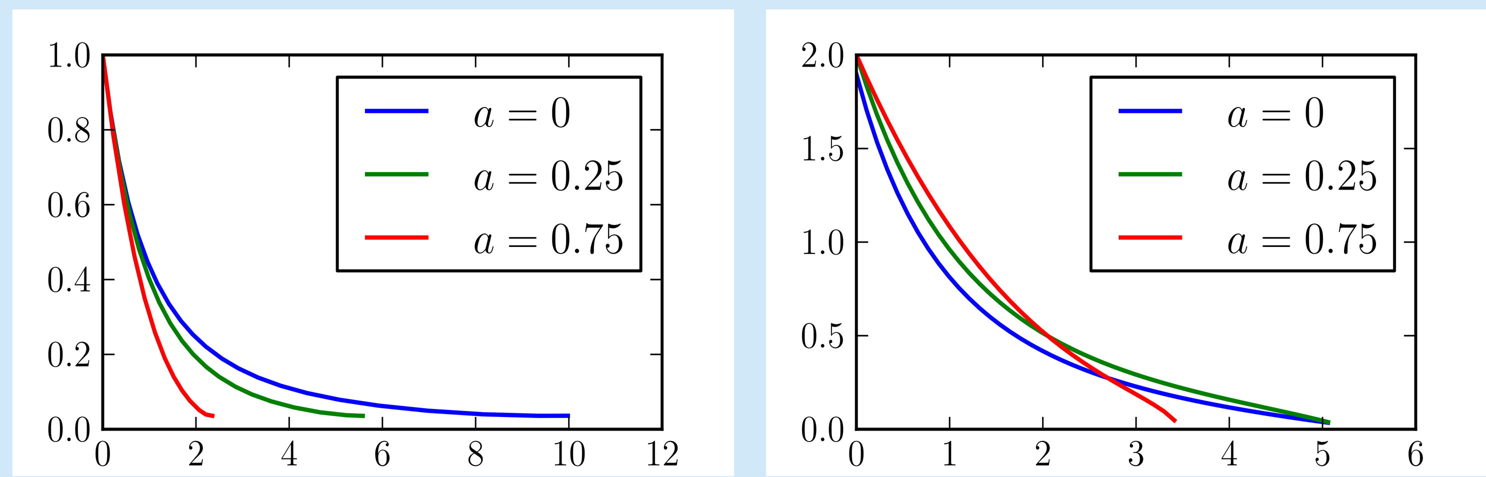
where the **kernel Fourier modes** $\hat{B}(l,m)$ are given using the weak form (2) of Q by

$$\hat{B}(l,m) = \frac{1}{2}\int_{\mathcal{B}(0,2\lambda V)} |u| e^{-i\frac{\pi}{V}\left(\left(\frac{3-e}{4}\right)u\cdot m - \left(\frac{1+e}{4}\right)u\cdot l\right)} \left(\int_{\mathbb{S}^{d-1}} e^{-i\frac{\pi}{V}\left(\frac{1+e}{4}\right)|u|(l+m)\cdot\sigma} d\sigma\right) du.$$

In this last expression, $\lambda = \frac{2}{3+\sqrt{2}}$ is a periodisation parameter.

Numerical experiments

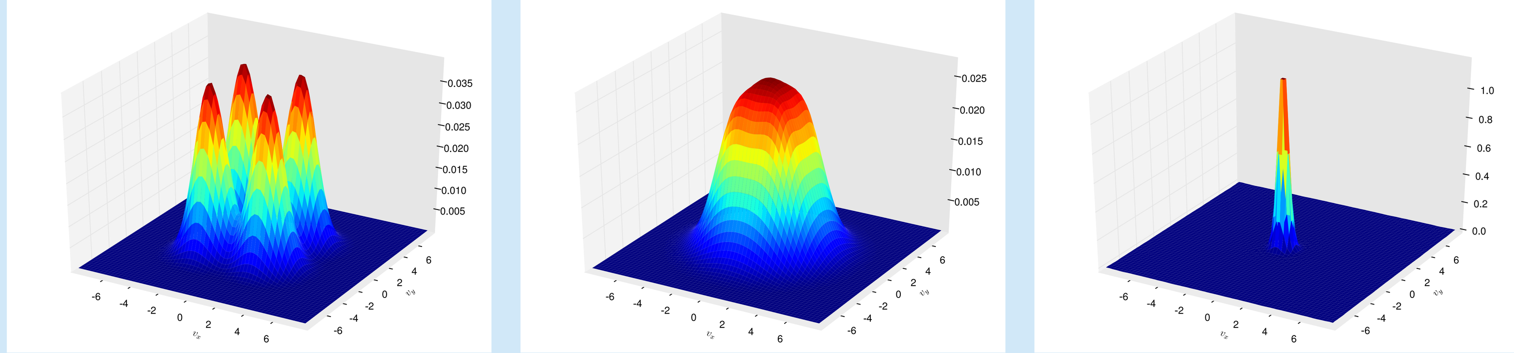
We present the evolution of the energy of f for a restitution coefficient $e = 0.9$ (true inelasticity). The system of o.d.e. fulfilled by the Fourier modes is solved using a **RK_2 scheme** with a time step $\Delta t = 0.01$. The initial condition is a reduced and centred Gaussian in velocity for three values of the parameter a: 0 and 0.25 are **infinite cooling time**, and 0.75 is **finite cooling** time.



$d = 1, N = 256$ — $d = 2, N = 32$

Time evolution of the energy for different values of a

The second experiment show the rapid evolution of a far-from-equilibrium initial distribution toward the Dirac mass centred in the mean momentum, for an **anomalous gas ($a = 0.75$)**. The parameters for the scheme are now $e = 0.9$, $d = 2$, $N = 32$ (there are then 64^2 Fourier modes) and $\Delta t = 0.01$.



$t = 0$ — $t = 0.5$ — $t = 4.5$

Time evolution of a $2d$ distribution toward the Dirac mass

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