Positive quantization in the presence of a variable magnetic field

Marius Măntoiu,[1,a)] Radu Purice,[2,b)] and Serge Richard[3,c)]
[1]*Departamento de Matemáticas, Universidad de Chile, Las Palmeras 3425, Casilla 653, Santiago, Chile*
[2]*Institute of Mathematics Simion Stoilow of the Romanian Academy, P.O. Box 1-764, Bucharest, RO-70700, Romania*
[3]*Graduate School of Pure and Applied Sciences, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan*

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Starting with a previously constructed family of coherent states, we introduce the Berezin quantization for a particle in a variable magnetic field and we show that it constitutes a strict quantization of a natural Poisson algebra. The phase-space reinterpretation involves a magnetic version of the Bargmann space and leads naturally to Berezin-Toeplitz operators. © *2011 American Institute of Physics*. [doi:10.1063/1.3656253]

I. INTRODUCTION

The mathematical literature treating Berezin-Toeplitz operators in phase space (also called anti-Wick operators, localization operators, etc.) and their connection with the pseudodifferential calculus in Weyl or Kohn-Nirenberg form is huge; we only cite some basic references as Refs. 5, 6, 11, 12, 19, and 33. One important *raison d'être* of this type of operators is the fact that they realize a quantization of certain classes of physical systems, the one consisting of a spinless non-relativistic particle being the basic one, a paradigm for the quantization of other systems. The Berezin-Toeplitz correspondence, sending classical observables (functions on phase-space) to quantum ones (self-adjoint operators in some Hilbert space), while less satisfactory than the Weyl correspondence from the point of view of composition properties, has the advantage of being positive, sending positive functions into positive operators. It is also very often handier for norm-estimates.

It is now known that if the particle is placed in a variable magnetic field, the Weyl form of the pseudodifferential calculus should be modified to insure gauge covariance and to cope with the changes in geometry and kinematics due to the presence of the magnetic field. Recent publications[13,14,16,21–26,29] introduced and developed a mathematical formalism for the observables naturally associated with such a system, both in a classical and in a quantum framework. The changes involve mathematical objects as group 2-cocycles with values in algebras of functions, twisted dynamical systems, and twisted crossed product C^*-algebras as well as an enlargement of the Weyl calculus, so their interest is not only related to the study of physical systems in magnetic fields. Recently, the magnetic calculus has been extended to the case of nilpotent Lie groups.[2–4]

Aside the quantization of observables, one must also perform the quantization of states. A convenient systematization of this topic is an axiomatic framework which can be found in Ref. 19, see also Refs. 17 and 18; it relies on seeing both the classical and the quantum pure states as forming Poisson spaces with a transition probability. The pure states of a classical particle are the points of the phase space Ξ (the symplectic form (1.1) takes the magnetic field into account). On another hand, the pure states space of $\mathbb{K}(\mathcal{H})$ (the C^*-algebra of all the compact operators in the Hilbert

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[a)]Electronic mail: marius.mantoiu@imar.ro.
[b)]Electronic mail: radu.purice@imar.ro.
[c)]On leave from Université de Lyon, Université Lyon 1, CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France. Electronic mail: richard@math.univ-lyon1.fr.

space $\mathcal{H}$) is homeomorphic to the projective space $\mathbb{P}(\mathcal{H})$. The latter space is also endowed with the $\hbar$-dependent Fubini-Study symplectic form.

Being guided by general prescriptions,[1,12,15,19,20] we defined in Ref. 27 a family of pure states (called *magnetic coherent states*), indexed by the points of the phase space and by Planck's constant $\hbar$. They satisfy certain structural requirements and a prescribed behavior in the limit $\hbar \to 0$. We would like now to complete the picture, indicating the appropriate modifications needed to obtain *magnetic Berezin operators* associated with the choice of a vector potential. The present article outlines this topic in the setting of quantization theory, but we hope to use the formalism in the future for concrete spectral problems involving magnetic operators.

Section II contains a brief recall of the magnetic Weyl calculus both in pseudodifferential and in twisted convolution form as well as a short description of the magnetic coherent states.

In Sec. III the magnetic Berezin quantization is defined on functions and distributions and its basic properties are studied. It is its fate to be (completely) positive, but in addition it has the important property of being *gauge covariant*: vector potentials corresponding to the same magnetic field lead to unitarily equivalent Berezin operators. We study the connection with magnetic Weyl operators and show that the two quantizations are equivalent in the limit $\hbar \to 0$. The magnetic Berezin-to-Weyl map depends intrinsically on the magnetic field B and not on the choice of a corresponding potential A satisfying $\mathrm{d}A = B$. A very convenient setting is obtained after making a unitary transformation, which is a generalization of the classical Bargmann transformation. The associated Bargmann-type space is a Hilbert space with reproducing kernel, and in this representation the Berezin quantization will consist of Toeplitz-type operators. The standard Bargmann transform is build on Gaussian coherent states and this has certain advantages, among which we quote a well-investigated holomorphic setting. The presence of the magnetic field seems to ruin such a possibility, so we are not going to privilege any *a priori* choice. It is also likely that the anti-Wick setting, involving creation and annihilation operators and a certain type of ordering, is no longer available for variable magnetic fields.

In Sec. IV, we prove that our framework provides a strict quantization of a natural Poisson algebra in the sense of Rieffel. One extends in this way some of the results of Refs. 7 and 8, see also Refs. 9, 10, and 19. We prove essentially that the $\hbar$-depending magnetic Berezin operators: (i) have continuously varying operator norms, (ii) mutually compose "in a classical commutative way" in the limit $\hbar \to 0$, (iii) have commutators which are governed by a magnetic Poisson bracket in the first order in $\hbar$. To do this, we use similar results proved in Ref. 23 for the magnetic Weyl calculus as well as the connection between the two magnetic quantizations that has been obtained in Sec. III. We notice that our procedure is not a *deformation* quantization in some obvious way, but this is not specific to the magnetic case. For the general theory of strict quantization and for many examples we refer to Refs. 19 and 31–32.

II. RECALL OF PREVIOUS CONSTRUCTIONS AND RESULTS

We start by briefly reviewing the geometry of the classical system with a variable magnetic field,[23,28] the structure of the twisted (magnetic) calculus,[13,16,22,25,29] and the natural form of the magnetic coherent states.[27] Details and developments are also included in the references cited above.

A. The geometry of the classical system with a variable magnetic field

The particle evolves in the Euclidean space $\mathcal{X} := \mathbb{R}^N$ under the influence of a smooth magnetic field, which is a closed 2-form B on $\mathcal{X}$ ($\mathrm{d}B = 0$), given by matrix-components

$$B_{jk} = -B_{kj} : \mathcal{X} \to \mathbb{R}, \qquad j, k = 1, \dots, N.$$

The phase space is denoted by $\Xi := T^*\mathcal{X} \equiv \mathcal{X} \times \mathcal{X}^*$, where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$; systematic notations as $X = (x, \xi)$, $Y = (y, \eta)$, $Z = (z, \zeta)$ will be used for its points.

The classical observables are given by real smooth functions on Ξ. They form a real vector space, which is also a Poisson algebra under the usual pointwise product $(f \cdot g)(X) \equiv (fg)(X)$

$:= f(X)g(X)$ and the Poisson bracket

$$\{f, g\}^B = \sum_{j=1}^{N}(\partial_{\xi_j} f\, \partial_{x_j} g - \partial_{\xi_j} g\, \partial_{x_j} f) + \sum_{j,k=1}^{N} B_{jk}(\cdot)\, \partial_{\xi_j} f\, \partial_{\xi_k} g.$$

For further use, we notice that $\{\,\cdot\,,\,\cdot\,\}^B$ is canonically generated by the symplectic form

$$(\sigma^B)_X(Y, Z) = z \cdot \eta - y \cdot \zeta + B(x)(y, z) = \sum_{j=1}^{N}(z_j\, \eta_j - y_j\, \zeta_j) + \sum_{j,k=1}^{N} B_{jk}(x)\, y_j\, z_k, \tag{1.1}$$

obtained by adding to the standard symplectic form

$$\sigma(X, Y) \equiv \sigma[(x, \xi), (y, \eta)] := y \cdot \xi - x \cdot \eta$$

a magnetic contribution.

B. The structure of the magnetic pseudodifferential calculus

The intrinsic way to turn to the quantum counter-part is to deform the pointwise product fg into a non-commutative product $f \sharp_\hbar^B g$ depending on the magnetic field B and the Planck constant $\hbar$. This is given by

$$\left(f \sharp_\hbar^B g\right)(X) := (\pi\hbar)^{-2N} \int_\Xi \int_\Xi \mathrm{d}Y\, \mathrm{d}Z\, e^{-\frac{2i}{\hbar}\sigma(X-Y, X-Z)}\, e^{-\frac{i}{\hbar}\Gamma^B\langle x-y+z,\, y-z+x,\, z-x+y\rangle}\, f(Y)\, g(Z), \tag{1.2}$$

and it involves fluxes of the magnetic field B through triangles. If $a, b, c \in \mathcal{X}$, then we denote by a, b, c the triangle in $\mathcal{X}$ of vertices a, b, and c and set

$$\Gamma^B\langle a, b, c\rangle := \int_{\langle a,b,c\rangle} B$$

for the invariant integration of the 2-form B through the 2-simplex a, b, c. For $B = 0$, (1.2) coincides with the Weyl composition of symbols in pseudodifferential theory. Also using complex conjugation $f \mapsto \overline{f}$ as involution, one gets various non-commutative *-algebras of functions on Ξ, some of them also admitting a natural C*-norm; they are regarded as *algebras of magnetic quantum observables*.

The full formalism also involves families of representations of these *-algebras in the Hilbert space $\mathcal{H} := L^2(\mathcal{X})$. They are defined by circulations $\Gamma^A[x, y] := \int_{[x,y]} A$ of vector potentials A through segments $[x, y] := \{ty + (1 - t)x | t \in [0, 1]\}$ for any $x, y \in \mathcal{X}$. We recall that, being a closed 2-form in $\mathcal{X} = \mathbb{R}^N$, the magnetic field is exact: it can be written as $B = \mathrm{d}A$ for some 1-form A. For such a vector potential A, we define

$$\left[\mathfrak{Op}_\hbar^A(f)u\right](x) := (2\pi\hbar)^{-N} \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}y\, \mathrm{d}\eta\, e^{\frac{i}{\hbar}(x-y)\cdot\eta}\, e^{-\frac{i}{\hbar}\Gamma^A[x,y]}\, f\left(\tfrac{x+y}{2}, \eta\right) u(y). \tag{1.3}$$

If $A = 0$, one recognizes the Weyl quantization, associating with functions or distributions on Ξ linear operators acting on function spaces on $\mathcal{X}$. For suitable functions f, g, one proves

$$\mathfrak{Op}_\hbar^A(f)\mathfrak{Op}_\hbar^A(g) = \mathfrak{Op}_\hbar^A(f \sharp_\hbar^B g), \qquad \mathfrak{Op}_\hbar^A(f)^* = \mathfrak{Op}_\hbar^A(\overline{f}).$$

The main interpretation of the operators defined in (1.3) is given by the formula $\mathfrak{Op}_\hbar^A(f) = f\left(Q; \Pi_\hbar^A\right)$, where $f\left(Q; \Pi_\hbar^A\right)$ should be regarded as the function f applied to the family of non-commuting self-adjoint operators

$$\left(Q, \Pi_\hbar^A\right) \equiv \left(Q_1, \ldots, Q_N, \Pi_{\hbar,1}^A, \ldots, \Pi_{\hbar,N}^A\right),$$

where Q_j is the operator of multiplication by the coordinate function x_j and $\Pi_{\hbar,j}^A := -i\hbar\partial_j - A_j$ is the jth component of the magnetic momentum. They satisfy the commutation relations

$$i[Q_j, Q_k] = 0, \quad i[\Pi_{\hbar,j}^A, Q_k] = \hbar\delta_{j,k}, \quad i[\Pi_{\hbar,j}^A, \Pi_{\hbar,k}^A] = -\hbar B_{jk}\,. \tag{1.4}$$

This stresses the interpretation of our twisted pseudodifferential theory as a *non-commutative functional calculus* constructed on the commutation relations (1.4). For $A = 0$, one gets $\Pi_\hbar^A = D_\hbar$

$:= -i\hbar\nabla$, so we recover the canonical commutation relations of non-relativistic quantum mechanics and the standard interpretation of the Weyl calculus.

C. The twisted crossed product representation

For conceptual and computational reasons a change of realization is useful; we obtain it composing the mapping $\mathfrak{Op}_\hbar^A$ with a partial Fourier transformation. Using the notation $\mathfrak{F} := 1 \otimes \mathcal{F}^*$, where $\mathcal{F}$ is the usual Fourier transform, we define $\mathfrak{Rep}_\hbar^A(F) := \mathfrak{Op}_\hbar^A(\mathfrak{F}F)$. This makes sense for various classes of functions. Here, we record the explicit formula

$$[\mathfrak{Rep}_\hbar^A(F)u](x) = \hbar^{-N} \int_{\mathcal{X}} \mathrm{d}y\, e^{-\frac{i}{\hbar}\Gamma^A[x,y]}\, F\left(\tfrac{x+y}{2}, \tfrac{y-x}{\hbar}\right) u(y). \tag{1.5}$$

Defining

$$(F \diamond_\hbar^B G)(x,y) := \int_{\mathcal{X}} \mathrm{d}z\, e^{-\frac{i}{\hbar}\Gamma^B\langle x-\frac{\hbar}{2}y,\, x-\frac{\hbar}{2}y+\hbar z,\, x+\frac{\hbar}{2}y\rangle}\, F\left(x - \tfrac{\hbar}{2}(y-z), z\right) G\left(x + \tfrac{\hbar}{2}z, y - z\right),$$

one gets

$$\mathfrak{Rep}_\hbar^A(F)\,\mathfrak{Rep}_\hbar^A(G) = \mathfrak{Rep}_\hbar^A(F \diamond_\hbar^B G).$$

In Refs. 23, 25, and 26, the representation $\mathfrak{Rep}_\hbar^A$ and the composition law $\diamond_\hbar^B$ have been used in connection with the C^*-algebraic twisted crossed product to quantize systems with magnetic fields. We are not going to use this systematically, but only state two basic results which are useful below: First, both the Schwartz space $\mathcal{S}(\mathcal{X} \times \mathcal{X})$ and the Banach space $L^1\big(\mathcal{X}_y; L^\infty(\mathcal{X}_x)\big)$ are stable under the multiplication $\diamond_\hbar^B$. Second, for each $F, G \in L^1\big(\mathcal{X}_y; L^\infty(\mathcal{X}_x)\big)$, one has

$$\|\mathfrak{Rep}_\hbar^A(F)\| \;\le\; \|F\|_{1,\infty} := \int_\Xi dy \|F(\cdot, y)\|_\infty \tag{1.6}$$

and

$$\|F \diamond_\hbar^B G\|_{1,\infty} \;\le\; \|F\|_{1,\infty} \|G\|_{1,\infty}\,.$$

D. Magnetic coherent states

Let us fix a unit vector $v \in \mathcal{H} := L^2(\mathcal{X})$, and for any $\hbar \in I := (0, 1]$ let us define the unit vector $v_\hbar \in \mathcal{H}$ by $v_\hbar(x) := \hbar^{-N/4} v\big(\frac{x}{\sqrt{\hbar}}\big)$. For any choice of a vector potential A generating the magnetic field B, we define *the family of magnetic coherent vectors associated with the pair* (A, v) by

$$\left[v_\hbar^A(Z)\right](x) = e^{\frac{i}{\hbar}(x-\frac{z}{2})\cdot\zeta}\, e^{\frac{i}{\hbar}\Gamma^A[z,x]}\, v_\hbar(x - z).$$

The pure state space of the C^*-algebra $\mathbb{K}(\mathcal{H})$ of compact operators can be identified with the projective space $\mathbb{P}(\mathcal{H})$. With the interpretation of the elements of $\mathbb{P}(\mathcal{H})$ as one-dimensional orthogonal projections on $\mathcal{H}$, it is natural to introduce for any $Z \in \Xi$ *the coherent states* $\mathfrak{v}_\hbar^A(Z) \in \mathbb{K}(\mathcal{H})$ by $\mathfrak{v}_\hbar^A(Z) := \big|v_\hbar^A(Z)\big\rangle\big\langle v_\hbar^A(Z)\big|$. With a slight abuse of notation, the identification with the pure states is then expressed by the relation

$$\left[\mathfrak{v}_\hbar^A(Z)\right](S) = \mathrm{Tr}\left(\big|v_\hbar^A(Z)\big\rangle\big\langle v_\hbar^A(Z)\big|\, S\right) \equiv \left\langle v_\hbar^A(Z), S\, v_\hbar^A(Z)\right\rangle.$$

We now state some properties related to the family of coherent vectors which have been proved in Ref. 27. These relations are used at various places in the sequel.

Proposition 2.1: Assume that the components of the magnetic field B belong to $BC^\infty(\mathcal{X})$ *(they are smooth and all the derivatives are bounded) and let v be an element of the Schwartz space* $\mathcal{S}(\mathcal{X})$*, satisfying* $\|v\| = 1$.

1. *For any* $\hbar \in I$ *and* $u \in \mathcal{H}$ *with* $\|u\| = 1$*, one has*

$$\int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N} \left|\langle v_\hbar^A(Y),\, u\rangle\right|^2 = 1. \tag{1.7}$$

2. *For any $Y, Z \in \Xi$, one has*

$$\lim_{\hbar\to 0} \left|\langle v_\hbar^A(Z),\ v_\hbar^A(Y)\rangle\right|^2 = \delta_{ZY}.$$

3. *If $g : \Xi \to \mathbb{C}$ is a bounded continuous function and $Z \in \Xi$, one has*

$$\lim_{\hbar\to 0} \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N} \left|\langle v_\hbar^A(Z), v_\hbar^A(Y)\rangle\right|^2 g(Y) \ =\ g(Z). \tag{1.8}$$

Furthermore, if $g \in \mathcal{S}(\Xi)$, then one has

$$\lim_{\hbar\to 0} \left[\mathfrak{v}_\hbar^A(Z)\right]\left[\mathfrak{Op}_\hbar^A(g)\right] = \delta_Z(g) = g(Z)\,. \tag{1.9}$$

III. MAGNETIC BEREZIN OPERATORS

Although not always necessary, for the sake of uniformity, we shall always assume that $v \in \mathcal{S}(\mathcal{X})$ and that $B_{jk} \in BC^\infty(\mathcal{X})$ for $j, k \in \{1, \ldots, N\}$. The components of the corresponding vector potentials A will always belong to $C^\infty_{\text{pol}}(\mathcal{X})$, i.e., they are smooth and all the derivatives are polynomially bounded. This can obviously be achieved under our assumption on B, and this will facilitate subsequent computations involving the Schwartz class.

A. The magnetic Berezin quantization

The following is an adaptation of Definition. II.1.3.4 of Ref. 19].

Definition 2.1: The magnetic Berezin quantization associated with the set of coherent states $\left\{\mathfrak{v}_\hbar^A(Z) \mid Z \in \Xi, \hbar \in I\right\}$ is the family of linear mappings $\left\{\mathfrak{B}_\hbar^A : L^\infty(\Xi) \to \mathbb{B}(\mathcal{H})\right\}_{\hbar\in I}$ given for any $f \in L^\infty(\Xi)$ by

$$\mathfrak{B}_\hbar^A(f) := \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}\, f(Y)\, \mathfrak{v}_\hbar^A(Y)\,, \tag{2.1}$$

where $\mathfrak{v}_\hbar^A(Y)$ is seen as the rank one projection $|v_\hbar^A(Y)\rangle\langle v_\hbar^A(Y)|$.

Note that for any unit vector $u \in \mathcal{H}$ and for the corresponding element $\mathfrak{u} \in \mathbb{P}(\mathcal{H})$, one has

$$\mathfrak{u}\big(\mathfrak{B}_\hbar^A(f)\big) = \mathrm{Tr}\big(|u\rangle\langle u|\mathfrak{B}_\hbar^A(f)\big) = \big\langle u, \mathfrak{B}_\hbar^A(f)u\big\rangle = \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}\, f(Y)\big|\langle v_\hbar^A(Y), u\rangle\big|^2\,.$$

By (1.7), this offers a rigorous interpretation of (2.1) as a weak integral; it can be regarded as a Bochner integral only under an integrability condition on *f*. The explicit function $H^A_{\hbar,u}(\cdot) := (2\pi\hbar)^{-N}\big|\langle v_\hbar^A(\cdot), u\rangle\big|^2$ deserves to be called *the magnetic Husimi function associated with the vector u*.[12,19] It is a positive phase space probability distribution.

Proposition 3.2: The following properties of the Berezin quantization hold.

1. *$\mathfrak{B}_\hbar^A$ is a linear map satisfying $\|\mathfrak{B}_\hbar^A(f)\| \le \|f\|_\infty$, $\forall f \in L^\infty(\Xi)$.*
2. *$\mathfrak{B}_\hbar^A$ is positive, i.e., for any $f \in L^\infty(\Xi)$ with $f \ge 0$ a.e., one has $\mathfrak{B}_\hbar^A(f) \ge 0$.*
3. *If $f \in L^1(\Xi) \cap L^\infty(\Xi)$, then $\mathfrak{B}_\hbar^A(f)$ is a trace-class operator and*

$$\mathrm{Tr}\big[\mathfrak{B}_\hbar^A(f)\big] = \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}\, f(Y)\,.$$

4. *For any $g \in L^1(\Xi)$, one has $\int_\Xi \mathrm{d}Y\, \mathfrak{v}_\hbar^A(Y)\big(\mathfrak{B}_\hbar^A(g)\big) = \int_\Xi \mathrm{d}Z\, g(Z)$.*
5. *Let us denote by $C_0(\Xi)$ the C^*-algebra of all complex continuous functions on Ξ vanishing at infinity. Then $\mathfrak{B}_\hbar^A[C_0(\Xi)] \subset \mathbb{K}(\mathcal{H})$.*

Proof: Most of the properties are quite straightforward, and they are true in a more abstract setting (Theorem II.1.3.5 of Ref. 19]). The fourth statement is a simple consequence of (1.7). By the point 3, one has $\mathfrak{B}_\hbar^A[C_c(\Xi)] \subset \mathbb{K}(\mathcal{H})$; we denoted by $C_c(\Xi)$ the space of continuous compactly supported functions on Ξ. This, the point 1 and the density of $C_c(\Xi)$ in $C_0(\Xi)$ imply that $\mathfrak{B}^A_\hbar[C_0(\Xi)] \subset \mathbb{K}(\mathcal{H})$. $\square$

Remark 2.3: To extend the weak definition of $\mathfrak{B}^A_\hbar(f)$ to distributions, remark that one can write

$$\left\langle u_1, \mathfrak{B}^A_\hbar(f)u_2\right\rangle = \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N} f(Y)\left[w^A_\hbar(u_1,v)\right](Y)\overline{\left[w^A_\hbar(u_2,v)\right](Y)},$$

where

$$\left[w^A_\hbar(u,v)\right](Y) := \left\langle u, v^A_\hbar(Y)\right\rangle = \int_\mathcal{X} \mathrm{d}x\, e^{\frac{i}{\hbar}(x-y/2)\cdot\eta}\, e^{\frac{i}{\hbar}\Gamma^A[y,x]}\,\overline{u(x)}\, v_\hbar(x-y)\,.$$

A simple computation shows that $w^A_\hbar(u,v)$ is obtained from $\overline{u}\otimes v$ by applying successively a linear change of variables, multiplication with a function belonging to $C^\infty_{\mathrm{pol}}(\mathcal{X}\times\mathcal{X})$, and a partial Fourier transform. All these operations are isomorphisms between the corresponding Schwartz spaces, so $w^A_\hbar(u,v)\in\mathcal{S}(\Xi)$ if $u,v\in\mathcal{S}(\mathcal{X})$, and the mapping $(u,v)\mapsto w^A_\hbar(u,v)$ is continuous. It follows that $w^A_\hbar(u_1,v)\,\overline{w^A_\hbar(u_2,v)}\in\mathcal{S}(\Xi)$, so one can define for $f\in\mathcal{S}'(\Xi)$ the linear continuous operator $\mathfrak{B}^A_\hbar(f):\mathcal{S}(\mathcal{X})\to\mathcal{S}'(\mathcal{X})$ by

$$\left\langle u_1, \mathfrak{B}^A_\hbar(f)u_2\right\rangle = (2\pi\hbar)^{-N}\left\langle \overline{w^A_\hbar(u_1,v)}\, w^A_\hbar(u_2,v),\, f\right\rangle,$$

using in the right-hand side the duality between $\mathcal{S}(\Xi)$ and $\mathcal{S}'(\Xi)$.

An important property that should be shared by any quantization procedure in the presence of a magnetic field is *gauge covariance*. Two vector potentials A and A' which differ only by the differential $\mathrm{d}\rho$ of a 1-form (function) will clearly generate the same magnetic field. It is already known[22] that the magnetic Weyl operators $\mathfrak{Op}^A_\hbar(f)$ and $\mathfrak{Op}^{A'}_\hbar(f)$ are unitarily equivalent. The next result expresses the gauge covariance of the magnetic Berezin quantization.

Proposition 3.4: If $A' = A + \mathrm{d}\rho$, then $\mathfrak{B}^{A'}_\hbar(f) = e^{\frac{i}{\hbar}\rho(Q)}\mathfrak{B}^A_\hbar(f)e^{-\frac{i}{\hbar}\rho(Q)}$.

Proof: A simple computation gives $v^{A'}_\hbar(Y) = e^{-\frac{i}{\hbar}\rho(y)}e^{\frac{i}{\hbar}\rho(Q)}v^A_\hbar(Y)$ for every $Y\in\Xi$. Then it follows that $\mathfrak{v}^{A'}_\hbar(Y) = e^{\frac{i}{\hbar}\rho(Q)}\mathfrak{v}^A_\hbar(Y)e^{-\frac{i}{\hbar}\rho(Q)}$ and this implies the result. $\square$

Some particular cases:

1. Clearly, we have for all $Z\in\Xi$,

$$\mathfrak{B}^A_\hbar(\delta_Z) = (2\pi\hbar)^{-N}|v^A_\hbar(Z)\rangle\langle v^A_\hbar(Z)|\,,$$

so the coherent states (seen as rank one projections) are magnetic Berezin operators in a very explicit way. Notice that $\mathfrak{B}^A_\hbar(\delta_Z)$ is a compact operator although δ_Z does not belong to $L^\infty(\Xi)$.

2. For $f := \varphi\otimes 1$, with $\varphi:\mathcal{X}\to\mathbb{C}$ (polynomially bounded), a simple computation leads to

$$\left\langle u, \mathfrak{B}^A_\hbar(f)u\right\rangle = \int_\mathcal{X}\int_\mathcal{X}\mathrm{d}x\,\mathrm{d}y\,\varphi(x-\sqrt{\hbar}y)\,|u(x)|^2\,|v(y)|^2\,.$$

Setting $\varphi(x) := x_j \equiv q_j(x,\xi)$ for some $j\in\{1,\dots,N\}$, one gets

$$\left\langle u, \mathfrak{B}^A_\hbar(f)u\right\rangle = \int_\mathcal{X}\mathrm{d}x\, x_j\,|u(x)|^2 - \sqrt{\hbar}\,\|u\|^2\int_\mathcal{X}\mathrm{d}y\, y_j\,|v(y)|^2.$$

Thus, if v is even, then $\mathfrak{B}^A_\hbar(q_j) = Q_j$. In general, we only get this in the limit $\hbar\to 0$.

3. If we set $f(x,\xi) := \xi_j \equiv p_j(x,\xi)$, then

$$\left\langle u, \mathfrak{B}^A_\hbar(p_j)u\right\rangle = \int_\mathcal{X}\int_\mathcal{X}\mathrm{d}x\mathrm{d}y\,\partial_{x_j}\left\{\Gamma^A[x,x-\sqrt{\hbar}y]\right\}|u(x)|^2\,|v(y)|^2 + i\sqrt{\hbar}\|u\|^2\int_\mathcal{X}\mathrm{d}y[\partial_j v](y)\overline{v(y)}$$
$$+\,i\hbar\int_\mathcal{X}\mathrm{d}x\,\overline{[\partial_j u](x)}\,u(x)\,.$$

B. Connection with the Weyl quantization

A natural question is to find the magnetic Weyl symbol of a Berezin operator. For computational reasons a change of realization is useful; we obtain it by composing the mapping $\mathfrak{B}_\hbar^A$ with a partial Fourier transformation. Using again the notation $\mathfrak{F} := 1 \otimes \mathcal{F}^*$, where $\mathcal{F}$ is the usual Fourier transform, we define $\mathfrak{D}_\hbar^A(F) := \mathfrak{B}_\hbar^A(\mathfrak{F}F)$. This makes sense for various classes of functions, but we are only going to use them for $F \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$. Here, we only record the explicit formula

$$\big[\mathfrak{D}_\hbar^A(F)u\big](x) = \hbar^{-N} \int_{\mathcal{X}} \int_{\mathcal{X}} \mathrm{d}y\, \mathrm{d}z\, F\left(z, \tfrac{y-x}{\hbar}\right) v_\hbar(x-z)\, \overline{v_\hbar(y-z)}\, e^{-\frac{i}{\hbar}\Gamma^A[x,z]}\, e^{-\frac{i}{\hbar}\Gamma^A[z,y]}\, u(y)\,. \tag{2.2}$$

It is easier to prove first the following.

Proposition 2.5: For any $F \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$, *one has* $\mathfrak{D}_\hbar^A(F) = \mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F)\big]$ *with* $\Sigma_\hbar^B(F)$ *given by*

$$\big[\Sigma_\hbar^B(F)\big](x, y) := \int_{\mathcal{X}} \mathrm{d}z\, F(x - \sqrt{\hbar} z, y)\, \overline{v\left(z + \tfrac{\sqrt{\hbar}}{2} y\right)}\, v\left(z - \tfrac{\sqrt{\hbar}}{2} y\right) e^{-\frac{i}{\hbar}\Gamma^B\langle x - \sqrt{\hbar} z, x + \frac{\hbar}{2} y, x - \frac{\hbar}{2} y\rangle}\,. \tag{2.3}$$

The mapping $\Sigma_\hbar^B$ *extends to a linear contraction of the Banach space* $L^1\big(\mathcal{X}_y; C_0(\mathcal{X}_x)\big)$.

Proof: By comparing (2.2) with (1.5) and using Stokes' theorem to write the sum of three circulations of A as the flux of $B = \mathrm{d}A$ through the corresponding triangle, one gets

$$\begin{aligned}\big[\Sigma_\hbar^B(F)\big]\left(\tfrac{x+y}{2}, \tfrac{y-x}{\hbar}\right) &= \int_{\mathcal{X}} \mathrm{d}z\, F\left(z, \tfrac{y-x}{\hbar}\right) v_\hbar(x-z)\, \overline{v_\hbar(y-z)}\, e^{-\frac{i}{\hbar}\Gamma_\hbar^A[x,z]}\, e^{-\frac{i}{\hbar}\Gamma_\hbar^A[z,y]}\, e^{-\frac{i}{\hbar}\Gamma_\hbar^A[y,x]} \\ &= \int_{\mathcal{X}} \mathrm{d}z\, F\left(z, \tfrac{y-x}{\hbar}\right) v_\hbar(x-z)\, \overline{v_\hbar(y-z)}\, e^{-\frac{i}{\hbar}\Gamma^B\langle z,y,x\rangle}\,.\end{aligned}$$

Then, some simple changes of variables lead to the above expression.

For any $x, y \in \mathcal{X}$, one clearly has

$$\Big|\big[\Sigma_\hbar^B(F)\big](x, y)\Big| \le \|F(\cdot, y)\|_\infty\, \|v\|_2^2 = \|F(\cdot, y)\|_\infty\,, \tag{2.4}$$

so $\Sigma_\hbar^B$ is a contraction if on $\mathcal{S}(\mathcal{X} \times \mathcal{X})$ we consider the norm of $L^1\big(\mathcal{X}_y; L^\infty(\mathcal{X}_x)\big)$. The function $x \mapsto \big[\Sigma_\hbar^B(F)\big](x, y)$ is continuous and vanishes as $x \to \infty$, by an easy application of the dominated convergence theorem. We conclude that $\Sigma_\hbar^B[\mathcal{S}(\mathcal{X} \times \mathcal{X})] \subset L^1\big(\mathcal{X}_y; C_0(\mathcal{X}_x)\big)$. Also using (2.4) and the density of $\mathcal{S}(\mathcal{X} \times \mathcal{X})$ in $L^1\big(\mathcal{X}_y; C_0(\mathcal{X}_x)\big)$, this completes the proof of the statement. □

Remark 2.6: With some extra work, one could show that $\Sigma_\hbar^B$ sends continuously $\mathcal{S}(\mathcal{X} \times \mathcal{X})$ into $\mathcal{S}(\mathcal{X} \times \mathcal{X})$. This relies on the assumption $v \in \mathcal{S}(\mathcal{X})$ and uses polynomial estimates on the magnetic phase factor in (2.3), which can be extracted rather easily from the fact that all the derivatives of B are bounded. We shall not use this result.

Corollary 2.7: For any $f \in \mathcal{S}(\Xi)$, one has $\mathfrak{B}_\hbar^A(f) = \mathfrak{Op}_\hbar^A\big[\mathfrak{S}_\hbar^B(f)\big]$, *with*

$$\big[\mathfrak{S}_\hbar^B(f)\big](x, \xi) := \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}z\, \mathrm{d}\zeta\, f(x - z, \xi - \zeta)\, \Upsilon_\hbar^B(x; z, \zeta)$$

and

$$\Upsilon_\hbar^B(x; z, \zeta) := (2\pi)^{-N} \int_{\mathcal{X}} \mathrm{d}y\, e^{-iy\cdot\zeta}\, \overline{v_\hbar\left(z + \tfrac{\hbar}{2} y\right)}\, v_\hbar\left(z - \tfrac{\hbar}{2} y\right) e^{-\frac{i}{\hbar}\Gamma^B\langle x - z, x + \frac{\hbar}{2} y, x - \frac{\hbar}{2} y\rangle}\,.$$

Proof: Our previous definitions imply that $\mathfrak{S}_\hbar^B(f) = \mathfrak{F}\left[\left(\Sigma_\hbar^B(\mathfrak{F}^{-1} f)\right)\right]$. The corollary follows from Proposition 3.5 by a straightforward computation. □

Remark 2.8: It is satisfactory that $\mathfrak{S}_\hbar^B(f)$ depends only on B and not on the vector potential A. If $B = 0$ (or if it is constant), then $\Upsilon_\hbar^B$ does not depend on x and the operation $\mathfrak{S}_\hbar^0$ is just a convolution, as expected.

Let us turn now to the study of the $\hbar \to 0$ behavior of the magnetic Berezin quantization. We record first the following simple consequence of (1.8) and (1.9).

Proposition 2.9: For any $X \in \Xi$ and any bounded continuous function $g : \Xi \to \mathbb{C}$, one has

$$\lim_{\hbar\to 0}\left\langle v_\hbar^A(X), \left[\mathfrak{B}_\hbar^A(g)\right] v_\hbar^A(X)\right\rangle = g(X)\ .$$

Furthermore, if $g \in \mathcal{S}(\Xi)$, then

$$\lim_{\hbar\to 0}\left\langle v_\hbar^A(X), \left[\mathfrak{Op}_\hbar^A(g) - \mathfrak{B}_\hbar^A(g)\right] v_\hbar^A(X)\right\rangle = 0.$$

Next, we would like to show that the representation $\mathfrak{Op}_\hbar^A$ and the Berezin quantization are equivalent in the limit $\hbar \to 0$, thus improving on the second statement of the previous proposition. We start with a result that will be used below and that might have some interest in its own. For that purpose, let us set $\overline{I} := \{0\} \cup I = [0, 1]$ and $\Sigma_0^B := \mathrm{id}$.

Proposition 2.10: The map $\overline{I} \ni \hbar \to \Sigma_\hbar^B \in \mathbb{B}\left[L^1\big(\mathcal{X}_y; C_0(\mathcal{X}_x)\big)\right]$ is strongly continuous.

Proof: We are going to check that for any $F \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$, one has

$$\|\Sigma_\hbar^B(F) - F\|_{1,\infty} \to 0 \quad \text{when} \quad \hbar \to 0. \tag{2.5}$$

By the density of $\mathcal{S}(\mathcal{X} \times \mathcal{X})$ in $L^1\big(\mathcal{X}_y; C_0(\mathcal{X}_x)\big)$, this will prove the continuity in $\hbar = 0$, which is the most interesting result. Continuity in other values $\hbar \in I$ is shown analogously and is left as an exercise.

Let us first observe that

$$\begin{aligned}
&\left[\Sigma_\hbar^B(F) - F\right](x, y)\\
&= \int_{\mathcal{X}} \mathrm{d}z\, F(x - \sqrt{\hbar}z, y)\overline{v\left(z + \tfrac{\sqrt{\hbar}}{2}y\right)}\, v\left(z - \tfrac{\sqrt{\hbar}}{2}y\right) e^{-\frac{i}{\hbar}\Gamma^B\langle x-\sqrt{\hbar}z, x+\frac{\hbar}{2}y, x-\frac{\hbar}{2}y\rangle} - F(x, y)\\
&= \int_{\mathcal{X}} \mathrm{d}z\Big[F(x - \sqrt{\hbar}z, y)\overline{v\left(z + \tfrac{\sqrt{\hbar}}{2}y\right)}\, v\left(z - \tfrac{\sqrt{\hbar}}{2}y\right) e^{-\frac{i}{\hbar}\Gamma^B\langle x-\sqrt{\hbar}z, x+\frac{\hbar}{2}y, x-\frac{\hbar}{2}y\rangle} - F(x, y)\overline{v(z)}\,v(z)\Big]\\
&=: \int_{\mathcal{X}} \mathrm{d}z\, J_\hbar(x, y; z)\ .
\end{aligned}$$

Furthermore, one clearly has

$$\left|\left[\Sigma_\hbar^B(F) - F\right](x, y)\right| \le 2\,\|F(\cdot, y)\|_\infty\, \|v\|_2^2 = 2\|F(\cdot, y)\|_\infty\ ,$$

with the right-hand side independent of x and which belongs to $L^1(\mathcal{X}_y)$. It then follows from the dominated convergence theorem that (2.5) holds if for each $y \in \mathcal{X}$, one has

$$\lim_{\hbar\to 0}\sup_{x\in\mathcal{X}}\left|\left[\Sigma_\hbar^B(F) - F\right](x, y)\right| = 0\ .$$

For that purpose, let $r \in \mathbb{R}_+$ and set B_r for the ball centered at $0 \in \mathcal{X}$ and of radius r and $B_r^\perp$ for the complement $\mathcal{X} \setminus B_r$. Observe then that for any fixed $y \in \mathcal{X}$ and for r large enough, one has

$$\begin{aligned}
\left|\left[\Sigma_\hbar^B(F) - F\right](x, y)\right| &\le \int_{B_r} \mathrm{d}z\, |J_\hbar(x, y; z)| + \int_{B_r^\perp} \mathrm{d}z\, |J_\hbar(x, y; z)|\\
&\le \int_{B_r} \mathrm{d}z\, |J_\hbar(x, y; z)| + \|F(\cdot, y)\|_\infty\, \|v\|_\infty \left(\|v\|_{L^1(B^\perp_{r-|y|/2})} + \|v\|_{L^1(B_r^\perp)}\right)\ .
\end{aligned} \tag{2.6}$$

Clearly, the second term of (2.6) is independent of x and $\hbar$ and can be made arbitrarily small by choosing r large enough. The hypothesis $F \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$ implies that $\int_{B_r} \mathrm{d}z\, |J_\hbar(x, y; z)|$ can also be made arbitrarily small (independently of $\hbar \in I$) by restricting x to the complement of a large compact subset of $\mathcal{X}$. Since F, v, and the magnetic phase factor are all continuous, for x and z restricted to compact subsets of $\mathcal{X}$ the integrant $J_\hbar(x, y; z)$ can be made arbitrarily small by choosing $\hbar$ small enough. Thus the first term of (2.6) also has a vanishing limit as $\hbar \to 0$. □

Corollary 2.11: For any $f \in \mathcal{S}(\Xi)$, one has

$$\lim_{\hbar\to 0} \left\| \mathfrak{B}^A_\hbar(f) - \mathfrak{Op}^A_\hbar(f) \right\| = 0 \, .$$

Proof: By using the notations above, Proposition 3.5 and the fact that $\mathfrak{F}$ is an isomorphism from $\mathcal{S}(\Xi)$ to $\mathcal{S}(\mathcal{X} \times \mathcal{X})$, one has to show for any $F \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$ that

$$\lim_{\hbar\to 0} \left\| \mathfrak{Rep}^A_\hbar \left[\Sigma^B_\hbar(F) - F \right] \right\| = 0 \, .$$

However, by (1.6), this follows from (2.5). □

C. Operators in the Bargmann representation

We now introduce the generalization to our framework of the Bargmann transform and consider the fate of the Berezin operators in the emerging realization. The proofs of the statements bellow are straightforward; most of them are not specific to our magnetic framework, see Sec. II.1.5 of Ref. 19]. In fact, the content of this section is not needed for the rest of the paper, but it opens the doors towards new perspectives or further investigations.

Definition 2.12

1. *The mapping* $\mathcal{U}^A_\hbar : L^2(\mathcal{X}) \to L^2_\hbar(\Xi) \equiv L^2\left(\Xi; \frac{\mathrm{d}X}{(2\pi\hbar)^N}\right)$ *given by* $\left(\mathcal{U}^A_\hbar u\right)(X) := \left\langle v^A_\hbar(X), u \right\rangle$ *is called the Bargmann transformation corresponding to the family of coherent vectors* $\left(v^A_\hbar(X)\right)_{X\in\Xi}$.
2. *The subspace* $\mathcal{K}^A_\hbar := \mathcal{U}^A_\hbar\left(L^2(\mathcal{X})\right) \subset L^2_\hbar(\Xi)$ *is called the magnetic Bargmann space corresponding to the family of coherent vectors* $\left(v^A_\hbar(X)\right)_{X\in\Xi}$.

First, we remark that $\mathcal{U}^A_\hbar$ is an isometry with adjoint

$$\left(\mathcal{U}^A_\hbar\right)^* : L^2_\hbar(\Xi) \to L^2(\mathcal{X}), \quad \left(\mathcal{U}^A_\hbar\right)^* \Phi := \int_\Xi \frac{\mathrm{d}X}{(2\pi\hbar)^N} \Phi(X) v^A_\hbar(X)$$

and final projection $P^A_\hbar := \mathcal{U}^A_\hbar \left(\mathcal{U}^A_\hbar\right)^* \in \mathbb{P}[L^2_\hbar(\Xi)]$, with $P^A_\hbar\left(L^2_\hbar(\Xi)\right) = \mathcal{K}^A_\hbar$. The integral kernel of this projection

$$K^A_\hbar : \Xi \times \Xi \to \mathbb{C}, \quad K^A_\hbar(Y, Z) := \left\langle v^A_\hbar(Y), v^A_\hbar(Z) \right\rangle \, ,$$

explicitly equal to

$$K^A_\hbar(Y, Z) = e^{\frac{i}{2\hbar}(y\cdot\eta - z\cdot\zeta)} \int_\mathcal{X} \mathrm{d}x \, e^{\frac{i}{\hbar} x\cdot(\zeta-\eta)} \, e^{-\frac{i}{\hbar}\Gamma^A[y,x]} \, e^{\frac{i}{\hbar}\Gamma^A[z,x]} \, \overline{v_\hbar(x-y)} \, v_\hbar(x-z) \, ,$$

is a continuous function and it is a reproducing kernel for $\mathcal{K}^A_\hbar$,

$$\Phi(Y) = \int_\Xi \frac{\mathrm{d}Z}{(2\pi\hbar)^N} K^A_\hbar(Y, Z)\Phi(Z), \quad \forall\, Y \in \Xi, \quad \forall\, \Phi \in \mathcal{K}^A_\hbar.$$

The magnetic Bargmann space is composed of continuous functions and contains all the vectors $K^A_\hbar(X, \cdot),\ X \in \Xi$. The evaluation maps $\mathcal{K}^A_\hbar \ni \Phi \to \Phi(X) \in \mathbb{C}$ are all continuous. Furthermore, the set of vectors $\Psi^A_\hbar(X) := \mathcal{U}^A_\hbar\left(v^A_\hbar(X)\right)$ with $X \in \Xi$ forms a family of coherent states in the magnetic Bargmann space.

Proposition 2.13: For $f \in L^\infty(\Xi)$, the operator

$$\mathfrak{T}_\hbar^A(f) := \mathcal{U}_\hbar^A \mathfrak{B}_\hbar^A(f)\big(\mathcal{U}_\hbar^A\big)^* \equiv \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N} f(Y)|\Psi_\hbar^A(Y)\rangle\langle\Psi_\hbar^A(Y)|$$

takes the form of a Toeplitz operator $\mathfrak{T}_\hbar^A(f) = P_\hbar^A M_f P_\hbar^A$, where M_f is the operator of multiplication by f in the Hilbert space $L^2_\hbar(\Xi)$.

Proof: Simple computation. □

Now, let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle_{(\hbar)}$ the scalar product of the space $L^2_\hbar(\Xi)$.

Definition 2.14: The magnetic covariant symbol of the operator $S \in \mathbb{B}[L^2_\hbar(\Xi)]$ is the function

$$s_\hbar^A(S) : \Xi \to \mathbb{C}, \quad \big[s_\hbar^A(S)\big](X) := \big\langle \Psi_\hbar^A(X), S\Psi_\hbar^A(X)\big\rangle_{(\hbar)}.$$

Of course, this can also be written as

$$\big[s_\hbar^A(S)\big](X) = \big\langle v_\hbar^A(X), \big(\mathcal{U}_\hbar^A\big)^* S\mathcal{U}_\hbar^A v_\hbar^A(X)\big\rangle,$$

which suggests the definition of the magnetic covariant symbol of an operator $T \in \mathbb{B}[L^2(\mathcal{X})]$ to be

$$\big[t_\hbar^A(T)\big](X) = \big\langle v_\hbar^A(X), T\, v_\hbar^A(X)\big\rangle = \big[\mathfrak{v}_\hbar^A(X)\big](T).$$

Sometimes $\mathfrak{B}_\hbar^A(f)$ and $\mathfrak{T}_\hbar^A(f)$ are called operators with *contravariant symbol f*. We avoided the Wick/anti-Wick terminology, since its full significance involving ordering is not clear here.

Remark 2.15: A straightforward calculation leads to the covariant symbol of a Toeplitz operator

$$\big(s_\hbar^A\big[\mathfrak{T}_\hbar^A(f)\big]\big)(X) = \int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N} f(Y)\, \big|\big\langle v_\hbar^A(X), v_\hbar^A(Y)\big\rangle\big|^2.$$

The relation (1.8) shows that *the magnetic Berezin transformation*, sending a continuous and bounded function f on Ξ to $s_\hbar^A\big[\mathfrak{T}_\hbar^A(f)\big]$, converges to the identity operator when $\hbar \to 0$.

IV. STRICT QUANTIZATION

This section is dedicated to a proof of the following.

Theorem 3.1: Assume that $v \in \mathcal{S}(\mathcal{X})$, that $B_{jk} \in BC^\infty(\mathcal{X})$ for $j, k \in \{1, \dots, N\}$ and that a corresponding vector potentials A with components in $C^\infty_{\mathrm{pol}}(\mathcal{X})$ has been chosen. Then, the magnetic Berezin quantization $\mathfrak{B}_\hbar^A$ is a strict quantization of the Poisson algebra $\big(\mathcal{S}(\Xi;\mathbb{R}), \cdot, \{\cdot,\cdot\}^B\big)$.

In technical terms this means that

1. for any real $f \in \mathcal{S}(\Xi)$, the map $I \ni \hbar \mapsto \big\|\mathfrak{B}_\hbar^A(f)\big\| \in \mathbb{R}_+$ is continuous and it extends continuously to $\overline{I} := [0, 1]$ if the value $\|f\|_\infty$ is assigned to $\hbar = 0$ (*Rieffel's axiom*).
2. For any $f, g \in \mathcal{S}(\Xi)$, the following property holds (*von Neumann's axiom*)

$$\lim_{\hbar\to 0} \big\|\tfrac{1}{2}\big[\mathfrak{B}_\hbar^A(f)\mathfrak{B}_\hbar^A(g) + \mathfrak{B}_\hbar^A(g)\mathfrak{B}_\hbar^A(f)\big] - \mathfrak{B}_\hbar^A(fg)\big\| = 0.$$

3. For any $f, g \in \mathcal{S}(\Xi)$, the following property holds (*Dirac's axiom*)

$$\lim_{\hbar\to 0} \big\|\tfrac{1}{i\hbar}\big[\mathfrak{B}_\hbar^A(f), \mathfrak{B}_\hbar^A(g)\big] - \mathfrak{B}_\hbar^A\big(\{f, g\}^B\big)\big\| = 0.$$

Equivalently, we intend to show that the map $\mathfrak{D}_\hbar^A$ defines a strict quantization of the Poisson algebra $\big(\mathcal{S}(\mathcal{X}\times\mathcal{X};\mathbb{R}), \diamond_0^B, \{\!\{\cdot,\cdot\}\!\}^B\big)$, where the product $\diamond_0^B$ and the Poisson bracket $\{\{\,\cdot\,,\,\cdot\,\}\}^B$ are deduced from the Poisson algebra $\big(\mathcal{S}(\Xi;\mathbb{R}), \cdot, \{\cdot,\cdot\}^B\big)$ through the partial Fourier transformation $\mathfrak{F}$. One obtains easily that

$$\big(F \diamond_0^B G\big)(x, y) = \tfrac{1}{(\sqrt{2\pi})^N} \int_{\mathcal{X}} \mathrm{d}z\, F(x, z)\, G(x, y - z).$$

Similarly, the Poisson bracket is given by

$$\{\!\{F,G\}\!\}^B = (2\pi)^{N/2}\Big[\sum_j \big[(Y_jF)\diamond_0^B \big(\tfrac{1}{i}\partial_{x_j}G\big) - \big(\tfrac{1}{i}\partial_{x_j}xF\big)\diamond_0^B (Y_jG)\big] - \sum_{j,k} B_{jk}(Y_jF)\diamond_0^B (Y_kG)\Big]$$

with $[Y_jF](x,y) = y_jF(x,y)$ and $[\partial_{x_j}F](x,y) = \frac{\partial F}{\partial x_j}(x,y)$.

Our approach relies on a similar proof[23] for the fact that $\mathfrak{Rep}_\hbar^A$ defines a strict quantization of the Poisson algebra $\big(\mathcal{S}(\mathcal{X}\times\mathcal{X};\mathbb{R}),\diamond_0^B,\{\!\{\cdot,\cdot\}\!\}^B\big)$. This and the results of Subsection III B will lead easily to the first two conditions. Dirac's axiom is more difficult to check; it relies on some detailed calculations and estimates.

A. Rieffel's condition

The most important information is of course

$$\lim_{\hbar\to 0}\|\mathfrak{B}_\hbar^A(f)\| \;=\; \|f\|_\infty .$$

This follows easily from the analog relation proved in Ref. 23 for the magnetic Weyl quantization $\mathfrak{Op}_\hbar^A$ and from Corollary 2.11. For convenience, we treat also continuity outside $\hbar = 0$.

Proposition 3.2: For any $F\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$, the map $I\ni\hbar\mapsto\big\|\mathfrak{D}_\hbar^A(F)\big\|\in\mathbb{R}_+$ is continuous.

Proof: We first recall that it has been proved in Ref. 23 that the map $I\ni\hbar\mapsto\big\|\mathfrak{Rep}_\hbar^A(F)\big\|\in\mathbb{R}_+$ is continuous for any $F\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$. Let $\hbar,\hbar'\in I$, and $F\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$. Then one has

$$\begin{aligned}\big\|\mathfrak{D}_\hbar^A(F)-\mathfrak{D}_{\hbar'}^A(F)\big\| &= \big\|\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F)\big]-\mathfrak{Rep}_{\hbar'}^A\big[\Sigma_{\hbar'}^B(F)\big]\big\| \\ &\le \big\|\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F)-\Sigma_{\hbar'}^B(F)\big]\big\| + \big\|\big(\mathfrak{Rep}_\hbar^A-\mathfrak{Rep}_{\hbar'}^A\big)\big[\Sigma_{\hbar'}^B(F)\big]\big\| . \end{aligned} \tag{3.1}$$

Since the inequality $\|\mathfrak{Rep}_\hbar^A(G)\|\le\|G\|_{1,\infty}$ always holds, the first term of (3.1) goes to 0 as $\hbar'\to\hbar$ by Proposition 3.10. The second term also vanishes as $\hbar'\to\hbar$ by the result of Ref. 23 recalled above and by a simple approximation argument. The statement then easily follows. □

B. Von Neumann's condition

One has to show that for any $F,G\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$, the following property holds

$$\lim_{\hbar\to 0}\big\|\tfrac{1}{2}\big[\mathfrak{D}_\hbar^A(F)\mathfrak{D}_\hbar^A(G)+\mathfrak{D}_\hbar^A(G)\mathfrak{D}_\hbar^A(F)\big]-\mathfrak{D}_\hbar^A(F\diamond_0^B G)\big\| = 0 .$$

In fact, since $F\diamond_0^B G = G\diamond_0^B F$, it is enough to show that

$$\lim_{\hbar\to 0}\big\|\mathfrak{D}_\hbar^A(F)\mathfrak{D}_\hbar^A(G)-\mathfrak{D}_\hbar^A(F\diamond_0^B G)\big\| = 0 .$$

By taking the previous results into account, one has

$$\begin{aligned}\big\|\mathfrak{D}_\hbar^A(F)\mathfrak{D}_\hbar^A(G)-\mathfrak{D}_\hbar^A(F\diamond_0^B G)\big\| &= \big\|\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F)\big]\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(G)\big]-\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F\diamond_0^B G)\big]\big\| \\ &= \big\|\mathfrak{Rep}_\hbar^A\big[\Sigma_\hbar^B(F)\diamond_\hbar^B\Sigma_\hbar^B(G)-\Sigma_\hbar^B(F\diamond_0^B G)\big]\big\| \\ &\le \big\|\Sigma_\hbar^B(F)\diamond_\hbar^B\Sigma_\hbar^B(G)-\Sigma_\hbar^B(F\diamond_0^B G)\big\|_{1,\infty} \\ &\le \big\|\Sigma_\hbar^B(F)-F\big\|_{1,\infty}\big\|\Sigma_\hbar^B(G)\big\|_{1,\infty}+\big\|\Sigma_\hbar^B(G)-G\big\|_{1,\infty}\|F\|_{1,\infty} \\ &\quad+\big\|\Sigma_\hbar^B\big(F\diamond_0^B G\big)-F\diamond_0^B G\big\|_{1,\infty}+\big\|F\diamond_\hbar^B G-F\diamond_0^B G\big\|_{1,\infty} .\end{aligned}$$

It has been shown in Proposition 3.10 that $\big\|\Sigma_\hbar^B(H)-H\big\|_{1,\infty}$ converges to 0 as $\hbar\to 0$ for any $H\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$. Using this and the fact that $\Sigma_\hbar^B$ is a contraction in $L^1\big(\mathcal{X}_y;C_0(\mathcal{X}_x)\big)$, it follows that the first three terms above vanish as $\hbar$ goes to 0. Finally, the convergence of $\big\|F\diamond_\hbar^B G-F\diamond_0^B G\big\|_{1,\infty}$ to 0 as $\hbar\to 0$ has been proved in Ref. 23 in a more general context.

C. Dirac's condition

One has to show that for any $F, G \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$, the following result holds

$$\lim_{\hbar \to 0} \left\| \tfrac{1}{i\hbar}\left[\mathfrak{D}^A_\hbar(F), \mathfrak{D}^A_\hbar(G)\right] - \mathfrak{D}^A_\hbar\left(\{\!\{F, G\}\!\}^B\right) \right\| = 0,$$

which is equivalent to

$$\lim_{\hbar \to 0} \left\| \tfrac{1}{i\hbar}\, \mathfrak{Rep}^A_\hbar\Big(\Sigma^B_\hbar(F) \diamond^B_\hbar \Sigma^B_\hbar(G) - \Sigma^B_\hbar(G) \diamond^B_\hbar \Sigma^B_\hbar(F)\Big) - \mathfrak{Rep}^A_\hbar\Big(\Sigma^B_\hbar\left(\{\!\{F, G\}\!\}^B\right)\right\| = 0\,.$$

By taking into account the previous results, this reduces to showing that

$$\lim_{\hbar \to 0} \left\| \tfrac{1}{i\hbar}\left(\Sigma^B_\hbar(F) \diamond^B_\hbar \Sigma^B_\hbar(G) - \Sigma^B_\hbar(G) \diamond^B_\hbar \Sigma^B_\hbar(F)\right) - \{\!\{F, G\}\!\}^B \right\|_{1,\infty} = 0\,.$$

For simplicity, let us denote by $V(a, b, c, d)$ the product $\overline{v(a)}v(b)\overline{v(c)}v(d)$ and let $\Gamma^B(a, b, c, d, e)$ be the flux of the magnetic field through the "pentagon" of vertices a, b, c, d, e. With these notations, one has

$$\begin{aligned}
&\left[\Sigma^B_\hbar(F) \diamond^B_\hbar \Sigma^B_\hbar(G)\right](x, y)\\
&= (2\pi)^{-N/2} \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}} \mathrm{d}z\, \mathrm{d}a\, \mathrm{d}b\, F\left(x - \sqrt{\hbar}a - \tfrac{\hbar}{2}(y - z), z\right) G\left(x - \sqrt{\hbar}b + \tfrac{\hbar}{2}z, y - z\right)\\
&\quad \cdot V\left(a + \tfrac{\sqrt{\hbar}}{2}z, a - \tfrac{\sqrt{\hbar}}{2}z, b + \tfrac{\sqrt{\hbar}}{2}(y - z), b - \tfrac{\sqrt{\hbar}}{2}(y - z)\right)\\
&\quad \cdot e^{-\frac{i}{\hbar}\Gamma^B(x - \frac{\hbar}{2}y, x - \sqrt{\hbar}a - \frac{\hbar}{2}(y - z), x - \frac{\hbar}{2}y + \hbar z, x - \sqrt{\hbar}b + \frac{\hbar}{2}z, x + \frac{\hbar}{2}y)}\,.
\end{aligned}$$

Then, with some simple changes of variables it follows that

$$\begin{aligned}
&\left[\Sigma^B_\hbar(F) \diamond^B_\hbar \Sigma^B_\hbar(G) - \Sigma^B_\hbar(G) \diamond^B_\hbar \Sigma^B_\hbar(F)\right](x, y)\\
&= (2\pi)^{-N/2} \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}} \mathrm{d}z\, \mathrm{d}a\, \mathrm{d}b\, V\left(a + \tfrac{\sqrt{\hbar}}{2}z, a - \tfrac{\sqrt{\hbar}}{2}z, b + \tfrac{\sqrt{\hbar}}{2}(y - z), b - \tfrac{\sqrt{\hbar}}{2}(y - z)\right)\\
&\quad \cdot \Big[F\left(x - \sqrt{\hbar}a - \tfrac{\hbar}{2}(y - z), z\right) G\left(x - \sqrt{\hbar}b + \tfrac{\hbar}{2}z, y - z\right) w^B_1(x, y, z, a, b; \hbar)\\
&\quad - F\left(x - \sqrt{\hbar}a + \tfrac{\hbar}{2}(y - z), z\right) G\left(x - \sqrt{\hbar}b - \tfrac{\hbar}{2}z, y - z\right) w^B_2(x, y, z, a, b; \hbar)\Big]
\end{aligned}$$

with

$$w^B_1(x, y, z, a, b; \hbar) := e^{-\frac{i}{\hbar}\Gamma^B(x - \frac{\hbar}{2}y, x - \sqrt{\hbar}a - \frac{\hbar}{2}(y - z), x - \frac{\hbar}{2}y + \hbar z, x - \sqrt{\hbar}b + \frac{\hbar}{2}z, x + \frac{\hbar}{2}y)}$$

and

$$w^B_2(x, y, z, a, b; \hbar) := e^{-\frac{i}{\hbar}\Gamma^B(x - \frac{\hbar}{2}y, x - \sqrt{\hbar}b - \frac{\hbar}{2}z, x + \frac{\hbar}{2}y - \hbar z, x - \sqrt{\hbar}a + \frac{\hbar}{2}(y - z), x + \frac{\hbar}{2}y)}\,.$$

By using the Taylor development for ε near 0,

$$\begin{aligned}
F(x + \varepsilon\, y, z) \;&=\; F(x, z) + \varepsilon \sum_j y_j \int_0^1 \mathrm{d}s\, [\partial^{x_j} F](x + s\, \varepsilon\, y, z)\\
&=: F(x, z) + \mathcal{L}\big(F; x, \varepsilon y, z\big)\,,
\end{aligned}$$

the term between square brackets can be rewritten as the sum of the following four terms:

$$I_1(x, y, z, a, b; \hbar) := F\left(x - \sqrt{\hbar}a, z\right) G\left(x - \sqrt{\hbar}b, y - z\right)\left[w^B_1(x, y, z, a, b; \hbar) - w^B_2(x, y, z, a, b; \hbar)\right],$$

$$\begin{aligned}
&I_2(x, y, z, a, b; \hbar) := F(x - \sqrt{\hbar}a, z)\\
&\cdot\Big[\mathcal{L}\big(G; x - \sqrt{\hbar}b, \tfrac{\hbar}{2}z, y - z\big) w^B_1(x, y, z, a, b; \hbar) - \mathcal{L}\big(G; x - \sqrt{\hbar}b, -\tfrac{\hbar}{2}z, y - z\big)\, w^B_2(x, y, z, a, b; \hbar)\Big],
\end{aligned}$$

$$
\begin{aligned}
&I_3(x,y,z,a,b;\hbar) := G(x-\sqrt{\hbar}b,\, y-z)\\
&\cdot\Big[\mathcal{L}\big(F;x-\sqrt{\hbar}a,-\tfrac{\hbar}{2}(y-z),z\big)\,w_1^B(x,y,z,a,b;\hbar)-\mathcal{L}\big(F;x-\sqrt{\hbar}a,\tfrac{\hbar}{2}(y-z),z\big)\,w_2^B(x,y,z,a,b;\hbar)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
I_4(x,y,z,a,b;\hbar) := {}&\mathcal{L}\big(F;x-\sqrt{\hbar}a,-\tfrac{\hbar}{2}(y-z),z\big)\mathcal{L}\big(G;x-\sqrt{\hbar}b,\tfrac{\hbar}{2}z,y-z\big)\,w_1^B(x,y,z,a,b;\hbar)\\
&-\mathcal{L}\big(F;x-\sqrt{\hbar}a,\tfrac{\hbar}{2}(y-z),z\big)\mathcal{L}\big(G;x-\sqrt{\hbar}b,-\tfrac{\hbar}{2}z,y-z\big)\,w_2^B(x,y,z,a,b;\hbar)\,.
\end{aligned}
$$

The term I_1 is going to be studied below. Then, observe that $I_2(x, y, z, a, b; \hbar) + I_3(x, y, z, a, b; \hbar)$ is equal to

$$
\begin{aligned}
&\tfrac{\hbar}{2}F(x-\sqrt{\hbar}a,z)\sum_j z_j\int_0^1 \mathrm{d}s\Big[[\partial_{x_j}G]\big(x-\sqrt{\hbar}b+\tfrac{\hbar}{2}sz,\,y-z\big)\,w_1^B(x,y,z,a,b;\hbar)\\
&\qquad+[\partial_{x_j}G]\big(x-\sqrt{\hbar}b-\tfrac{\hbar}{2}sz,\,y-z\big)\,w_2^B(x,y,z,a,b;\hbar)\Big]\\
&-\tfrac{\hbar}{2}G(x-\sqrt{\hbar}b,y-z)\sum_j(y_j-z_j)\int_0^1 \mathrm{d}s\Big[[\partial_{x_j}F]\big(x-\sqrt{\hbar}a-\tfrac{\hbar}{2}s(y-z),z\big)\,w_1^B(x,y,z,a,b;\hbar)\\
&\qquad+[\partial_{x_j}F]\big(x-\sqrt{\hbar}a+\tfrac{\hbar}{2}s(y-z),z\big)\,w_2^B(x,y,z,a,b;\hbar)\Big]\,.
\end{aligned}
$$

Furthermore, the term $I_4(x, y, z, a, b; \hbar)$ clearly belongs to $O(\hbar^2)$, for fixed x, y, z, a, and b. So, let us now concentrate on the main part of I_1.

Lemma 3.3: For fixed x, y, z, a, and b, one has

$$
\lim_{\hbar\to 0}\tfrac{1}{i\hbar}\big[w_1^B(x,y,z,a,b;\hbar)-w_2^B(x,y,z,a,b;\hbar)\big] = -\sum_{j,k} z_j\,(y_k-z_k)\,B_{jk}(x)\,. \tag{3.2}
$$

Proof: Since $|w_j^B| = 1$, one has $w_1^B - w_2^B = w_1^B(1-(w_1^B)^{-1}\,w_2^B)$. Furthermore, one has

$$
\begin{aligned}
&w_1^B(x,y,z,a,b;\hbar)^{-1}\,w_2^B(x,y,z,a,b;\hbar)\\
&= e^{-\frac{i}{\hbar}\Gamma^B\langle x+\frac{\hbar}{2}y,\,x-\frac{\hbar}{2}y+\hbar z,\,x-\frac{\hbar}{2}y\rangle}\, e^{-\frac{i}{\hbar}\Gamma^B\langle x-\frac{\hbar}{2}y,\,x+\frac{\hbar}{2}y-\hbar z,\,x+\frac{\hbar}{2}y\rangle}\\
&\quad\cdot e^{-\frac{i}{\hbar}\Gamma^B\langle x-\frac{\hbar}{2}y+\hbar z,\,x-\sqrt{\hbar}a-\frac{\hbar}{2}(y-z),\,x-\frac{\hbar}{2}y\rangle}\, e^{-\frac{i}{\hbar}\Gamma^B\langle x+\frac{\hbar}{2}y-\hbar z,\,x-\sqrt{\hbar}a+\frac{\hbar}{2}(y-z),\,x+\frac{\hbar}{2}y\rangle}\\
&\quad\cdot e^{-\frac{i}{\hbar}\Gamma^B\langle x+\frac{\hbar}{2}y,\,x-\sqrt{\hbar}b+\frac{\hbar}{2}z,\,x-\frac{\hbar}{2}y+\hbar z\rangle}\, e^{-\frac{i}{\hbar}\Gamma^B\langle x-\frac{\hbar}{2}y,\,x-\sqrt{\hbar}b-\frac{\hbar}{2}z,\,x+\frac{\hbar}{2}y-\hbar z\rangle}\\
&=: [L_1^B\cdot L_2^B\cdot L_3^B](x,y,z,a,b;\hbar)\,.
\end{aligned}
$$

By using the standard parametrization of the flux through triangles, one then obtains

$$
\begin{aligned}
L_1^B(x,y,z,a,b;\hbar) := \exp\Big\{&-i\hbar\sum_{j,k}(y_j-z_j)\,z_k\int_0^1\mathrm{d}\mu\int_0^1\mathrm{d}\nu\,\mu\\
&\cdot\Big[B_{jk}\big(x+\tfrac{\hbar}{2}y-\mu\hbar(y-z)-\mu\nu\hbar z\big)+B_{jk}\big(x-\tfrac{\hbar}{2}y+\mu\hbar(y-z)-\mu\nu\hbar z\big)\Big]\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
L_2^B(x,y,z,a,b;\hbar) := \exp\Big\{&i\sum_{j,k}\big(a_j+\tfrac{\sqrt{\hbar}}{2}z_j\big)\big(a_k-\tfrac{\sqrt{\hbar}}{2}z_k\big)\int_0^1\mathrm{d}\mu\int_0^1\mathrm{d}\nu\,\mu\\
&\cdot\Big[B_{jk}\big(x-\mu\sqrt{\hbar}a(1-\nu)-\tfrac{\hbar}{2}\big[y-z(2-\mu-\mu\nu)\big]\big)-B_{jk}\big(x-\mu\sqrt{\hbar}a(1-\nu)+\tfrac{\hbar}{2}\big[y-z(2-\mu-\mu\nu)\big]\big)\Big]\Big\},
\end{aligned}
$$

and

$$L_3^B(x,y,z,a,b;\hbar) := \exp\Big\{i\sum_{j,k}\big(b_j+\tfrac{\sqrt{\hbar}}{2}(y_j-z_j)\big)\big(b_k-\tfrac{\sqrt{\hbar}}{2}(y_k-z_k)\big)\int_0^1 \mathrm{d}\mu\int_0^1\mathrm{d}\nu\,\mu$$

$$\cdot\Big[B_{jk}\big(x-\mu\sqrt{\hbar}b(1-\nu)+\tfrac{\hbar}{2}\big[y-(y-z)\mu(1+\nu)\big]\big)-B_{jk}\big(x-\mu\sqrt{\hbar}b(1-\nu)-\tfrac{\hbar}{2}\big[y-(y-z)\mu(1+\nu)\big]\big)\Big]\Big\}.$$

Now, let us observe that

$$\begin{aligned}\tfrac{1}{i\hbar}[w_1^B-w_2^B] &= (w_1^B)^{-1}\tfrac{1}{i\hbar}[1-L_1^B L_2^B L_3^B]\\ &= (w_1^B)^{-1}\tfrac{1}{i\hbar}[1-L_1^B]+(w_1^B)^{-1}L_1^B\tfrac{1}{i\hbar}[1-L_2^B]+(w_1^B)^{-1}L_1^B L_2^B\tfrac{1}{i\hbar}[1-L_3^B]\,.\end{aligned}$$

By taking the limit $\hbar\to 0$ and by taking the equality $B_{jk}=-B_{kj}$ into account, the first term leads to the right-hand side of (3.2). For the other two terms, by a Taylor development of the magnetic field, one easily obtains that their limit as $\hbar\to 0$ is null. □

By adding these different results, one can now prove the following.

Proposition 3.4 (Dirac's condition): For any $F,G\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$, the following property holds

$$\lim_{\hbar\to 0}\big\|\tfrac{1}{i\hbar}\big[\mathfrak{D}_\hbar^A(F),\mathfrak{D}_\hbar^A(G)\big]-\mathfrak{D}_\hbar^A\big(\{\!\{F,G\}\!\}^B\big)\big\|=0\,.$$

Proof: By considering the results obtained above, the proof simply consists in numerous applications of the dominated convergence theorem and in various approximations as in Proposition 3.10. The normalization $\|v\|_{L^2(\mathcal{X})}=1$ should also been taken into account. □

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