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Absence of singular spectrum for Schrödinger operators with anisotropic potentials and magnetic fields

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We study magnetic Schrödinger operators of the form $H=(P-a)^2+V$ in $L^2(\mathbb{R}^{m+p})$, with $m \geqslant 2$. We get a limiting absorption principle and the absence of singular spectrum under rather mild and especially anisotropic hypothesis. The magnetic field B and the potential V will be connected by some $\mathbb{R}^m$-conditions, but in the $\mathbb{R}^p$-variable there will be almost no constraints. If $m=2$ and $p=0$, our results contrast with the known fact that P^2+V always has bound states if V is negative. © *2000 American Institute of Physics.* [S0022-2488(00)05105-7]

I. INTRODUCTION

We shall study spectral properties of the magnetic Schrödinger operator $H=(P-a)^2+V$ acting in the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^n)$, where $a:\mathbb{R}^n \to \mathbb{R}^n$ is the vector potential generating the magnetic field $B_{jk}=\partial_j a_k-\partial_k a_j$ and V is a multiplication operator. It is an established fact that the magnetic Hamiltonians have specific spectral properties distinguishing them from the usual Schrödinger operators (see, for example, Refs. 1 and 2). We intend to present some new results in this direction.

Let us begin with the two-dimensional case. It is well known[3] that if V is not identically zero, is small at infinity in a suitable sense, and if $\int_{\mathbb{R}^2} V(x)\,dx \leqslant 0$, then the operator $P^2+\beta V$ has bound states however small $\beta>0$ may be. We shall show that a magnetic field can change this picture drastically. For potentials V satisfying some mild conditions, there exists a rather rich class of magnetic fields such that $(P-a)^2+\beta V$ is purely absolutely continuous and unitary equivalent to $(P-a)^2$. We shall show the absence of singular spectrum even for V's that do not decay at infinity. Roughly, it will be enough for V to have radial limits, possibly direction dependent, and to converge towards them with a rate which is dictated by the magnetic field. This anisotropic phenomenon has been put into evidence in Ref. 4 for the Schrödinger operator P^2+V in dimension $n \geqslant 3$, the convergence rate towards the radial limits being of the form c/r^2, with a suitable constant c. In the presence of a magnetic field this becomes possible in dimension two.

A new effect appears in dimension n greater than 2. Let us decompose $\mathbb{R}^n$ as a direct sum $Y \oplus Z$ with $\dim Y \equiv m \geqslant 2$. Suppose that the magnetic field has a two-block decomposition: $B_{jk}=0$ if j labels a variable in Y and k a variable in Z. Some mild conditions on the Y-magnetic components and some constraints on V in the Y-variable are enough to get absolutely continuous spectrum for the corresponding operator $(P-a)^2+V$. The Z-components of the magnetic field and the Z-variable dependence of the potential are essentially unrestricted (we impose only smoothness assumptions for convenience). In Ref. 4 the same type of high-dimension results were obtained for the Schrödinger Hamiltonian $H_\beta=P^2+\beta V$. If V behaves well in a subspace of $\mathbb{R}^n$ of dimension *of at least 3*, there will be no singular spectrum for β small enough even if V acts very badly in the remaining variables. It seems that good properties of the potential in three dimensions are enough to allow the particle to propagate to infinity. We shall show now that, in the presence

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of a Y-well-behaved magnetic field, *the critical dimension is reduced to 2*. And this is compatible with very wild Z-components of the magnetic field.

The results we described are stated in a precise form for $m=2$ and n arbitrary in Sec. II and proved in Sec. IV. In Sec. V we comment briefly on some extensions for $m\geqslant 3$. The abstract method used in the proofs is exposed in Sec. III. It originates in Ref. 4 and the (stronger) form-version we need here is proved in Ref. 5 in which perturbations of convolution operators and partial differential operators with variable coefficients are treated as applications. It may be considered a generalization of the classical Kato–Putnam theorem, also relying on the positivity of the commutator between H and a supplementary self-adjoint operator A. It is no longer asked that A be H-bounded. Since it is close in spirit (and in proof) to the Mourre theory, which appears occasionally under the name ''the method of the conjugate operator,'' we called it ''the method of the weakly conjugate operator.'' It lacks the main qualities of Mourre's method (localization in energy, the compact operator), but it relies on a weaker positivity assumption: $i[H,A]>0$ and this is crucial for the type of results we obtain. In particular, remark that all our resolvent estimates are global, i.e., also valid at ''threshold points.''

II. THE MAIN RESULTS

Let us consider the decomposition $\mathbb{R}^n=Y\oplus Z$ with $\dim Y\equiv m\geqslant 2$ and the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^n)$, with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For $j=1,\ldots,n$, Q_j will be the operator of multiplication by the variable x_j and P_j the self-adjoint extension of $-i(\partial/\partial x_j)$ defined on $C_0^\infty(\mathbb{R}^n)\equiv\mathcal{D}$. We shall write freely $x=(y,z)$, with $y\in Y$, $z\in Z$, and $Q=(Q^Y,Q^Z)$.

We consider a self-adjoint operator H in $\mathcal{H}$, $\mathcal{G}^2$ its domain and $\mathcal{G}^1$ its form-domain with the corresponding graph-norms; both are Hilbert spaces. Identifying $\mathcal{H}$ with its topological dual and setting $\mathcal{G}^{-s}$ the dual of $\mathcal{G}^s(s=1,2)$ with its canonical structure of Hilbert space, we get continuous, dense embeddings $\mathcal{G}^2\hookrightarrow\mathcal{G}^1\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{G}^{-1}\hookrightarrow\mathcal{G}^{-2}$. Obviously, H extends to an element of $B(\mathcal{G}^1,\mathcal{G}^{-1})$, the Banach space of all linear, bounded operators: $\mathcal{G}^1\to\mathcal{G}^{-1}$.

We shall introduce now our magnetic Hamiltonians. We look to Ref. 1 or 2 for the proof of the assertions and for further details. Note, however, that we work under various smoothness assumptions which are not essential, but which simplify our arguments. So let $a\in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ be the vector potential and $V\in C^\infty(\mathbb{R}^n,\mathbb{R})\cap L^\infty(\mathbb{R}^n,\mathbb{R})$ the electromagnetic potential. The magnetic field is given by $B_{jk}=\partial_j a_k-\partial_k a_j$, $j,k=1,\ldots,n$. We denote by Π_j the magnetic momentum, which is the closure of $P_j-a_j(Q)$ defined on $\mathcal{D}$. The differential operator

$$H=H_0+V=\sum_{j=1}^n \Pi_j^2+V(Q)$$

is essentially self-adjoint on $\mathcal{D}$ and its form-domain is $\mathcal{G}^1=\{f\in\mathcal{H}|\Pi_j f\in\mathcal{H},\ j=1,\ldots,n\}$. Further on, we shall rely heavily on the following lemma.

Lemma II.1: For every $f\in\mathcal{D}$ and $j,k=1,\ldots,n$ one has

$$\langle f,(\Pi_j^2+\Pi_k^2)f\rangle\geqslant\pm\langle f,B_{jk}f\rangle.$$

The two alternative signs will come into play frequently in the sequel.

Let us assume that the magnetic field B has a split form: $B_{jk}=0$ if $j\in\{1,\ldots,m\}$ and $k\in\{m+1,\ldots,n\}$. By the cocycle condition $\partial_i B_{jk}+\partial_j B_{ki}+\partial_k B_{ij}=0$ it follows that the Y-components of B depend only on y and the Z-components only on z. To implement this, we shall suppose that a has the form $a=(a^Y,a^Z)$, with $a^Y\in C^\infty(Y,Y)$ and $a^Z\in C^\infty(Z,Z)$. In consequence, $H_0=H_0^Y\otimes 1+1\otimes H_0^Z$, where $H_0^Y=\Sigma_{j=1}^m\Pi_j^2$ acts in $\mathcal{H}^Y=L^2(Y)$ and $H_0^Z=\Sigma_{j=m+1}^n\Pi_j^2$ acts in $\mathcal{H}^Z=L^2(Z)$.

We focus now on the special case $m=2$ and n arbitrary. The general case will be discussed in Sec. V. The magnetic component $B_{12}=-B_{21}$ will be written simply B^Y; if $h\in C^\infty(\mathbb{R}^n)$, we set $(D^Y h)(y,z)=y\cdot(\nabla_y h)(y,z)=\Sigma_{j=1}^2 y_j(\partial_j h)(y,z)$. We state our main result:

Theorem II.2: *Assume that the following conditions are verified:*

(i) $|B^Y(y)| \leqslant c/(1+|y|^{1+\eta})$, *where* $\eta>0$.

(ii) $D^Y V \in L^\infty(\mathbb{R}^n)$.

(iii) *There exist positive constants* $\gamma, \varepsilon, \delta$ *with* $(2-1/\gamma)>\varepsilon>0$ *and* $\delta>0$, *so that one of the inequalities*

$$G_\pm(y,z) \equiv \left(2-\frac{1}{\gamma}-\varepsilon\right)[\pm B^Y(y)] - (D^Y V)(y,z) - (\gamma+\delta)|y|^2 (B^Y)^2(y) \geqslant 0 \tag{1}$$

is satisfied. Let us denote simply by G the function verifying one of the above inequalities.

(iv) *There are positive constants* c_1, c_2, c_3 *such that*

$$|D^Y(D^Y V)(y,z)| + |y|^2 (D^Y B^Y)^2(y) \leqslant c_1[\pm B^Y(y)] + c_2 G(y,z) + c_3 |y|^2 (B^Y)^2(y).$$

Then, we have the following.

(a) $\|(H-\lambda \mp i\nu)^{-1}\|_{B(\mathcal{A},\mathcal{A}^*)} \leqslant c$, *uniformly in* $\lambda \in \mathbb{R}$, $\nu>0$.

(b) *All the elements of* $B(\mathcal{A}^*,\mathcal{H})$ *are (globally) H-smooth operators.*

(c) *H has only purely absolutely continuous spectrum.*

Remark II.3: In their greatest generality, the spaces $\mathcal{A}$ and $\mathcal{A}^*$ which appear in the limiting absorption principle will be constructed in Sec. IV. But under one not too restrictive extra condition, we can give a very transparent version. Assume that $B^Y(y)>0$ for all $y \in \mathbb{R}^2$. Then we can define $F(y) = \max\{B^Y(y)^{-1/2}, |y|\}$ and set $\mathcal{F}$ the completion of $\mathcal{D}$ with respect to the norm $\|f\|_{\mathcal{F}} = \|F(Q)f\|$. In the points (a) and (b) of Theorem II.2, the space $\mathcal{A}$ can be replaced by $\mathcal{F}$ and $\mathcal{A}^*$ by its dual $\mathcal{F}^*$. It follows in particular that, under the stated hypothesis on B^Y, the operator of multiplication by the function L is H-smooth if one has $|L(y,z)| \leqslant c \min\{B^Y(y)^{1/2}, 1/|y|\}$. Similar results are true if $B^Y(y)<0$ for all $y \in \mathbb{R}^2$.

We give now a corollary including a result on the existence and completeness of some suitable wave operators. There will be more restrictive conditions on the magnetic field B^Y, but we get rid of the hypothesis on the smoothness and on the Y-derivatives of the potential. The proof of the corollary is given at the end of Sec. IV.

Corollary II.4: Assume that the Y-component of the magnetic field satisfies

(i) $0<B^Y(y) \leqslant c/(1+|y|^2)$, *for a subunitary constant* c, *and*

(ii) $|y|^2 (D^Y B^Y)^2(y) \leqslant c' B^Y(y)$, *for a finite constant* c'.

Suppose that the Borel function V has radial limits in the directions contained in Y: for any $\omega \in Y, |\omega|=1$ *and any* $z \in Z$ *there exists* $V_o(\omega,z) = \lim_{r\to\infty} V(r\omega,z)$. *Assume in addition that*

$$|V(r\omega,z) - V_o(\omega,z)| \leqslant \beta B^Y(r\omega).$$

Then, for β *small enough, the operator* $H=H_0+V$ *is purely absolutely continuous and unitary equivalent to* $H_{(o)}=H_0+V_o$ *through the wave operators*

$$\Omega_\pm = s-\lim_{t\to\pm\infty} e^{itH} e^{-itH_{(o)}}.$$

Remark II.5: In both Theorem II.2 and Corollary II.4 very few constraints were imposed in the subspace Z. The smoothness condition left apart, B_{jk} $(j,k=3,\ldots,n)$ are completely arbitrary; they may even grow at infinity. To have some insight into the way V may behave, let us make the very particular assumption that it factorizes: $V(y,z) = V^Y(y)\cdot V^Z(z)$. The function V^Z must only be smooth and bounded. The hypothesis essentially connect V^Y to B^Y; V^Z comes into play only through its L^∞-norm.

Remark II.6: If $n=m=2$ (hence $Z=\{0\}$), the results contrast with the zero magnetic field case. Let us work, for instance, in the framework of the Corollary II.4 We shall neglect the Y-anisotropy which is permitted and set $V_o=0$. Choose a function V which is negative and decay at infinity at least as $|y|^{-2}$. It is known that the operator P^2+V will surely have bound states. But if $\|V\|_{L^\infty}$ is small enough, there will always exist a magnetic field such that the associated Hamiltonian $(P-a)^2+V$ is purely absolutely continuous and unitarily equivalent to $(P-a)^2$.

Remark II.7: Assume that B^Y behaves as $|y|^{-\beta}$ at infinity. It is easy to see that, for $\beta \in (1,2)$, the magnetic part of the lhs of the inequality (1) is negative. Hence, in order to verify (1), the function V must be repulsive with respect to the Y-variable. Let us stick now to the case $n=2$. It is known that for $\beta \leqslant 1$, special spectral phenomena are possible for $V=0$. In some situations, dense pure point spectrum appears in an interval $[0, d]$ or on the entire positive real axis.[6] If the magnetic field is constant, the spectrum is composed of equidistant, infinitely degenerated eigenvalues. For B diverging at infinity there is no essential spectrum. However, we may suspect that by introducing a strongly repulsive potential one restores purely absolutely continuous spectrum. This is shown to be true in some situations (see Ref. 7 and references cited therein), but it seems that there is still some work to do in this direction. A short glimpse at the formula (1) shows that for $\beta<1$ the function V must be (negative and) unbounded at infinity. We were not able to adapt our method to this kind of situation.

III. THE METHOD OF THE WEAKLY CONJUGATE OPERATOR

Let H be a self-adjoint operator in the Hilbert space $\mathcal{H}$, $\mathcal{G}^1$ its form-domain and $\mathcal{G}^{-1}$ the topological dual of $\mathcal{G}^1$. We consider a unitary group $\{W_t=e^{itA} | t\in\mathbb{R}\}$ in $\mathcal{H}$ and we assume that for all t, W_t leaves $\mathcal{G}^1$ invariant. It is a standard result that $\{W_t|_{\mathcal{G}^1}\}_{t\in\mathbb{R}}$ is a C_0-group in $\mathcal{G}^1$ and $\{(W_{-t}|_{\mathcal{G}^1})^*\}_{t\in\mathbb{R}}$ a C_0-group in $\mathcal{G}^{-1}$ (see Ref. 8, Proposition 6.3.1 for instance). We will use for all of them the simple notation W_t. By $D(A;\mathcal{G}^1)$ we denote the domain of the generator of the C_0-group acting in $\mathcal{G}^1$.

Definition III.1: We say that $H\in C^1(A;\mathcal{G}^1,\mathcal{G}^{-1})$ if the map

$$\mathbb{R}\ni t\to W_{-t}HW_t\in B(\mathcal{G}^1,\mathcal{G}^{-1})$$

is strongly C^1 or, equivalently, if the sesquilinear form

$$D(A;\mathcal{G}^1)\times D(A;\mathcal{G}^1)\ni (f,g)\to i\langle Hf,Ag\rangle - i\langle Af,Hg\rangle \in\mathbb{C}$$

is continuous when we consider on $D(A;\mathcal{G}^1)$ the topology of $\mathcal{G}^1$.

The equivalence follows easily from Ref. 8, Proposition 5.1.2 (b).

Let us denote by T the strong derivative $(d/dt)\,(W_{-t}HW_t)|_{t=0}$ or, equivalently, the extension of the sesquilinear form to an element of $B(\mathcal{G}^1,\mathcal{G}^{-1})$. One might think of T as a rigorous form of the commutator $i[H,A]$.

Definition III.2: The self-adjoint operator A is said to be weakly conjugate to H if $H\in C^1(A;\mathcal{G}^1,\mathcal{G}^{-1})$ and $T>0$ (i.e., $\langle f,Tf\rangle>0$ for all $f\in\mathcal{G}^1\setminus\{0\}$).

Let $\mathcal{T}$ be the completion of $\mathcal{G}^1$ with respect to the norm $\|f\|_{\mathcal{T}}=\langle f,Tf\rangle^{1/2}$ and $\mathcal{T}^*$ its topological dual (T extends to an isometric operator: $\mathcal{T}\to\mathcal{T}^*$). $\mathcal{T}^*$ can be identified with the completion of $T\mathcal{G}^1$ with respect to the norm $\|g\|_{\mathcal{T}^*}=\langle g,T^{-1}g\rangle^{1/2}$. $\mathcal{T}$ and $\mathcal{T}^*$ are Hilbert spaces which are generally not comparable with $\mathcal{H}$. However, since $\mathcal{G}^1\hookrightarrow\mathcal{T}$ and $\mathcal{T}^*\hookrightarrow\mathcal{G}^{-1}$, it makes sense to assume that $\{W_t\}_{t\in\mathbb{R}}$ restricts to a C_0-group in $\mathcal{T}^*$ or, equivalently, that it extends to a C_0-group in $\mathcal{T}$. Under this assumption, we denote by $\mathcal{A}=D(A;\mathcal{T}^*)$ the domain of the generator of the C_0-group in $\mathcal{T}^*$ endowed with the graph-norm. We are now in a position to state the abstract result (see Ref. 5):

Theorem III.3: *Let H be a self-adjoint operator in $\mathcal{H}$ having a spectral gap (i.e., a real point outside its spectrum). Assume that A is weakly conjugate to H and that $T=i[H,A]\in C^1(A;\mathcal{T},\mathcal{T}^*)$. Then we have the following.*

(a) $\|(H-\lambda\mp i\nu)^{-1}\|_{B(\mathcal{A},\mathcal{A}^*)}\leqslant c$, *uniformly in* $\lambda\in\mathbb{R},\nu>0$.
(b) *Any operator L belonging to $B(\mathcal{A}^*,\mathcal{H})$ is H-smooth.*
(c) *H has purely absolutely continuous spectrum.*

Notice that $B(\mathcal{G}^{-1},\mathcal{G}^1)\subset B(\mathcal{A},\mathcal{A}^*)$; this gives sense to (a).

By combining the point (b) with standard results on smooth operators[9] and by taking into account the embeddings $\mathcal{G}^1 \hookrightarrow \mathcal{A}^*$ and $\mathcal{A} \hookrightarrow \mathcal{G}^{-1}$, we can state the next result on the existence and completeness of the wave operators:

Corollary III.4: Under the hypothesis of Theorem III.3 we assume, in addition, that H is lower semibounded. Let U be a symetric operator in $\mathcal{H}$ which extends to an element of $B(\mathcal{A}^*,\mathcal{A})$. Then there exists $\beta_o>0$ such that for any $\beta\in(-\beta_o,\beta_o)$ the operator $H_\beta=H+\beta U$ is well defined in form-sense, self-adjoint, and is unitary equivalent to H through the wave operators $\Omega_\pm=s-\lim_{t\to\pm\infty} e^{itH_\beta}e^{-itH}$.

IV. PROOFS

Proof of Theorem II.2: The proof consists in verifying the hypothesis in Theorem III.3 and will be divided into several parts.

(i) The weakly conjugate operator A is introduced through its unitary group in $\mathcal{H}$ given by (see also Ref. 10)

$$(W_t f)(y,z)=e^t \exp(-i\Phi_t(y))f(e^t y,z)$$

with

$$\Phi_t(y)=\int_1^{e^t}\left(\sum_{j=1}^{2} y_j a_j(sy)\right) ds.$$

In relation with the splitting $\mathcal{H}=L^2(\mathbb{R}^2)\otimes L^2(\mathbb{R}^{n-2})$ we have $W_t=W_t^Y\otimes 1$, with an obvious meaning for W_t^Y. It is easy to check that $\{W_t\}_{t\in\mathbb{R}}$ is indeed an evolution group in $\mathcal{H}$. Its generator A is essentially self-adjoint on $\mathcal{D}$ and for $f\in\mathcal{D}$ one has

$$Af=\frac{1}{2}\sum_{j=1}^{2}(\Pi_j Q_j+Q_j\Pi_j)f.$$

(ii) Let us verify that for all $t\in\mathbb{R}$ the operator W_t leaves invariant the magnetic Sobolev space $\mathcal{G}^1=\{f\in\mathcal{H}|\Pi_j f\in\mathcal{H},\ \ j=1,\dots,n\}$. A straightforward calculation on $\mathcal{D}$ shows that for $j=1,2$

$$W_{-t}\Pi_j W_t=e^t\left(\Pi_j-(-1)^k\int_{e^{-t}}^{1} sQ_k B^Y(sQ^Y)\,ds\right)\quad (k=j+1 \bmod 2).$$

However, by hypothesis (i), the second term defines a bounded operators in $\mathcal{H}$, hence $W_{-t}\Pi_j W_t$ extends to a bounded operator: $\mathcal{G}^1\to\mathcal{H}$. For $j\geqslant 3$ one has $W_{-t}\Pi_j W_t=\Pi_j$. All these imply the invariance of $\mathcal{G}^1$ under W_t.

(iii) We verify now the second condition in Definition III.1. Because of our smoothness assumptions, one gets easily on the invariant domain $\mathcal{D}$ the formula

$$i[H,A]=2H_0^Y\otimes 1-D^Y V+\{B^Y(Q_1\Pi_2-Q_2\Pi_1)+(\Pi_2 Q_1-\Pi_1 Q_2)B^Y\}. \tag{2}$$

By applying to the last term, which we denote by H_c, the Cauchy–Schwarz inequality, and the formula $2ab\leqslant(1/\gamma)\,a^2+\gamma b^2(\gamma>0)$, we get for $f\in\mathcal{D}$:

$$|\langle f,H_c f\rangle|\leqslant\frac{1}{\gamma}\langle f,H_0^Y\otimes 1 f\rangle+\gamma\langle f,(|Q^Y|B^Y)^2 f\rangle. \tag{3}$$

Since $D^Y V$ and $(|Q^Y|B^Y)^2$ are bounded, the required continuity of $i[H,A]$ is obtained.

(iv) We denote by $T\in B(\mathcal{G}^1,\mathcal{G}^{-1})$ the continuous extension of $i[H,A]$ defined initially on $\mathcal{D}$. Let us show that $T>0$. From (2), (3), and Lemma II.1 we obtain on $\mathcal{D}$

$$T \geqslant \varepsilon H_0^Y \otimes 1 + \left(2 - \frac{1}{\gamma} - \varepsilon\right)[\pm B^Y] - D^Y V - (\gamma + \delta)(|Q^Y|B^Y)^2 + \delta(|Q^Y|B^Y)^2.$$

By hypothesis (iii) of the theorem, we obtain the following three inequalities on $\mathcal{D}$, which extend in form-sense to $\mathcal{G}^1$:

$$T \geqslant \varepsilon H_0^Y \otimes 1 \geqslant \pm \varepsilon B^Y, \tag{4}$$

$$T \geqslant G(Q), \tag{5}$$

$$T \geqslant \delta(|Q^Y|B^Y)^2. \tag{6}$$

Obviously, $T \geqslant 0$. However, $f \in \mathcal{G}^1$ satisfies $\langle f, H_0^Y \otimes 1 f\rangle = 0$ if and only if $\Pi_j f = 0$ for all index j corresponding to Y. Since Π_j is unitary equivalent with P_j, by a simple gauge transformation, this implies $f = 0$. Therefore, $T > 0$, hence A is weakly conjugate to H.

(v) Some more calculations on the invariant domain $\mathcal{D}$ give

$$\begin{aligned} i[T,A] = {} & 4H_0^Y \otimes 1 + D^Y(D^Y V) + 2(|Q^Y|B^Y)^2 + 2H_c \\ & + \{D^Y B^Y (Q_2 \Pi_1 - Q_1 \Pi_2) + (\Pi_1 Q_2 - \Pi_2 Q_1) D^Y B^Y\}. \end{aligned}$$

By using for the last term the same strategy that gave (3), we obtain

$$|\langle f, i[T,A] f\rangle| \leqslant c_1 \langle f, H_0^Y \otimes 1 f\rangle + c_2 \langle f, |Q^Y|^2 \{(B^Y)^2 + (D^Y B^Y)^2\} f\rangle + |\langle f, D^Y(D^Y V) f\rangle|.$$

Then the hypothesis (iv) and the inequalities (4)–(6) show that $i[T,A] \leqslant cT$ on $\mathcal{D}$, for a suitable positive constant c. Let $\mathcal{T}$ be the completion of $\mathcal{G}^1$ with respect to the norm $\|f\|_{\mathcal{T}} = \langle f, Tf\rangle^{1/2}$ and denote by $\mathcal{T}^*$ its topological dual. Since $\mathcal{D}$ is dense in $\mathcal{T}$, the preceding step shows that $i[T,A]$ extends to an operator belonging to $B(\mathcal{T},\mathcal{T}^*)$.

(vi) We check now that W_t can be extended to a bounded operator in $\mathcal{T}$. It will be enough to prove that $\|W_t f\|_{\mathcal{T}} \leqslant c(t)\|f\|_{\mathcal{T}}$ for all $f \in \mathcal{D}$. One has, by (v),

$$\begin{aligned} \|W_t f\|_{\mathcal{T}}^2 &= \langle f, Tf\rangle + \int_0^t ds\, \langle W_s f, i[T,A] W_s f\rangle \\ &\leqslant \|f\|_{\mathcal{T}}^2 + c\left|\int_0^t ds\, \langle W_s f, T W_s f\rangle\right| \\ &= \|f\|_{\mathcal{T}}^2 + c\left|\int_0^t ds\, \left\| W_s f \right\|_{\mathcal{T}}^2\right|. \end{aligned}$$

The function $(0,t) \ni s \to \|W_s f\|_{\mathcal{T}}^2 \in \mathbb{R}$ is bounded (since $\mathcal{G}^1 \hookrightarrow \mathcal{T}$), hence, by a simple form of the Gronwall lemma, we obtain the inequality $\|W_t f\|_{\mathcal{T}} \leqslant e^{c/2\,|t|}\|f\|_{\mathcal{T}}$.

(vii) By duality, we define operators $(W_{-t})^*$ in $\mathcal{T}^*$ which are restrictions of the elements of the C_0-group $\{W_t\}_{t \in \mathbb{R}}$ acting in $\mathcal{G}^{-1}$. In this way we automatically get a C_0-group in $\mathcal{T}^*$. We finished proving the property $T \in C^1(A;\mathcal{T},\mathcal{T}^*)$ and all the conditions in Theorem III.3 are checked. Hence the conclusions of Theorem II.2 are true, where for $\mathcal{A}$ we take the domain of the generator (also denoted by A) of the C_0-group $\{W_t\}_{t \in \mathbb{R}}$ in $\mathcal{T}^*$ endowed with the graph-norm $\|f\|_{\mathcal{A}} = (\|f\|_{\mathcal{T}^*}^2 + \|Af\|_{\mathcal{T}^*}^2)^{1/2}$. $\mathcal{A}^*$ stands for its topological dual. ■

The spaces $\mathcal{A}$ and $\mathcal{A}^*$ introduced in the proof are generally quite intricate. We shall prove now the assertions made in Remark II.3, which give a simple form of the limiting absorption principle. This is contained in the following proposition.

Proposition IV.1: Let us place ourselves in the framework of Theorem II.2 but also suppose that $B^Y(y) > 0$ *for all* $y \in Y$. *Let* $F(y) = \max\{B^Y(y)^{-1/2}, |y|\}$ *and* $\mathcal{F}$ *the completion of* $\mathcal{D}$ *with respect to the norm* $\|f\|_{\mathcal{F}} = \|F(Q)f\|$. *Then* $\mathcal{F}$ *is continuously embedded into* $\mathcal{A}$.

Proof: We need to show that $\langle f,T^{-1}f\rangle+\langle Af,T^{-1}Af\rangle\leqslant c\langle f,F^2(Q)f\rangle$ for all $f\in\mathcal{D}$. To bound the first term we use (4) and Corollary 1 of Ref. 11 to get $\langle f,T^{-1}f\rangle\leqslant(1/\varepsilon)\langle f,(B^Y)^{-1}f\rangle$. For the second one, we write $A=\Sigma_{j=1}^2\Pi_jQ_j+i1$. From the first part of the inequality (4) it follows easily that Π_j extends to a bounded operator: $\mathcal{H}\to\mathcal{T}^*$. Then

$$\|Af\|_{\mathcal{T}^*}\leqslant\frac{1}{\sqrt{\varepsilon}}\sum_{j=1}^{2}\|Q_jf\|+\|f\|_{\mathcal{T}^*}\leqslant\frac{2}{\sqrt{\varepsilon}}\||Q^Y|f\|+\frac{1}{\sqrt{\varepsilon}}\|(B^Y)^{-1/2}f\|\leqslant c\|F(Q)f\|,$$

and the proof is finished. ∎

Proof of Corollary II.4: We may interpret V_o as a real function defined on $\mathbb{R}^n=Y\oplus Z$, which is homogeneous of degree 0 in Y. Therefore $W_{-t}V_o(Q^Y,Q^Z)W_t=V_o(Q^Y,Q^Z)$ and $D^YV_o=0$. Theorem II.2 can be applied with V replaced by V_o. The lack of regularity of V_o can be easily overcome using the first condition of Definition III.1 in the point (iii) of the proof of the Theorem. Since B^Y is everywhere strictly positive, we can also apply Proposition IV.1. It follows that the operator of multiplication with $(B^Y)^{1/2}$ is globaly H-smooth. Then the result is a consequence of Corollary III.4, or we apply directly the standard Kato's smoothness theory; see, for example, Ref. 9. ∎

V. SOME FURTHER DEVELOPMENTS

The case $m\geqslant3$ is not more difficult than the previous one, apart from the notations. In fact, some new opportunities are available. We shall state a result and make some brief comments on the changes needed in the proof and on its special features. The setting will be exactly that exposed in the beginning of Sec. II, before the statement of Theorem II.2, except that now we assume that $m\geqslant3$. By $\dot{\mathcal{H}}_1^Y$ we denote *the homogeneous Lebesgue space of order* 1 *in* Y, i.e., the completion of $C_0^\infty(Y)$ in the homogeneous norm $\|f\|_{\dot{\mathcal{H}}_1^Y}=\||Q^Y|f\|_{L^2(Y)}$. Its topological dual will be denoted by $\dot{\mathcal{H}}_{-1}^Y$. Both these spaces are incomparable with $L^2(Y)$. We set $\dot{\mathcal{H}}_{\pm1}=\dot{\mathcal{H}}_{\pm1}^Y\otimes L^2(Z)$.

Theorem V.1: *Assume that*

(i) $|B_{jk}(y)|\leqslant c/(1+y_j^2+y_k^2)^{1/2+\eta},\quad j,k=1,\dots,m,\quad \eta>0.$

(ii) $D^YV=y\cdot(\nabla_yV)\in L^\infty(\mathbb{R}^n)$.

(iii) *There exist constants* $\gamma,\varepsilon,\delta,\alpha$, $\{\alpha_{jk}\}_{j,k=1,\dots,m}$ *satisfying* $(2-(1/\gamma)(m-1))>\varepsilon>0$, $\delta>0$, $\alpha\in[0,1]$, $\alpha_{jk}=\alpha_{kj}$, $\alpha_{jj}=0$, $\Sigma_{k=1}^m|\alpha_{jk}|\leqslant(1-\alpha)$ *for every* $j\leqslant m$ *such that*

$$G(y,z)=\left(2-\frac{1}{\gamma}(m-1)-\varepsilon\right)\left\{\sum_{1\leqslant j<k\leqslant m}\alpha_{jk}B_{jk}+\alpha\left(\frac{m-2}{2}\right)^2\frac{1}{|y|^2}\right\}$$

$$-D^YV-(\gamma+\delta)\sum_{1\leqslant j<k\leqslant m}(y_j^2+y_k^2)B_{jk}^2\geqslant0.$$

(iv) The functions $|D^Y(D^YV)|$ *and* $(y_j^2+y_k^2)(D^YB_{jk})^2$, $(j,k=1,\dots,m)$, *can be dominated by linear combinations of* G, $|y|^{-2}$, B_{pq}, *and* $(y_p^2+y_q^2)B_{pq}^2$ $(p,q=1,\dots,m)$.

Then we have the following.

(a) $\|(H-\lambda\mp i\nu)^{-1}\|_{B(\dot{\mathcal{H}}_1,\dot{\mathcal{H}}_{-1})}\leqslant c$, *uniformly in* $\lambda\in\mathbb{R}$, $\nu>0$.

(b) *Any bounded operator* $L:\dot{\mathcal{H}}_{-1}\to\mathcal{H}$ *is* H*-smooth.*

(c) H *is purely absolutely continuous.*

Proof: The proof is very similar to that given for Theorem II.2, so we will just point out some adaptations. The main new fact is that aside from Lemma II.1, we have a new positivity result:

$$H_0^Y\geqslant\left(\frac{m-2}{2}\right)^2|Q^Y|^{-2}. \tag{7}$$

For zero magnetic field this is the classical Hardy inequality; it stays true in the magnetic case by Kato's inequality.

The unitary group is now given by $(W_t f)(y,z)=e^{mt/2}\exp(-i\Phi_t(y))f(e^t y,z)\ \forall f\in\mathcal{H}$, with $\Phi_t(y)=\int_1^{e^t}(\Sigma_{j=1}^m y_j a_j(sy))\,ds$. The invariance of $\mathcal{G}^1$ is proved straightforwardly by calculating $W_{-t}\Pi_j W_t=e^t(\Pi_j-\int_{e^{-t}}^1\Sigma_{k=1}^m sQ_kB_{jk}(sQ)ds)$ and by using hypothesis (i).

Simple calculations performed on the invariant domain $\mathcal{D}$ lead to

$$i[H,A]=2H_0^Y\otimes 1-D^YV+\sum_{j,k=1}^m (\Pi_jQ_kB_{kj}+Q_kB_{kj}\Pi_j),$$

which can be extended continuously to $T\in B(\mathcal{G}^1,\mathcal{G}^{-1})$ by using the Cauchy–Schwartz inequality and hypothesis (i) and (ii). The strict positivity of T is obtained from hypothesis (iii), which includes all inequalities connecting H_0^Y and a multiplicative operator in a profitable way by making appropriate linear combinations. Moreover, we collect the following inequalities [similar to (4) and (6)]: $T\geqslant\varepsilon B_{jk},\ \ j,k\leqslant m$, and $T\geqslant\delta\Sigma_{1\leqslant j<k\leqslant m}(Q_j^2+Q_k^2)B_{jk}^2$.

Analogous calculations on $\mathcal{D}$ give

$$i[T,A]=4H_0^Y\otimes 1+D^Y(D^YV)+2\sum_{j=1}^m\left(\sum_{k=1}^m Q_kB_{kj}\right)^2$$
$$+\sum_{j,k=1}^m (2\Pi_jQ_kB_{kj}+2Q_kB_{kj}\Pi_j-\Pi_jQ_k(D^YB_{kj})-Q_k(D^YB_{kj})\Pi_j),$$

which can be handled similary as for T. An upper bound can be obtained by using the above inequalities and hypothesis (iv). The remainder of the proof is exactly similar to that for Theorem II.2.

For simplicity, the limiting absorption principle is expressed between spaces $\dot{\mathcal{H}}_1$ and $\dot{\mathcal{H}}_{-1}$ and not $\mathcal{A}$ and $\mathcal{A}^*$. This follows from a calculation similar to the one in Proposition IV.1 and by using the magnetic Hardy inequality (7). ■

Remark V.2: Remark II.5 has a counterpart here which is easy to formulate, but notice that some extra anisotropy is allowed for the magnetic field inside Y. A very simple, relevant example is as follows: take $Y=\mathbb{R}^3,\ \ Z=\{0\}$ and set $B_{jk}(y)=c_{jk}/(1+y_j^2+y_k^2)$. Let V be homogeneous of degree 0, for simplicity. Then, if the constants c_{jk} are not too large, the hypothesis of Theorem V.1 are fulfilled; in conclusion H is purely absolutely continuous. However, B_{12} does not decay when we go to infinity staying close to the third axis and analogously for the other two components.

Remark V.3: In general, *global H*-smooth operators are not easy to obtain. By the point (b) of Theorem V.1, $|Q^Y|^{-1}\otimes 1$ is H-smooth. In the particular case of the Laplace operator, which is obtained by taking $Z=\{0\},\ \ B=0$ and $V=0$, we get a result which was not known for quite a long time. We look to Refs. 12 and 13 where, however, more general and precise results are presented. In Ref. 14 the method of the weakly conjugate operator is used to deduce global smooth operators for operators of multiplication by a rather large class of functions in a unified manner.

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