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COHERENT STATES AND PURE STATE QUANTIZATION IN THE PRESENCE OF A VARIABLE MAGNETIC FIELD

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We introduce magnetic coherent states for a particle in a variable magnetic field. They provide a pure state quantization of the phase space $\mathbb{R}^{2N}$ endowed with a magnetic symplectic form.

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0. Introduction

This article is concerned with a spinless non-relativistic particle placed in a variable magnetic field. Recent publications [3, 5, 6, 13–15, 17, 20] introduced and developed a mathematical formalism for the observables naturally associated with such a system, both in a classical and in a quantum framework. We now would like to complete the picture, indicating an appropriate way to model the states of these systems.

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Classically, the magnetic field is described by a change in the geometry of the phase-space and is implemented by a modification of the standard symplectic form. Consequently, it also modifies the Poisson algebra structure of the smooth functions on phase-space, which are interpreted as classical observables, see [19, 15] for details. At the quantum level, one introduces algebras of observables defined only in terms of the magnetic field [5, 6, 14, 20]. The main new feature is a composition law on symbols defined in terms of fluxes of the magnetic field through triangles. With a proper implementation of Planck's constant $\hbar$, it has been proved in [15] that the quantum algebra of observables converges to the classical one in the sense of strict deformation quantization [11, 22–24].

To get the traditional setting involving self-adjoint operators, the quantum algebra can be represented in Hilbert spaces. This is realized by choosing any vector potential defining the magnetic field. In such a way one gets essentially a new pseudodifferential calculus [3, 4, 13, 14] which can be interpreted as a functional calculus for the family of non-commuting operators composed of positions and magnetic momenta. When no magnetic field is present, it coincides with the Weyl quantization. It has been adapted in [2] to the framework of nilpotent Lie groups. One of the main virtues of this construction is gauge-covariance: equivalent choices of vector potentials lead to unitarily equivalent representations. Both the intrinsic and the represented version admit C^*-algebraic reformulations [16, 17]. They were essential in [15] to prove the classical norm-sense limit of the quantum structure and are also useful in the spectral analysis of magnetic Schrödinger operators [18].

Now, the complete formalism involves coupling the classical and the quantum observables to states. We refer to [9–12] and to references therein for a general presentation and justification of the concept of state quantization. We simply recall that the space of pure states of both classical and quantum mechanical systems are Poisson spaces with a transition probability [11, Definition I.3.1.4]. In the magnetic case, the classical setting consists in the phase space $\Xi = \mathbb{R}^{2N}$ endowed with the magnetic symplectic form σ^B and the transition probability defined by

$$p^{\mathrm{cl}} : \Xi \times \Xi \to [0,1], \quad p^{\mathrm{cl}}(X,Y) := \delta_{XY}.$$

In the quantum case, the pure states space of $\mathcal{K}(\mathcal{H})$ (the C^*-algebra of all the compact operators in the Hilbert space $\mathcal{H}$) with the w^*-topology is homeomorphic to the projective space $\mathbb{P}(\mathcal{H})$ with its natural topology, see [11, Proposition I.2.5.2] and [11, Corollary I.2.5.3]. The latter space is also endowed with the $\hbar$-dependent Fubini–Study symplectic form $\Sigma'_\hbar$. With the interpretation of elements $\mathfrak{v}$ of $\mathbb{P}(\mathcal{H})$ as one-dimensional orthogonal projections $|v\rangle\langle v|$ defined by unit vectors $v \in \mathcal{H}$, the quantum transition probability is given by

$$p^{\mathrm{qu}} : \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0,1], \quad p^{\mathrm{qu}}(\mathfrak{u},\mathfrak{v}) := \mathrm{Tr}(|u\rangle\langle u| \cdot |v\rangle\langle v|) = |\langle u, v\rangle|^2. \tag{0.1}$$

A pure state quantization corresponds then to a family of injective embeddings $\{\mathfrak{v}_\hbar : \Xi \to \mathbb{P}(\mathcal{H})\}_{\hbar\in(0,1]}$ satisfying a certain set of axioms [11, Definition II.1.3.3]. In particular, the transition probabilities and the symplectic structures of Ξ and $\mathbb{P}(\mathcal{H})$

are respectively connected in the limit $\hbar \to 0$. In our case the embeddings $\mathfrak{v}_\hbar \equiv \mathfrak{v}_\hbar^A$ are defined in Definition 2.1 by the choice of a vector potential A generating the magnetic field. When composed with the magnetic Weyl calculus, they furnish pure states $\mathfrak{v}_\hbar^B$ on a C^*-algebra defined intrinsically by the magnetic composition law. At this level the pure states only depend on the magnetic field and not on any vector potential. In Theorem 3.2 we sum up how such a pure state quantization can be achieved in the magnetic case.

In conformity with [11, Definition II.1.5.1] this pure state quantization is coherent, i.e. it can be deduced from a family of continuous injections $v_\hbar^A : \Xi \to \mathcal{H}$. In Definition 2.2, this family is referred to as a family of magnetic coherent states. In a future article we are going to pursue this, introducing and studying the Berezin–Toeplitz type quantization associated with these coherent states as well as the corresponding Bargmann transform.

The structure of this article is the following: In the first section we make a short survey of the quantization of observables. In Sec. 2 we define coherent vectors and coherent states and study their properties, especially those connected to the limit $\hbar \to 0$. In the last section we put the results of Sec. 2 in the perspective of deformation quantization of states and observables.

1. Quantization of Observables

In this section we recall the structure of the observable algebras of a particle in a variable magnetic field, both from a classical and a quantum point of view. We follow [14, 15] which contain further details and technical developments. Our main purpose is to introduce the basic objects that will be used subsequently and to give motivations.

Let us consider the physical system consisting in a spinless particle moving in the Euclidean space $\mathcal{X} := \mathbb{R}^N$ under the influence of a magnetic field. We denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$. The duality is simply given by $\mathcal{X} \times \mathcal{X}^* \ni (x, \xi) \mapsto x \cdot \xi$. The phase space is denoted by $\Xi := T^*\mathcal{X} \equiv \mathcal{X} \times \mathcal{X}^*$; systematic notations as $X = (x, \xi)$, $Y = (y, \eta)$, $Z = (z, \zeta)$ will be used for its points. If no magnetic field is present, the standard symplectic form $\sigma(X, Y) \equiv \sigma[(x, \xi), (y, \eta)] := y \cdot \xi - x \cdot \eta$ prepares Ξ for doing classical mechanics.

The magnetic field is a closed 2-form B on $\mathcal{X}$ ($\mathrm{d}B = 0$), given by the matrix-components $B_{jk} : \mathcal{X} \to \mathbb{R}$ satisfying $B_{jk} = -B_{kj}$ for $j, k = 1, \ldots, N$. Suitable smoothness will be indicated when necessary; for the moment we simply assume that the components of the magnetic field are continuous. The effect of B is to change the geometry of the phase space by adding an extra term to the standard symplectic form: $\sigma^B := \sigma + \pi^* B$, where π^* is the pull-back associated with the cotangent bundle projection $\pi : T^*\mathcal{X} \to \mathcal{X}$. In coordinates

$$(\sigma^B)_X(Y, Z) = z \cdot \eta - y \cdot \zeta + B(x)(y, z) = \sum_j (z_j \eta_j - y_j \zeta_j) + \sum_{j,k} B_{jk}(x) y_j z_k. \quad (1.1)$$

If the components of the magnetic field are smooth, one can associate with this new symplectic form a canonical Poisson bracket acting on functions $f, g \in C^\infty(\Xi)$ by the formula:

$$\{f, g\}^B = \sum_j (\partial_{\xi_j} f \partial_{x_j} g - \partial_{\xi_j} g \partial_{x_j} f) + \sum_{j,k} B_{jk}(\cdot)\, \partial_{\xi_j} f \partial_{\xi_k} g. \tag{1.2}$$

It is a standard fact that $C^\infty(\Xi; \mathbb{R})$ endowed with $\{\cdot, \cdot\}^B$ and with the usual product of functions is *a Poisson algebra*. This means that $\{\cdot, \cdot\}^B$ is a Lie bracket, $(f, g) \mapsto fg$ is bilinear, associative and commutative and the Leibnitz rule $\{f, gh\}^B = \{f, g\}^B h + g\{f, h\}^B$ holds for all $f, g, h \in C^\infty(\Xi; \mathbb{R})$.

The point of view of deformation quantization is to suitably modify the classical structure of a Poisson algebra to get a quantum structure of observables for values of some Planck constant $\hbar \in I := (0, 1]$. Eventually, for $\hbar \mapsto 0$, the Poisson algebra will re-emerge in some sense.

The magnetic field B comes into play in defining the observables composition in terms of its fluxes through triangles. If $a, b, c \in \mathcal{X}$, then we denote by $\langle a, b, c\rangle$ the triangle in $\mathcal{X}$ of vertices a, b and c and set $\Gamma^B\langle a, b, c\rangle := \int_{\langle a,b,c\rangle} B$ for the flux of B through this triangle (invariant integration of a 2-form through a 2-simplex). With this notation and for $f, g : \Xi \to \mathbb{C}$, the formula

$$\begin{aligned}(f \sharp^B_\hbar g)(X) \\ := (\pi\hbar)^{-2N} \int_\Xi \int_\Xi \mathrm{d}Y \mathrm{d}Z e^{-\frac{2i}{\hbar}\sigma(X-Y, X-Z)} e^{-\frac{i}{\hbar}\Gamma^B\langle x-y+z, y-z+x, z-x+y\rangle} f(Y) g(Z)\end{aligned} \tag{1.3}$$

defines a formal associative composition law on functions. For $B = 0$ it coincides with the Weyl composition of symbols in pseudodifferential theory.

The formula (1.3) makes sense and has nice properties under various circumstances. For example, if the components B_{jk} belong to $C^\infty_{\mathrm{pol}}(\mathcal{X})$, the class of smooth functions on $\mathcal{X}$ with polynomial bounds for all the derivatives, then the Schwartz space $\mathcal{S}(\Xi)$ is stable under $\sharp^B_\hbar$. Denoting by $\mathcal{M}^B_\hbar(\Xi)$ the largest space of tempered distributions for which

$$\sharp^B_\hbar : \mathcal{S}(\Xi) \times \mathcal{M}^B_\hbar(\Xi) \to \mathcal{S}(\Xi) \quad \text{and} \quad \sharp^B_\hbar : \mathcal{M}^B_\hbar(\Xi) \times \mathcal{S}(\Xi) \to \mathcal{S}(\Xi),$$

it has been shown in [14] that $\mathcal{M}^B_\hbar(\Xi)$ is a *-algebra under $\sharp^B_\hbar$ and under complex conjugation. This is a large class of distributions, containing all the bounded measures as well as the class $C^\infty_{\mathrm{pol,u}}(\Xi)$ of all smooth functions for which all the derivatives are bounded by some polynomial (depending on the function, but not on the order of the derivative). In addition, if we assume that all the derivatives of the functions B_{jk} are bounded, then the Hörmander classes of symbols $S^m_{\rho,\delta}(\Xi)$ compose in the usual way under $\sharp^B_\hbar$.

Being a closed 2-form in $\mathcal{X} = \mathbb{R}^N$, the magnetic field is exact: it can be written as $B = \mathrm{d}A$ for some 1-form A (called *vector potential*). It is easy to see that if B is of class $C^\infty_{\mathrm{pol}}(\mathcal{X})$, then A can be chosen in the same class and we shall assume this

in the sequel. The vector potentials enter into the construction by their circulations $\Gamma^A[x,y] := \int_{[x,y]} A$ along segments $[x,y] := \{tx+(1-t)y \,|\, t \in [0,1]\}$ for any $x, y \in \mathcal{X}$. For a vector potential A with $\mathrm{d}A = B$ and for $u : \mathcal{X} \to \mathbb{C}$, let us define

$$[\mathfrak{Op}_\hbar^A(f)u](x) := (2\pi)^{-N} \int_\Xi \mathrm{d}Y [\mathfrak{F}_\Xi f](Y)[W_\hbar^A(Y)u](x) \tag{1.4}$$

$$= (2\pi\hbar)^{-N} \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}y\mathrm{d}\eta e^{\frac{i}{\hbar}(x-y)\cdot\eta} e^{-\frac{i}{\hbar}\Gamma^A[x,y]} f\left(\frac{x+y}{2}, \eta\right) u(y) \tag{1.5}$$

where $\mathfrak{F}_\Xi f$ is *the symplectic Fourier transform of* f and $W_\hbar^A$ is the *magnetic Weyl system* acting on $u \in \mathcal{H}$ and for $x \in \mathcal{X}$ as

$$[W_\hbar^A(Y)u](x) = e^{-i(x+\frac{\hbar}{2}y)\cdot\eta} e^{-\frac{i}{\hbar}\Gamma^A[x,x+\hbar y]} u(x+\hbar y).$$

For $A = 0$ one recognizes the Weyl quantization, associating with functions or distributions on Ξ linear operators acting on function spaces on $\mathcal{X}$.

It has been shown that $\mathfrak{Op}_\hbar^A$, suitably interpreted (by using rather simple duality arguments) defines a representation of the *-algebra $\mathcal{M}_\hbar^B(\Xi)$ by linear continuous operators: $\mathcal{S}(\mathcal{X}) \to \mathcal{S}(\mathcal{X})$. This means, of course, that

$$\mathfrak{Op}_\hbar^A(f \sharp_\hbar^B g) = \mathfrak{Op}_\hbar^A(f)\mathfrak{Op}_\hbar^A(g) \quad \text{and} \quad \mathfrak{Op}_\hbar^A(\overline{f}) = \mathfrak{Op}_\hbar^A(f)^*$$

for any $f, g \in \mathcal{M}_\hbar^B(\Xi)$. In addition, $\mathfrak{Op}_\hbar^A$ restricts to an isomorphism from $\mathcal{S}(\Xi)$ to $\mathcal{B}[\mathcal{S}^*(\mathcal{X}), \mathcal{S}(\mathcal{X})]$ and extends to an isomorphism from $\mathcal{S}^*(\Xi)$ to $\mathcal{B}[\mathcal{S}(\mathcal{X}), \mathcal{S}^*(\mathcal{X})]$ (we set $\mathcal{B}(\mathcal{R}, \mathcal{T})$ for the family of all linear continuous operators between the topological vector spaces $\mathcal{R}$ and $\mathcal{T}$). The class of Hilbert-Schmidt operators in the Hilbert space $\mathcal{H} := L^2(\mathcal{X})$ coincides with the class of operators of the form $\mathfrak{Op}_\hbar^A(f)$ with $f \in L^2(\Xi)$.

An important property of (1.4) is *gauge covariance*. If $A' = A + \mathrm{d}\rho$ for a smooth function $\rho : \mathcal{X} \to \mathbb{R}$ (the equivalent choices A and A' would give the same magnetic field), then $\mathfrak{Op}_\hbar^{A'}(f) = e^{\frac{i}{\hbar}\rho}\mathfrak{Op}_\hbar^A(f)e^{-\frac{i}{\hbar}\rho}$. Such a unitary equivalence would not hold for the wrong quantization appearing in the literature

$$[\mathcal{O}p_{A,\hbar}(f)u](x) := (2\pi\hbar)^{-N} \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}y\mathrm{d}\eta e^{\frac{i}{\hbar}(x-y)\cdot\eta} f\left(\frac{x+y}{2}, \eta - A\left(\frac{x+y}{2}\right)\right) u(y).$$

The operator norm $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$ being relevant in Quantum Mechanics, we pull it back by setting

$$\|\cdot\|_\hbar^B : \mathcal{S}(\Xi) \to \mathbb{R}_+ \quad \text{with} \quad \|f\|_\hbar^B := \|\mathfrak{Op}_\hbar^A(f)\|.$$

By gauge covariance, it is clear that $\|\cdot\|_\hbar^B$ only depends on the magnetic field B and not on the vector potential A. We denote by $\mathfrak{A}_\hbar^B$ the completion of $\mathcal{S}(\Xi)$ under $\|\cdot\|_\hbar^B$. It is a C^*-algebra that can be identified with a vector subspace of $\mathcal{S}^*(\Xi)$ and $\mathfrak{Op}_\hbar^A : \mathfrak{A}_\hbar^B \to \mathcal{B}(\mathcal{H})$ is a faithful *-representation, with $\mathfrak{Op}_\hbar^A[\mathfrak{A}_\hbar^B] = \mathcal{K}(\mathcal{H})$, the C^*-algebra of compact operators in $\mathcal{H}$. Let us mention that another approach to introduce norms would be by using twisted C^*-dynamical systems and twisted crossed products, as in [15–17]. To summarize, for any $\hbar \in (0, 1]$ we have defined a

C^*-algebra $\mathfrak{A}_\hbar^B$ embedded into $\mathcal{S}^*(\Xi)$ and isomorphic by $\mathfrak{Op}_\hbar^A$ to $\mathcal{K}(\mathcal{H})$. The product is essentially given by (1.3).

2. Magnetic Coherent States and Pure States

Let us fix a unit vector $v \in \mathcal{H} := L^2(\mathcal{X})$, and for any $\hbar \in I := (0,1]$ we define the unit vector $v_\hbar \in \mathcal{H}$ by $v_\hbar(x) := \hbar^{-\frac{N}{4}} v(\frac{x}{\sqrt{\hbar}})$. Using the non-magnetic Weyl system, for every $Y \in \Xi$ we set $v_\hbar(Y) := W_\hbar(-\frac{Y}{\hbar})v_\hbar$. For any choice of a continuous vector potential A generating the magnetic field B, we then define

$$v_\hbar^A(Z) := e^{\frac{i}{\hbar}\Gamma^A[z,Q]} v_\hbar(Z) = e^{\frac{i}{\hbar}\Gamma^A[z,Q]} W_\hbar\left(-\frac{Z}{\hbar}\right) v_\hbar,$$

where $Q = (Q_1, \ldots, Q_N)$ and Q_j is the operator of multiplication by the coordinate function x_j. Explicitly, this means

$$[v_\hbar^A(Z)](x) = e^{\frac{i}{\hbar}(x-\frac{z}{2})\cdot\zeta} e^{\frac{i}{\hbar}\Gamma^A[z,x]} v_\hbar(x-z). \tag{2.1}$$

The pure state space of $\mathcal{K}(\mathcal{H})$ can be identified with the projective space $\mathbb{P}(\mathcal{H})$; considering the isomorphism $\mathfrak{Op}_\hbar^A : \mathfrak{A}_\hbar^B \to \mathcal{K}(\mathcal{H})$, it is natural to introduce the following families of pure states on the two C^*-algebras:

Definition 2.1. For any $Z \in \Xi$ we define $\mathfrak{v}_\hbar^A(Z) : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ by

$$[\mathfrak{v}_\hbar^A(Z)](S) := \mathrm{Tr}(|v_\hbar^A(Z)\rangle\langle v_\hbar^A(Z)|S) \equiv \langle v_\hbar^A(Z), S v_\hbar^A(Z)\rangle,$$

for any $S \in \mathcal{K}(\mathcal{H})$, and $\mathfrak{v}_\hbar^B(Z) : \mathfrak{A}_\hbar^B \to \mathbb{C}$ by

$$[\mathfrak{v}_\hbar^B(Z)](f) := [\mathfrak{v}_\hbar^A(Z)](\mathfrak{Op}_\hbar^A(f)) = \langle v_\hbar^A(Z), \mathfrak{Op}_\hbar^A(f) v_\hbar^A(Z)\rangle$$

for any $f \in \mathfrak{A}_\hbar^B$.

The intrinsic notation $\mathfrak{v}_\hbar^B(Z)$ is justified by a straightforward computation leading for $Z = (z, \zeta)$ to

$$\begin{aligned}
[\mathfrak{v}_\hbar^B(Z)](f) &= (2\pi\hbar)^{-N} \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}^*} \mathrm{d}x\mathrm{d}y\mathrm{d}\eta\, e^{\frac{i}{\hbar}(x-y)\cdot(\eta-\zeta)} f\left(\frac{x+y}{2}, \eta\right) \\
&\quad \cdot e^{-\frac{i}{\hbar}\Gamma^B\langle z,x,y\rangle} \overline{v_\hbar(x-z)} v_\hbar(y-z) \\
&= (2\pi\hbar)^{-N} \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}^*} \mathrm{d}x\mathrm{d}y\mathrm{d}\eta\, e^{\frac{i}{\hbar}(x-y)\cdot\eta} f\left(z+\frac{x+y}{2}, \zeta+\eta\right) \\
&\quad \cdot e^{-\frac{i}{\hbar}\Gamma^B\langle z,x+z,y+z\rangle} \overline{v_\hbar(x)} v_\hbar(y) \\
&= (2\pi)^{-N} \int_{\mathcal{X}}\int_{\mathcal{X}}\int_{\mathcal{X}^*} \mathrm{d}x\mathrm{d}y\mathrm{d}\eta\, e^{i(x-y)\cdot\eta} f\left(z+\frac{\sqrt{\hbar}}{2}(x+y), \zeta+\sqrt{\hbar}\eta\right) \\
&\quad \cdot e^{-\frac{i}{\hbar}\Gamma^B\langle z,z+\sqrt{\hbar}x,z+\sqrt{\hbar}y\rangle} \overline{v(x)} v(y).
\end{aligned} \tag{2.2}$$

Definition 2.2. The family $\{v_\hbar^A(Z) \,|\, Z \in \Xi, \hbar \in I\}$ given by (2.1) is called *the family of magnetic coherent vectors associated with the pair* (A, v). The elements

of the families $\{\mathfrak{v}_\hbar^A(Z)\,|\,Z\in\Xi,\hbar\in I\}$ and $\{\mathfrak{v}_\hbar^B(Z)\,|\,Z\in\Xi,\hbar\in I\}$ will be called *the coherent states.*

Remark 2.3. Our strategy for defining coherent states is quite remote of the standard one, consisting in propagating a given state along the orbit of a (projective) representation, cf. for example [21]. Our main concern was to define a A-independent family of states on the intrinsic C^*-algebras $\mathfrak{A}_\hbar^B$ and to end up with objects which converge to their classical magnetic analogues in the limit $\hbar\to 0$. It is not clear to us how to put our approach in the perspective of covariant quantization of phase space [1, 12] or to compare our coherent states with the ones proposed in [8] in the special case of a constant magnetic field.

Remark 2.4. Note, however, that for the standard Gaussian $v(x)=\pi^{-\frac{N}{4}}e^{-\frac{x^2}{2}}$ and for $A=0$ one gets the usual coherent states of Quantum Mechanics (see for example [1]). We insist on the fact that the state corresponding to $Z=0$ is built upon an arbitrary unit vector of $\mathcal{H}$. The standard Gaussian choice (generating a holomorphic setting in the absence of a magnetic field) has no relevance at this stage. For part of our results, however, some smoothness or decay properties will be needed.

The first result, basic to any theory involving coherent states, says that

$$\int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}|v_\hbar^A(Y)\rangle\langle v_\hbar^A(Y)|=1.$$

For convenience, we shall sometimes use the shorter notation $\gamma_\hbar^A(x,y)$ for $e^{-\frac{i}{\hbar}\Gamma^A[x,y]}$.

Proposition 2.5. *Assume that the magnetic field B is continuous and let v be a unit vector in $\mathcal{H}$. For any $\hbar\in I$ and $u\in\mathcal{H}$ with $\|u\|=1$, one has*

$$\int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}|\langle v_\hbar^A(Y),u\rangle|^2=1. \tag{2.3}$$

Proof. One has to show that

$$\int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}|\langle v_\hbar^A(Y),u\rangle|^2=\|v_\hbar^A\|^2\|u\|^2,$$

which follows if the mapping

$$\mathcal{H}\otimes\mathcal{H}\ni v\otimes u\mapsto\langle e^{\frac{i}{\hbar}\Gamma^A[\cdot,Q]}W_\hbar\left(-\frac{\cdot}{\hbar}\right)v,u\rangle\in L^2\left(\Xi;\frac{\mathrm{d}Y}{(2\pi\hbar)^N}\right) \tag{2.4}$$

is proved to be isometric.

By using (2.1) and after a simple change of variables, one gets:

$$\begin{aligned}\left\langle\bar\gamma_\hbar^A(y,Q)W_\hbar\left(-\frac{Y}{\hbar}\right)v,u\right\rangle&=\int_{\mathcal{X}}\mathrm{d}x\,e^{-\frac{i}{\hbar}x\cdot\eta}\gamma_\hbar^A\left(y,x+\frac{y}{2}\right)u\left(x+\frac{y}{2}\right)\overline{v}\left(x-\frac{y}{2}\right)\\&=(2\pi)^{N/2}[(1\otimes\mathcal{F})\circ C][\beta_\hbar^A\cdot(u\otimes\overline{v})]\Big(y,\frac{\eta}{\hbar}\Big),\end{aligned}$$

where $1 \otimes \mathcal{F}$ is a unitary partial Fourier transformation, C is the operator "change of variables"

$$(CF)(y,x) := F\Big(x + \frac{y}{2},\, x - \frac{y}{2}\Big),$$

which is also unitary in $L^2(\mathcal{X}\times\mathcal{X}) \simeq \mathcal{H}\otimes\mathcal{H}$ and $\beta^A_\hbar(x,y) = \gamma^A_\hbar(x-y,x)$. From this, and since $|\beta^A_\hbar(x,y)| = 1$ for all $x, y \in \mathcal{X}$, the isometry of (2.4) follows immediately. □

Proposition 2.6. *Assume that the magnetic field B is continuous and let v be a unit vector in $\mathcal{H}$. For any Y, $Z \in \Xi$, one has*

$$\lim_{\hbar\to 0} |\langle v^A_\hbar(Z), v^A_\hbar(Y)\rangle|^2 = \delta_{ZY}.$$

Proof. Since the case $Z = Y$ is trivial, we can assume that $Z \neq Y$. A short computation shows that the expression $\langle v^A_\hbar(Z), v^A_\hbar(Y)\rangle$ is equal to

$$e^{\frac{i}{2\hbar}(z\cdot\zeta - y\cdot\eta)}\gamma^A_\hbar(z,y)\int_{\mathcal{X}} \mathrm{d}x e^{\frac{i}{\hbar}x\cdot(\eta-\zeta)} e^{-\frac{i}{\hbar}\Gamma^B\langle x,y,z\rangle}\hbar^{-\frac{N}{2}}\overline{v\left(\frac{x-z}{\sqrt{\hbar}}\right)}v\left(\frac{x-y}{\sqrt{\hbar}}\right).$$

After the change of variables $\frac{x-z}{\sqrt{\hbar}} = t$, one gets

$$\big|\langle v^A_\hbar(Z), v^A_\hbar(Y)\rangle\big| = \left|\int_{\mathcal{X}} \mathrm{d}t e^{it\cdot\frac{\eta-\zeta}{\sqrt{\hbar}}} e^{-\frac{i}{\hbar}\Gamma^B\langle z+\sqrt{\hbar}t,y,z\rangle}\overline{v(t)}v\left(t+\frac{z-y}{\sqrt{\hbar}}\right)\right|. \tag{2.5}$$

Now, if $z \neq y$, the R.H.S. of (2.5) is dominated by $\int_{\mathcal{X}} \mathrm{d}t|v(t)||v(t+\frac{z-y}{\sqrt{\hbar}})|$. It is easily shown that this one converges to 0 as $\hbar \to 0$ by a simple approximation argument using functions with compact support. On the other hand, if $z = y$ but $\eta \neq \zeta$, then the R.H.S. of (2.5) is equal to

$$\left|\int_{\mathcal{X}} \mathrm{d}t e^{it\cdot\frac{\eta-\zeta}{\sqrt{\hbar}}}|v(t)|^2\right| = (2\pi)^{N/2}\left|(\mathcal{F}^*|v|^2)\left(\frac{\eta-\zeta}{\sqrt{\hbar}}\right)\right|$$

which converges to 0 as $\hbar \to 0$ by the Riemann–Lebesgue lemma. □

The next property of the magnetic coherent states is an important one, having consequences on the behavior of the magnetic Berezin quantization for $\hbar \to 0$. Unfortunately, some extra assumptions on the magnetic field B and on the state v will be needed. On the other hand, notice that the result is valid for any bounded continuous function g; no compact support assumption is required (see [11, Sec. II.1.3]).

Proposition 2.7. *Let $B_{jk} \in BC^\infty(\mathcal{X})$ for $j,k \in \{1,\dots,N\}$ and assume that $v \in \mathcal{S}(\mathcal{X})$. For any $g : \Xi \to \mathbb{C}$ bounded continuous function and any $Z \in \Xi$ one has*

$$\lim_{\hbar\to 0}\int_\Xi \frac{\mathrm{d}Y}{(2\pi\hbar)^N}|\langle v^A_\hbar(Z), v^A_\hbar(Y)\rangle|^2 g(Y) = g(Z).$$

Proof. By taking (2.3) into account one has to show that

$$\int_{\Xi} \frac{\mathrm{d}Y}{(2\pi\hbar)^N} |\langle v_\hbar^A(Z), v_\hbar^A(Y)\rangle|^2 [g(Y) - g(Z)] \tag{2.6}$$

converges to 0 for $\hbar \to 0$. By using the expression already obtained in (2.5) together with the change of variables $x = \frac{(z-y)}{\sqrt{\hbar}}$ and $\xi = \frac{(\eta-\zeta)}{\sqrt{\hbar}}$, one gets that (2.6) is equal to

$$(2\pi)^{-N} \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}x\mathrm{d}\xi [g(z - \sqrt{\hbar}x, \zeta + \sqrt{\hbar}\xi) - g(z,\zeta)] |F_{\hbar,z}(x,\xi)|^2, \tag{2.7}$$

with

$$F_{\hbar,z}(x,\xi) := \int_{\mathcal{X}} \mathrm{d}t e^{it\cdot\xi} e^{-\frac{i}{\hbar}\Gamma^B\langle z, z+\sqrt{\hbar}t, z-\sqrt{\hbar}x\rangle} \overline{v(t)} v(t+x).$$

We are now going to apply the Dominated Convergence theorem to show that (2.7) converges to 0 when $\hbar \to 0$. First, since g is continuous one has for any fixed $(x,\xi) \in \Xi$ that

$$\lim_{\hbar\to 0} [g(z - \sqrt{\hbar}x, \zeta + \sqrt{\hbar}\xi) - g(z,\zeta)] = 0.$$

In addition, for any $\hbar \in I$ and any $(x,\xi) \in \Xi$ one has $|F_{\hbar,z}(x,\xi)|^2 \leq \|v\|^4$, so the integrand of (2.7) converges to 0.

Since g is also bounded, it will be enough to find functions $a \in L^1(\mathcal{X}; \mathbb{R}_+)$, $b \in L^1(\mathcal{X}^*; \mathbb{R}_+)$ such that for all $\hbar, x$ and ξ

$$|F_{\hbar,z}(x,\xi)| \leq a(x) \quad \text{and} \quad |F_{\hbar,z}(x,\xi)| \leq b(\xi). \tag{2.8}$$

The first estimation is simple. For arbitrary positive numbers n, m one has

$$\begin{aligned} |F_{\hbar,z}(x,\xi)| &\leq \int_{\mathcal{X}} \mathrm{d}t |v(t)||v(t+x)| \leq C \int_{\mathcal{X}} \mathrm{d}t \langle t\rangle^{-m} \langle t+x\rangle^{-n} \\ &\leq C' \left[\int_{\mathcal{X}} \mathrm{d}t \langle t\rangle^{-m} \langle t\rangle^{n}\right] \langle x\rangle^{-n}, \end{aligned}$$

and it is then enough to choose $n > N$ and $m > N + n$.

The second inequality in (2.8) is more involved. Relying on a basic property of Fourier transformation, it consists essentially in showing that the function

$$w_\hbar(t; z, x) := e^{-\frac{i}{\hbar}\Gamma^B\langle z, z+\sqrt{\hbar}t, z-\sqrt{\hbar}x\rangle} \overline{v(t)} v(t+x)$$

belongs to the Schwartz's space with respect to t, uniformly with respect to the parameters $z, x, \hbar$. First of all, since $v \in \mathcal{S}(\mathcal{X})$, for any $\alpha, \beta \in \mathbb{N}^N$ and any $k, l \in \mathbb{N}$ one has

$$|\overline{(\partial^\alpha v)(t)} (\partial^\beta v)(t+x)| \leq C_{k,l}^{\alpha,\beta} \langle t\rangle^{-k} \langle x\rangle^{-l}.$$

Thus it will be enough to show that the map $e^{-\frac{i}{\hbar}\Gamma^B\langle z, z+\sqrt{\hbar}\cdot, z-\sqrt{\hbar}x\rangle} : \mathcal{X} \to \mathbb{C}$ is in $C^\infty_{\mathrm{pol}}(\mathcal{X})$, and that the polynomial bounds on each of its derivatives are uniform

in $\hbar \in I$ and $z \in \mathcal{X}$, with growth at most polynomial in $x \in \mathcal{X}$. For this, we need to recall the natural parametrizations of the flux of the magnetic field B:

$$\Gamma^B\langle a,b,c\rangle = \sum_{j,k}(b_j-a_j)(c_k-b_k)\int_0^1 \mathrm{d}\mu \int_0^1 \mathrm{d}\nu\mu B_{jk}(a+\mu(b-a)+\mu\nu(c-b)).$$

By using this parametrization, one obtains

$$e^{-\frac{i}{\hbar}\Gamma^B\langle z,z+\sqrt{\hbar}t,z-\sqrt{\hbar}x\rangle} = \exp\left\{ i\sum_{j,k} t_j(x_k+t_k)\int_0^1 \mathrm{d}\mu\int_0^1 \mathrm{d}\nu\mu B_{jk}(z+\mu\sqrt{\hbar}t-\mu\nu\sqrt{\hbar}(x+t))\right\}.$$

The needed estimates follow then quite straightforwardly from this representation and from the assumption that all the derivatives of the magnetic field are bounded. □

We now take into consideration the maps $v_\hbar^A : (\Xi,\sigma^B) \to (\mathcal{H},\Sigma_\hbar)$ and $\mathfrak{v}_\hbar^A : (\Xi,\sigma^B) \to (\mathbb{P}(\mathcal{H}),\Sigma'_\hbar)$ between symplectic manifolds. We refer to [11, Sec. I.2.5] for a detailed presentation of the symplectic structures of $\mathcal{H}$ and of the projective space $\mathbb{P}(\mathcal{H})$ and simply recall some key elements. Note that our convention differs from that reference by a minus sign.

On the Hilbert space $\mathcal{H}$, the (constant) symplectic form is defined at the point $w \in \mathcal{H}$ by

$$\Sigma_{\hbar,w}(u,v) := -2\hbar\,\mathsf{Im}\langle u,v\rangle$$

for each $u,v \in \mathcal{H}$ and $\hbar \in I$. For the space $\mathbb{P}(\mathcal{H})$, recall first that each of its elements can be identified with the one-dimensional orthogonal projections $\mathfrak{v} = |v\rangle\langle v|$ defined by some unit vector $v \in \mathcal{H}$, with the known phase ambiguity. Then, the Fubini–Study symplectic form $\Sigma'_\hbar$ is explicitly given at the point $\mathfrak{v} \in \mathbb{P}(\mathcal{H})$ by

$$\Sigma'_{\hbar,\mathfrak{v}}(iSv,iTv) = i\hbar\mathfrak{v}([S,T]) = i\hbar\langle v,[S,T]v\rangle$$

for any self-adjoint element S,T of $\mathcal{B}(\mathcal{H})$. This relies among others on identifying the tangent space $T_\mathfrak{v}\mathbb{P}(\mathcal{H})$ to a quotient (depending on v) of the real vector space $\{iSv \mid S = S^* \in \mathcal{B}(\mathcal{H})\}$ of Hermitian bounded operators.

We would like now to show that the pull-back by $\mathfrak{v}_\hbar^A$ of the form $\Sigma'_\hbar$ converges to σ^B when $\hbar \to 0$. But $\Sigma_\hbar$ is already the pull-back of $\Sigma'_\hbar$ by the canonical map $\mathcal{H} \to \mathbb{P}(\mathcal{H})$, so we only need to show the next result:

Proposition 2.8. *Let $B_{jk} \in BC^\infty(\mathcal{X})$ for $j,k \in \{1,\dots,N\}$ and assume that $v \in \mathcal{S}(\mathcal{X})$. The pull-back by $v_\hbar^A$ of $\Sigma_\hbar$ converges to σ^B in the limit $\hbar \to 0$.*

Proof. One has to calculate, for any $X,Y,Z \in \Xi$, the expression

$$\Sigma_{\hbar,v_\hbar^A(X)}(T[v_\hbar^A(X)](Y),T[v_\hbar^A(X)](Z)) = -2\hbar\,\mathsf{Im}\langle T[v_\hbar^A(X)](Y),T[v_\hbar^A(X)](Z)\rangle$$

where T denotes the tangent map (total derivative). For that purpose, let us recall that

$$v_\hbar^A(X) = e^{\frac{i}{\hbar}\Gamma^A[x,Q]} W_\hbar \left(-\frac{X}{\hbar}\right) v_\hbar = e^{\frac{i}{\hbar}(Q-\frac{x}{2})\cdot\xi} e^{\frac{i}{\hbar}\Gamma^A[x,Q]} e^{-ix\cdot D} v_\hbar,$$

with $D = (D_1, \ldots, D_N)$ and $D_j := -i\partial_j$. It then follows that

$$T[v_\hbar^A(X)](Y) = ie^{\frac{i}{\hbar}(Q-\frac{x}{2})\cdot\xi} e^{\frac{i}{\hbar}\Gamma^A[x,Q]} e^{-ix\cdot D} M_\hbar(Y;Q,D) v_\hbar$$

with (obvious formal notations)

$$M_\hbar(Y;Q,D) := \frac{1}{\hbar}\left[-\frac{y}{2}\cdot\xi + \left(Q+\frac{x}{2}\right)\cdot\eta + \left.\frac{\partial\Gamma^A[x,q]}{\partial x}\right|_{q=Q+x}(Y)\right] - y\cdot D$$

and with

$$\left.\frac{\partial\Gamma^A[x,q]}{\partial x}\right|_{q=Q+x}(Y) = -y\cdot\int_0^1 \mathrm{d}s A(x+sQ) + \sum_{j,k} Q_j y_k \int_0^1 \mathrm{d}s(1-s)\partial_k A_j(x+sQ).$$

A similar expression holds for $T[v_\hbar^A(X)](Z)$. Thus, one has to calculate

$$\begin{aligned}
&-2\hbar\,\mathsf{Im}\langle T[v_\hbar^A(X)](Y), T[v_\hbar^A(X)](Z)\rangle \\
&\quad = -2\hbar\,\mathsf{Im}\langle M_\hbar(Y;Q,D)v_\hbar, M_\hbar(Z;Q,D)v_\hbar\rangle \\
&\quad = -2\hbar\,\mathsf{Im}\left\langle M_\hbar\left(Y;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right)v, M_\hbar\left(Z;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right)v\right\rangle \\
&\quad = i\hbar\left\langle v, \left[M_\hbar\left(Y;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right), M_\hbar\left(Z;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right)\right]v\right\rangle. \qquad (2.9)
\end{aligned}$$

A rather lengthy calculation then gives

$$\begin{aligned}
&\left[M_\hbar\left(Y;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right), M_\hbar\left(Z;\sqrt{\hbar}Q,\frac{1}{\sqrt{\hbar}}D\right)\right] \\
&\quad = -\frac{i}{\hbar}\sigma(Y,Z) - \frac{i}{\hbar}\sum_{j,k} y_j z_k \int_0^1 \mathrm{d}s s B_{jk}(x+s\sqrt{\hbar}Q) \\
&\qquad - \frac{i}{\hbar}\sum_{j,k} y_j z_k \int_0^1 \mathrm{d}s(1-s) B_{jk}(x+s\sqrt{\hbar}Q) \\
&\qquad + \frac{i}{\hbar}\sqrt{\hbar}\sum_{j,k,\ell}(y_\ell z_k - y_k z_\ell) Q_j \int_0^1 \mathrm{d}s s(1-s)\partial_{\ell k}^2 A_j(x+s\sqrt{\hbar}Q).
\end{aligned}$$

Finally, by inserting these expressions into (2.9) and by an application of the Dominated Convergence theorem, one obtains the statement of the proposition. □

The following result can be interpreted as the convergence of the quantum pure state $\mathfrak{v}_\hbar^B(Z)$ to a corresponding classical pure state in the semiclassical limit.

For that purpose, we shall consider functions $g : \overline{I} \times \Xi$ and write $g_\hbar(X)$ for $g(\hbar, X)$. We assume that $\hbar \mapsto g_\hbar(X)$ is continuous for any $X \in \Xi$ and that $g_\hbar \in \mathcal{S}(\Xi)$ for all $\hbar$.

Proposition 2.9. *Let $B_{jk} \in BC^\infty(\mathcal{X})$ for $j, k \in \{1, \ldots, N\}$ and assume that $v \in \mathcal{S}(\mathcal{X})$. Then for any g as above and any $Z \in \Xi$ one has*

$$\lim_{\hbar \to 0} [\mathfrak{v}_\hbar^B(Z)](g_\hbar) = \delta_Z(g_0) = g_0(Z). \tag{2.10}$$

Proof. Let $\hbar \in I$. Starting from the last expression obtained in (2.2) and performing the change of variables $y' = x - y$, one finds that $\left[\mathfrak{v}_\hbar^B(Z)\right](g_\hbar)$ is equal to

$$\begin{aligned} &\frac{1}{(2\pi)^N} \int_{\mathcal{X}} \int_{\mathcal{X}} \int_{\mathcal{X}^*} \mathrm{d}x \mathrm{d}y \mathrm{d}\eta \, e^{iy\cdot\eta} g_\hbar \left(z + \sqrt{\hbar} \left(\frac{x-y}{2} \right), \zeta + \sqrt{\hbar}\eta \right) \\ &\qquad \times \varphi_\hbar(z; x, y) v(x-y) \overline{v(x)} \end{aligned} \tag{2.11}$$

with $\varphi_\hbar(z; x, y) := e^{-\frac{i}{\hbar}\Gamma^B\langle z, z+\sqrt{\hbar}x, z+\sqrt{\hbar}(x-y)\rangle}$. The expression (2.11) can be rewritten as

$$\frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \frac{\mathrm{d}y}{\hbar^{\frac{N}{2}}} G_\hbar \left(x, y, \frac{y}{\sqrt{\hbar}} \right) \varphi_\hbar(z; x, y) v(x-y) \overline{v(x)}, \tag{2.12}$$

where

$$G_\hbar(x, y, \mathrm{Y}) := (1 \otimes \mathcal{F}^*)[\Theta(z, \zeta) g_\hbar] \left(\sqrt{\hbar} \left(x - \frac{y}{2} \right), \mathrm{Y} \right)$$

and $\Theta(z, \zeta) g_\hbar = g_\hbar(\cdot + y, \cdot + \zeta)$. The map $\mathcal{X} \ni \mathrm{Y} \mapsto G_\hbar(x, y, \mathrm{Y}) \in \mathbb{C}$ clearly belongs to $L^1(\mathcal{X})$.

Now, in order to have a better understanding of the expression (2.12), let us observe that

$$\varphi_\hbar(z; x, y) = \exp \left\{ i \sum_{j,k} x_j y_k \int_0^1 \mathrm{d}\mu \int_0^1 \mathrm{d}\nu \mu B_{jk}(z + \mu\sqrt{\hbar}x - \mu\nu\sqrt{\hbar}y) \right\}.$$

Clearly, this functions has a limit as $\hbar \to 0$. More precisely, one has

$$\lim_{\hbar \to 0} \varphi_\hbar(z; x, y) = \exp \left\{ \frac{i}{2} \sum_{j,k} y_k z_j B_{jk}(z) \right\}$$

but one also has $\lim_{\hbar \to 0} \varphi_\hbar(z; x, \sqrt{\hbar}y) = 1$, both convergences being locally uniform.

By taking these information into account, one easily shows that

$$\begin{aligned} &\frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathcal{X}} \frac{\mathrm{d}y}{\hbar^{\frac{N}{2}}} G_\hbar \left(x, y, \frac{y}{\sqrt{\hbar}} \right) \varphi_\hbar(z; x, y) v(x-y) \\ &\quad = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathcal{X}} \mathrm{d}y G_\hbar(x, \sqrt{\hbar}y, y) \varphi_\hbar(z; x, \sqrt{\hbar}y) v(x - \sqrt{\hbar}y) \end{aligned} \tag{2.13}$$

converges as $\hbar \to 0$ to

$$(2\pi)^{-\frac{N}{2}} \int_{\mathcal{X}} \mathrm{d}y G_0(0,0,y)v(x) = g_0(z,\zeta)v(x),$$

locally uniformly in x.

In order to conclude, one still has to show that for the expression (2.12) the limit $\hbar \to 0$ and the integration with respect to x can be exchanged. However, this follows from the Dominated Convergence theorem and the observation that the expression (2.13) is bounded in $x \in \mathcal{X}$ and in $\hbar \in \overline{I}$. □

3. Strict Quantization

We start by recalling from [15] the classical limit of the magnetic Weyl calculus. With the definitions and the notations introduced in Sec. 1, and in the language of [11, 23, 24], a particular case of the results of [15] states that

Theorem 3.1. *If $B_{jk} \in BC^\infty(\mathcal{X})$ for $j,k = 1,\ldots,N$, then the embeddings $(\mathfrak{Q}_\hbar^B : \mathcal{S}(\Xi;\mathbb{R}) \to [\mathfrak{A}_\hbar^B]_\mathbb{R})_{\hbar\in\overline{I}}$ form a strict deformation quantization of the Poisson algebra $(\mathcal{S}(\Xi;\mathbb{R}), \{\cdot,\cdot\}^B)$.*

To explain this, observe first that $\mathcal{S}(\Xi;\mathbb{R})$ is really a Poisson subalgebra of $C^\infty(\Xi;\mathbb{R})$ for the pointwise product and the Poisson bracket $\{\cdot,\cdot\}^B$ defined in (1.2). The embedding $\mathfrak{Q}_\hbar^B$ just interprets $f \in \mathcal{S}(\Xi;\mathbb{R})$ as a self-adjoint element in the C^*-algebra $\mathfrak{A}_\hbar^B$. The self-adjoint part $[\mathfrak{A}_\hbar^B]_\mathbb{R}$ is a Jordan–Lie–Banach algebra under the C^*-norm and the operations

$$f \circ_\hbar^B g := \frac{1}{2}(f\sharp_\hbar^B g + g\sharp_\hbar^B f) \quad \text{and} \quad [f,g]_\hbar^B := \frac{1}{i\hbar}(f\sharp_\hbar^B g - g\sharp_\hbar^B f).$$

Then the above theorem says that the following are verified for any $f,g \in \mathcal{S}(\Xi;\mathbb{R})$:

(1) $\overline{I} \ni \hbar \mapsto \|f\|_\hbar^B \in \mathbb{R}_+$ is continuous (Rieffel's axiom),
(2) $\|f \circ_\hbar^B g - fg\|_\hbar^B \to 0$ as $\hbar \to 0$ (von Neumann's axiom),
(3) $\|[f,g]_\hbar^B - \{f,g\}^B\|_\hbar^B \to 0$ as $\hbar \to 0$ (Dirac's axiom).

In our framework, a pure state quantization would be a family of smooth injections $\{\mathfrak{v}_\hbar : \Xi \to \mathbb{P}(\mathcal{H})\}_{\hbar\in I}$ satisfying the three axioms stated in [11, Definition II.1.3.3]. In fact, by setting

$$\mathfrak{v}_\hbar(X) := \mathfrak{v}_\hbar^A(X) \equiv |v_\hbar^A(X)\rangle\langle v_\hbar^A(X)| \tag{3.1}$$

for any suitable unit vector $v \in \mathcal{H}$ and any $X \in \Xi$, the three axioms correspond to our Propositions 2.5, 2.7 and 2.8. Actually, it seems to us that the content of Proposition 2.6 is more intuitive than the one of Proposition 2.7, and could replace the latter statement.

Recalling the definition (0.1) of the quantum transition probabilities, let us still rewrite some of the results obtained so far in this language. In that framework, Proposition 2.5 reads

$$\int_{\Xi} \frac{\mathrm{d}Y}{(2\pi\hbar)^N} p^{\mathrm{qu}}(\mathfrak{v}^A_\hbar(Y), \mathfrak{u}) = 1 \tag{3.2}$$

for any unit vector $v \in \mathcal{H}$, any $\hbar \in I$ and any $\mathfrak{u} \in \mathbb{P}(\mathcal{H})$. Proposition 2.6 is then equivalent to

$$\lim_{\hbar\to 0} p^{\mathrm{qu}}(\mathfrak{v}^A_\hbar(Z), \mathfrak{v}^A_\hbar(Y)) = p^{\mathrm{cl}}(Z, Y) \tag{3.3}$$

for any $Y, Z \in \Xi$. Finally, under the stated conditions on B_{jk} and v, Proposition 2.7 reads

$$\lim_{\hbar\to 0} \int_{\Xi} \frac{\mathrm{d}Y}{(2\pi\hbar)^N} p^{\mathrm{qu}}(\mathfrak{v}^A_\hbar(Z), \mathfrak{v}^A_\hbar(Y)) g(Y) = g(Z)$$

for any $Z \in \Xi$ and $g \in BC(\Xi)$. Obviously, these relations could be rewritten on the pure states space of the intrinsic algebra $\mathfrak{A}^B_\hbar$ simply by transporting the transition probability on this structure and by replacing $\mathfrak{v}^A_\hbar$ with $\mathfrak{v}^B_\hbar$.

By collecting these results together with Proposition 2.8 one has obtained:

Theorem 3.2. *If $B_{jk} \in BC^\infty(\mathcal{X})$ and $v \in \mathcal{S}(\mathcal{X})$, the family $\{\mathfrak{v}^A_\hbar\}_{\hbar\in I}$ forms a pure state quantization of the Poisson space with transition probabilities $(\Xi, \sigma^B, p^{\mathrm{cl}})$.*

Remark 3.3. We stress that the conditions (3.2) and (3.3) have been obtained for every continuous magnetic field and for every unit vector v. We also mention that Ref. [11] imposes the axiom on the symplectic forms as a limit for $\hbar \to 0$, but says on p. 114 that in all the examples of this book the equality holds without the limit. In might be interesting that we really need a limit.

We also set $\mathfrak{v}^B_{\hbar=0}(Z) := \delta_Z$. Let us now show that the family $\{\mathfrak{v}^B_\hbar(Z) \,|\, \hbar \in \overline{I}, Z \in \Xi\}$ forms a continuous field of pure states associated with a continuous field of C^*-algebras, see [11, Secs. II.1.2 and II.1.3] for the abstract presentation.

We recall some results from [15] and hinted in Theorem 3.1. For $\hbar \in I$ the C^*-algebra $\mathfrak{A}^B_\hbar$ is isomorphic to $\mathfrak{C}^B_\hbar$, the twisted crossed product algebras $C_0(\mathcal{X}) \times^{\omega^B_\hbar}_{\theta_\hbar} \mathcal{X}$, where the group 2-cocycle $\omega^B_\hbar$ is defined in terms of Γ^B (cf. [15] for details) and $[\theta_\hbar(x)f](y) = f(y + \hbar x)$ for any $x, y \in \mathcal{X}$ and $f \in C_0(\mathcal{X})$. Furthermore, let us consider the twisted action (Θ, Ω^B) of $\mathcal{X}$ on $C_0(\overline{I} \times \mathcal{X})$, where $[\Theta(x)g](\hbar, y) := g(\hbar, y + \hbar x)$ for all $g \in C_0(\overline{I} \times \mathcal{X})$ and $[\Omega^B(x, y)](\hbar, z) := \omega^B_\hbar(z; x, y)$. The corresponding twisted crossed product algebra $C_0(\overline{I} \times \mathcal{X}) \rtimes^{\Omega^B}_{\Theta} \mathcal{X}$ is simply denoted by $\mathfrak{C}^B$. Now, it is proved in [15, Sec. VI] that $(\mathfrak{C}^B, \{\mathfrak{C}^B_\hbar, \varphi_\hbar\}_{\hbar\in\overline{I}})$ is a continuous field of C^*-algebras, where $\varphi_\hbar : \mathfrak{C}^B \to \mathfrak{C}^B_\hbar$ is the surjective morphism corresponding to the evaluation map $[\varphi_\hbar(\Phi)](x) = \Phi(x, \hbar) \in C_0(\mathcal{X})$ for any $\Phi \in L^1(\mathcal{X}; C_0(\overline{I} \times \mathcal{X}))$. By performing a partial Fourier transform, with respect to the variable in $L^1(\mathcal{X})$,

one again obtains a continuous field of C^*-algebras $(\mathfrak{A}^B, \{\mathfrak{A}^B_\hbar, \psi_\hbar\}_{\hbar\in\overline{I}})$. In this representation, the C^*-algebra $\mathfrak{A}^B_0$ corresponding to $\hbar = 0$ is simply equal to $C_0(\Xi)$.

Proposition 3.4. *Let $B_{jk} \in BC^\infty(\mathcal{X})$ for $j,k \in \{1,\ldots,N\}$ and assume that $v \in \mathcal{S}(\mathcal{X})$. Then the family $\{\mathfrak{v}^B_\hbar(Z) \,|\, \hbar \in \overline{I}, Z \in \Xi\}$ forms a continuous field of pure states relative to the continuous field of C^*-algebras $(\mathfrak{A}^B, \{\mathfrak{A}^B_\hbar, \psi_\hbar\}_{\hbar\in\overline{I}})$.*

Proof. Since $\mathfrak{v}^B_\hbar(Z)$ is a pure state on $\mathfrak{A}^B_\hbar$ for each $Z \in \mathcal{X}$ and $\hbar \in I$ and since δ_Z is a pure state on $\mathfrak{A}^B_0 \equiv C_0(\Xi)$, the proof simply consists in verifying that the two conditions stated in [11, Definition II.1.3.1] are satisfied.

The first condition stipulates that for any $g \in \mathfrak{A}^B$, the map $\hbar \mapsto [\mathfrak{v}^B_\hbar(Z)](\psi_\hbar(g))$ belongs to $C(\overline{I})$. If $g_\hbar \in \mathcal{S}(\Xi), \forall\, \hbar$, then the continuity in $\hbar \in I$ follows from the explicit formula (2.2) and the continuity at $\hbar = 0$ has been proved in Proposition 2.9. The general case follows then by density and approximation.

The second condition requires that $\cap_{Z\in\Xi} \ker[\pi^B_\hbar(Z)] = \{0\}$ for any $\hbar \in \overline{I}$, where $\pi^B_\hbar(Z)$ is the GNS representation associated with $\mathfrak{v}^B_\hbar(Z)$. The GNS representation of a pure state is irreducible, every irreducible representation of $\mathfrak{A}^B_\hbar \cong \mathcal{K}(\mathcal{H})$ is unitarily equivalent to the identity representation, which is faithful. So the mentioned condition holds for $\hbar \in I$. For $\hbar = 0$, this condition is also clearly satisfied. □

Remark 3.5. According to the terminology of [7], the magnetic Weyl quantization is *positive*, meaning that any pure state δ_Z of the classical C^*-algebra $C_0(\Xi)$ can be deformed for each $\hbar$ to a pure state $\mathfrak{v}^B_\hbar(Z)$ of the quantum C^*-algebra $\mathfrak{A}^B_\hbar$ in the precise sense given by Proposition 3.4 (cf. also (2.10)).

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