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The topological meaning of Levinson's theorem, half-bound states included

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Abstract
We propose to interpret Levinson's theorem as an index theorem. This exhibits its topological nature. It furthermore leads to a more coherent explanation of the corrections due to resonances at thresholds.

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1. Introduction

Levinson's theorem is a relation between the number of bound states of a quantum mechanical system and an expression related to the scattering part of that system. The latter can be written either in terms of an integral over the time delay, or as an evaluation of the spectral shift function (see the review papers [3, 7, 11] and references therein). In the simplest situations, the relation is an equality, but that is not always the case. Depending on the space dimension and the existence of resonances at thresholds, also called half-bound states, corrections to the former equality have to be taken into account. Different explanations for these corrections can be found in the literature, but they often have the flavor of a case-by-case study. It has not been realized that Levinson's theorem is a topological theorem, and this includes the corrections. We propose here a topological explanation by interpreting it as an index theorem. This does not only shed new light on it, but also provides a more coherent and natural way to take the corrections into account. It is inspired by Bellissard's approach to topological phenomena in solid states physics [2] and was first proposed for systems without resonances in [10]. The proof relies on evaluating the index of the wave operator by the winding number of an expression involving not only the scattering operator, but also new operators that describe the system at zero energy and large energy.

The content of this paper is the following: first we recall one of the forms in which Levinson's theorem can be commonly found in the literature. Then, we expose our topological approach and introduce the framework for potential scattering on $\mathbb{R}^n$. Finally, we illustrate our ideas with one dimensional scattering systems.

2. Levinson's theorem

We consider a quantum-mechanical system described by a Hamiltonian H in a Hilbert space $\mathcal{H}$. The spectrum of H is composed of a discrete part, the bound state energies and a continuous part. The projection onto the subspace spanned by the bound states is denoted by P_p. The continuous part of the system is commonly described by scattering theory. For example, in stationary scattering theory a system of generalized eigenfunctions of H is constructed by solving the Lippmann–Schwinger equation for outgoing spherical wave asymptotics [12]. The wave operators $\Omega_\pm$ can then be defined as the linear operators that map the generalized eigenfunction of a comparison operator H_0 to a certain wave vector to the generalized eigenfunction of H to the same wave vector. In time-dependent scattering theory, which seems more adapted to our approach, the wave operators are defined by the strong limits

$$\Omega_\pm = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}$$

and satisfy

$$\Omega_\pm^*\Omega_\pm = 1, \qquad \Omega_\pm\Omega_\pm^* = 1 - P_p. \tag{1}$$

The scattering operator is given by $S = \Omega_+^*\Omega_-$, which can also be written as $\Omega := \Omega_-$ evolved to time $+\infty$: $S = \mathrm{s} - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_0}\Omega\,\mathrm{e}^{-\mathrm{i}tH_0}$. Since S commutes with H_0, it is unitarily equivalent to an operator-valued function $\lambda \mapsto S(\lambda)$ where λ ranges over all values of the spectrum $\sigma(H_0)$ of H_0. $S(\lambda)$ is referred to as the S-matrix at energy λ and $\mathrm{i}S^*(\lambda)S'(\lambda)$ as the time delay operator at energy λ. Levinson's theorem can then be expressed as

$$\frac{1}{2\pi}\int_{\sigma(H_0)} (\mathrm{tr}_\lambda[\mathrm{i}S^*(\lambda)S'(\lambda)] - c(\lambda))\,\mathrm{d}\lambda = \mathrm{Tr}(P_p) + \nu. \tag{2}$$

Here S' denotes the derivative of S w.r.t. energy, tr_λ is the trace on the generalized eigenspace of H_0 to energy λ and Tr is the trace on $\mathcal{H}$. In particular $\mathrm{Tr}(P_p)$ is the number of bound states. The regularizing term $c(\lambda)$ is necessary if the map $\lambda \mapsto \mathrm{tr}_\lambda[\mathrm{i}S^*(\lambda)S'(\lambda)]$ is not integrable. The correction term ν is related to resonances at thresholds in the spectrum of H.

For example, for $H_0 = -\Delta$, the Laplacian on $\mathbb{R}^n$, and $H = -\Delta + V$, a perturbation by a rapidly vanishing potential, the correction depends on the existence of 0-energy resonances and on the dimension n. 0-energy resonances are solutions of the equation $H\Psi = 0$ with Ψ not in $L^2(\mathbb{R}^n)$ but in some suitable larger space. If $n = 3$ and if such a 0-energy resonance exists, the correction ν is equal to $1/2$. In other dimensions the picture is different.

3. Topological approach

In this section, we show how to rewrite (2) as an index theorem. Our approach is based on the construction of a norm-closed algebra $\mathcal{E}$ which sits in between the algebra of compact operators $\mathcal{K}(\mathcal{H})$ and that of bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$: $\mathcal{K}(\mathcal{H}) \subset \mathcal{E} \subset \mathcal{B}(\mathcal{H})$. $\mathcal{K}(\mathcal{H})$ is the norm-closed algebra generated by operators with finite-dimensional image. It forms an ideal in $\mathcal{B}(\mathcal{H})$. $F \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if its kernel and cokernel are finite-dimensional. This is the case whenever it is invertible modulo a compact operator, that is, its image $q(F)$ in the quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible. The difference of the dimensions of its kernel and its cokernel is its index. The index is a topological (even homotopy) invariant, namely it is stable against perturbations of F along continuous paths of Fredholm operators. By (1) the wave operator Ω is a Fredholm operator of index $-\mathrm{Tr}(P_p)$.

Suppose F belongs to $\mathcal{E}$ and that $\mathcal{E}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C(\mathbb{S}, \mathcal{K}(\mathcal{H}'))^\sim$, where $C(\mathbb{S}, \mathcal{K}(\mathcal{H}'))$ is the algebra of continuous functions over the circle with values in $\mathcal{K}(\mathcal{H}')$, and $\sim$ means that a unit has been added. Here $\mathcal{H}'$ is some possibly finite-dimensional separable

Hilbert space. Then, viewing $q(F)$ as such a function we can take pointwise its determinant (if needed regularized) to obtain a non-vanishing function over the circle. The winding number of that latter function is another topological invariant of F. We denote it by $w(q(F))$. The index theorem of Krein and Gohberg states [6] that these two invariants are equal for a suitable choice of $\mathcal{E}$. We wish to apply this to $F = \Omega$ and hence need to make sure that Ω belongs to $\mathcal{E}$. We will observe that in some cases $q(\Omega)$ is essentially given by S and this implies that the correction ν in (2) vanishes, but in general $q(\Omega)$ incorporates besides S other terms. Our formulation of Levinson's theorem is therefore

$$w(q(\Omega)) = -\mathrm{Tr}(P_p), \tag{3}$$

and we shall see below how the winding number $w(q(\Omega))$ is related to the time delay.

We note that (3) can be refined: if P is a projection which commutes with Ω, then restricting the analysis to the Hilbert space $P\mathcal{H}$ results in $w(q(\Omega P)) = -\mathrm{Tr}(P_p P)$. For example, choosing for P a projection on an angular momentum sector leads to a Levinson's theorem for that sector.

3.1. Potential scattering on $\mathbb{R}^n$

When $H_0 = -\Delta$ in $\mathcal{H} := L^2(\mathbb{R}^n)$, we construct $\mathcal{E}$ with the help of H_0 and with the generator $A = \frac{1}{2\mathrm{i}}(\nabla \cdot X + X \cdot \nabla)$ of dilations. The crucial property is that A and $B := \frac{1}{2}\ln(H_0)$ satisfy the canonical commutation relation $[A, B] = \mathrm{i}$ so that A generates translations in B and vice versa,

$$\mathrm{e}^{\mathrm{i}tB} A\, \mathrm{e}^{-\mathrm{i}tB} = A + t, \qquad \mathrm{e}^{\mathrm{i}sA} B\, \mathrm{e}^{-\mathrm{i}sA} = B - s. \tag{4}$$

The spectrum of H_0 is $\mathbb{R}_+$ and that of A is $\mathbb{R}$.

Let $\mathcal{E}'$ be the closure in $\mathcal{B}(\mathcal{H})$ of the algebra generated by elements of the form $\eta(A)\psi(H_0)$, where η is a continuous function on $\mathbb{R}$ which converges at $\pm\infty$, and ψ is a continuous function $\mathbb{R}_+$ which converges at 0 and $+\infty$. Stated differently, $\eta \in C(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$, and $\psi \in C(\overline{\mathbb{R}_+})$ with $\overline{\mathbb{R}_+} = [0, +\infty]$. Let $\mathcal{J}'$ be the norm closed algebra generated by $\eta(A)\psi(H_0)$ with functions η and ψ for which the above limits vanish. $\mathcal{J}'$ can be identified with $\mathcal{K}(L^2(\mathbb{R}_+))$. It is an ideal of $\mathcal{E}'$.

To describe the quotient $\mathcal{E}'/\mathcal{J}'$ we consider the square $\square := \overline{\mathbb{R}_+} \times \overline{\mathbb{R}}$ whose boundary $\partial\square$ is the union of four parts: $\partial\square = B_1 \cup B_2 \cup B_3 \cup B_4$, with $B_1 = \{0\} \times \overline{\mathbb{R}}$, $B_2 = \overline{\mathbb{R}_+} \times \{+\infty\}$, $B_3 = \{+\infty\} \times \overline{\mathbb{R}}$ and $B_4 = \overline{\mathbb{R}_+} \times \{-\infty\}$. We can also view $C(\partial\square)$ as the subalgebra of $C(\overline{\mathbb{R}}) \oplus C(\overline{\mathbb{R}_+}) \oplus C(\overline{\mathbb{R}}) \oplus C(\overline{\mathbb{R}_+})$ given by elements $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ which coincide at the corresponding end points, that is, for instance, $\Gamma_1(+\infty) = \Gamma_2(0)$. Then $\mathcal{E}'/\mathcal{J}'$ is isomorphic to $C(\partial\square)$ and the image $q'(\eta(A)\psi(H_0))$ in $C(\overline{\mathbb{R}}) \oplus C(\overline{\mathbb{R}_+}) \oplus C(\overline{\mathbb{R}}) \oplus C(\overline{\mathbb{R}_+})$ is given by $\Gamma_1(A) = \eta(A)\psi(0)$, $\Gamma_2(H_0) = \eta(+\infty)\psi(H_0)$, $\Gamma_3(A) = \eta(A)\psi(+\infty)$ and $\Gamma_4(H_0) = \eta(-\infty)\psi(H_0)$.

The generalized eigenspaces of H_0 can be naturally identified with $L^2(\mathbb{S}^{n-1})$, the square-integrable functions on the sphere in momentum space, and $\mathcal{H}$ can be written as the tensor product $L^2(\mathbb{R}_+) \otimes L^2(\mathbb{S}^{n-1})$. Thus $\mathcal{K}(\mathcal{H}) = \mathcal{J}' \otimes \mathcal{K}(L^2(\mathbb{S}^{n-1}))$. We define $\mathcal{E}$ as the tensor product of $\mathcal{E}'$ with $\mathcal{K}(L^2(\mathbb{S}^{n-1}))$, and add a unit. As a result, $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{E}$ and $\mathcal{E}/\mathcal{K}(\mathcal{H}) \cong C(\partial\square, \mathcal{K}(L^2(\mathbb{S}^{n-1})))^{\sim}$. Thus, $\mathcal{E}$ is suitable for the index theorem of Krein and Gohberg, and in fact $\mathcal{E}'$ is the Toeplitz extension of $C(\mathbb{S})$. Let us furthermore observe that for any $F \in \mathcal{E}$, it follows from (4) that $\Gamma_{2,4} \in C(\overline{\mathbb{R}}_+, \mathcal{K}(L^2(\mathbb{S}^{n-1})))^{\sim}$ are obtained by $\mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tB} F\, \mathrm{e}^{-\mathrm{i}tB}$.

Our basic assumption is that Ω belongs to $\mathcal{E}$. We will verify it below for one-dimensional systems. From the above observation we get from the intertwining relation and the invariance

principle that $q(\Omega) = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ with $\Gamma_2(H_0) = s - \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tB}\Omega\,\mathrm{e}^{-\mathrm{i}tB} = S$ and similarly $\Gamma_4(H_0) = 1$. We will determine below $\Gamma_1(A)$ and $\Gamma_3(A)$.

The winding number $w(q(\Omega))$ is the sum of four terms, each side of the square contributing for one. In regular cases, the winding along B_j contributes with $w_j = \frac{1}{2\pi\mathrm{i}}\int_{B_j} \mathrm{tr}[\Gamma_j^{-1}\,\mathrm{d}\Gamma_j]$. Then $w_2 = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{tr}_\lambda[S^*(\lambda)S'(\lambda)]\,\mathrm{d}\lambda$ and $w_4 = 0$. Comparing (2) with (3) we see therefore that the correction term arises now on the lhs of the equality from the possible contribution of Γ_1 and Γ_3 to the winding number. If $c(\lambda) \neq 0$ the above formulae have to be regularized.

We believe that the above construction is sufficient to describe scattering for perturbations of $-\Delta$ by a large class of potentials vanishing at infinity. But the algebraic framework is flexible enough to cover all kinds of situations. In fact, there is a very well-developed index theory which is based on non-commutative topology and allows for situations in which not only $\mathcal{E}$ can be an arbitrary Banach algebra but also $\mathcal{K}(\mathcal{H})$ can be replaced by any ideal of $\mathcal{E}$. We expect that these generalizations will lead to new Levinson's type identities.

4. One-dimensional scattering

We illustrate our approach with one-dimensional systems, first for so-called point interactions, and second for ordinary potential scattering. In both cases, the decomposition of $\mathcal{H} = L^2(\mathbb{R})$ into even and odd functions leads to $\mathcal{H}' = \mathbb{C}^2$ and $\mathcal{E}/\mathcal{K}(\mathcal{H}) \cong C(\partial\square, M_2(\mathbb{C}))$. Our aim is to obtain a formula for Ω which shows that it belongs to $\mathcal{E}$, to determine each Γ_j and to show how they contribute to $w(q(\Omega))$. For this purpose, the following observation taken from [9] is essential: let g be a smooth rapidly vanishing function on $\mathbb{R}$ and T be the operator defined by $[Tg](r\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{\mathrm{i}\kappa r}\hat{g}(\kappa\omega)\,\mathrm{d}\kappa$, with $r \geqslant 0$, $\omega \in \{-1, 1\}$ and $\hat{g}$ the Fourier transform of g. Then one has $T = \frac{1}{2}(1 - R)$ with

$$R = r_e(A)P_e + r_o(A)P_o$$

where P_e and P_o are the projections onto the even (symmetric) and odd functions, respectively, and

$$r_e(x) := -\tanh(\pi x) - \mathrm{i}[\cosh(\pi x)]^{-1}, \qquad r_o := \overline{r_e}.$$

Clearly, r_e and r_o belong to $C(\overline{\mathbb{R}})$.

4.1. One-dimensional point interactions

Here we consider two families of point interactions, the so-called δ-interactions at 0 and the δ'-interactions at 0. They can be described by boundary conditions at 0 for solutions of the free Schrödinger equation on $\mathbb{R}\backslash\{0\}$.

A δ-interaction at 0 can formally be interpreted as arising from a potential $V = \alpha\delta$ (δ-function at 0). The solutions of the Schrödinger equation for a δ-interaction of strength $\alpha \in \mathbb{R} \cup \{\infty\}$ at energy λ are continuous functions Ψ satisfying $-\Psi'' = \lambda\Psi$ on $\mathbb{R}\backslash\{0\}$ and the boundary condition $\Psi'(0_+) - \Psi'(0_-) = \alpha\Psi(0)$ (this is to be interpreted as $\Psi(0) = 0$ if $\alpha = \infty$). In particular $\alpha = 0$ corresponds to the free Laplacian and $\alpha = \infty$ to the Laplacian on $\mathbb{R}\backslash\{0\}$ with Dirichlet boundary conditions at the origin. There is a single bound state if $\alpha < 0$ and no bound state if $\alpha \in [0, \infty]$. The scattering operator is given by $S = s^\alpha(H_0)P_e + P_o$ with $s^\alpha(\lambda) = \frac{2\sqrt{\lambda}-\mathrm{i}\alpha}{2\sqrt{\lambda}+\mathrm{i}\alpha}$.

The solutions of the Schrödinger equation for a δ'-interaction of scattering length β^{-1}, $\beta \in \mathbb{R} \cup \{\infty\}$ at energy λ are functions Ψ with continuous first derivative satisfying $-\Psi'' = \lambda\Psi$ on $\mathbb{R}\backslash\{0\}$ and the boundary condition $\Psi(0_+) - \Psi(0_-) = \beta\Psi'(0)$ ($\Psi'(0) = 0$ if $\beta = \infty$). Now $\beta = 0$ corresponds to the free Laplacian and $\beta = \infty$ to the Laplacian on $\mathbb{R}\backslash\{0\}$

with Neumann boundary conditions at the origin. There is a single bound state if $\beta < 0$ but no bound state if $\beta \in [0, \infty]$. The scattering operator is $S = P_e + s^\beta(H_0)P_o$ with $s^\beta(\lambda) = \frac{2+i\beta\sqrt{\lambda}}{2-i\beta\sqrt{\lambda}}$.

In all cases the wave operator has the form [9]:

$$\Omega = 1 + \tfrac{1}{2}(1 - R)(S - 1)$$

so that in particular it belongs to $\mathcal{E}$. We point out that R is universal as it does not depend on the choice of point interaction, only the S-term depends on it. We shall see later that a similar form holds for the wave operator in the case of potential scattering. The contributions to the winding number corresponding to Γ_1 and Γ_3 depend on the different behaviour of the matrix $S(\lambda)$ at $\lambda = 0$ or $\lambda = +\infty$. For example if $S(0) = 1$, then $\Gamma_1 = 1$. More interesting phenomena arise if $P_e S(0) \neq P_e$ or $P_o S(0) \neq P_o$, as exhibited in the following situations.

Since Ω commutes with P_e and P_o we obtain a Levinson's theorem for each sector separately. But for δ-interactions $\Omega P_o = P_o$ and hence the odd sector theorem is trivial. Likewise the even sector theorem is trivial for a δ'-interaction. We present the non-trivial results in the two tables below with the notations $\Omega_{o/e} = \Omega P_{o/e}$, $\Gamma_j^{o/e} := \Gamma_j P_{o/e}$ and $w_j^{o/e} := w\big(\Gamma_j^{o/e}\big)$.

δ-interaction	Γ_1^e	Γ_2^e	Γ_3^e	Γ_4^e	w_1^e	w_2^e	w_3^e	w_4^e	$w(q(\Omega_e))$
$\alpha < 0$	r_e	s^α	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1
$\alpha = 0$	1	1	1	1	0	0	0	0	0
$\alpha > 0$	r_e	s^α	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\alpha = \infty$	r_e	-1	r_e	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0

δ'-interaction	Γ_1^o	Γ_2^o	Γ_3^o	Γ_4^o	w_1^o	w_2^o	w_3^o	w_4^o	$w(q(\Omega_o))$
$\beta < 0$	1	s^β	r_o	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1
$\beta = 0$	1	1	1	1	0	0	0	0	0
$\beta > 0$	1	s^β	r_o	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\beta = \infty$	r_o	-1	r_o	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0

We thus see that both w_1 and w_3 contribute to the correction term ν in (2).

4.2. One-dimensional potential scattering

In this section, we consider $H = -\Delta + V$ with a sufficiently regular potential V given by a multiplication operator. The wave operator Ω can be expressed with the help of the solutions Ψ_k of the Lippmann–Schwinger equation for all wave vectors $k \in \mathbb{R}$:

$$[\Omega g](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi_k(x)\hat{g}(k)\,\mathrm{d}k,$$

with g as above. They have asymptotic behaviour

$$\Psi_k(x) \overset{|x|\to\infty}{\sim} \mathrm{e}^{\mathrm{i}k\cdot x} + \mathrm{e}^{\mathrm{i}|k||x|} f(k^2, \omega_k, \omega_x), \tag{5}$$

where f is the scattering amplitude, $x = r\omega_x$, $k = |k|\omega_k$ with $\omega_x, \omega_k \in \{-1, 1\}$. Consider the integral operator

$$\begin{aligned}[\tilde{\Omega} g](x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}|k|r} f(k^2, \omega_k, \omega_x)\hat{g}(k)\,\mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}\kappa r} \left[\sum_{\omega_k = \pm 1} f(\kappa^2, \omega_k, \omega_x)\hat{g}(\kappa\omega_k)\right] \mathrm{d}\kappa\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}\kappa r}[(S(\kappa^2) - 1)\hat{g}](\kappa\omega_x)\,\mathrm{d}\kappa$$
$$= \left[\frac{1}{2}(1 - R)(S - 1)g\right](x).$$

Then, it follows that

$$\Omega = 1 + \tfrac{1}{2}(1 - R)(S - 1) + K$$

with $[Kg](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \rho(k, x)\hat{g}(k)\,\mathrm{d}k$ where $\rho(k, x)$ is the difference between the lhs and the rhs in (5). In particular Ω belongs to $\mathcal{E}$ provided that the map $\lambda \mapsto S(\lambda)$ is continuous and has limits at 0 and $+\infty$, and K is a compact operator. Both conditions require assumptions on the potential, which go beyond the ones implicitly assumed for the validity of the above approach. Without aiming at the most general case here, we can say that ρ is square integrable and hence K compact provided $|V(x)| \leqslant C(1 + |x|)^{-\frac{7}{2}-\epsilon}$ for some $\epsilon > 0$ and $C > 0$. This condition is sufficient to conclude that Ω belongs to $\mathcal{E}$.

We finally explain how the correction term ν of (2) arises in our approach. For this purpose we use a basis for $M_2(\mathbb{C})$ in which $R = \left(\begin{smallmatrix} r_e & 0 \\ 0 & r_o \end{smallmatrix}\right)$. It corresponds to the decomposition of $L^2(\mathbb{R})$ into even and odd sectors. The form of $S(0)$ falls into two cases, characterized by the value of $\det(S(0))$. One finds accordingly [1]

$$S(0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \frac{1}{\gamma^2 + 1}\begin{pmatrix} 2\gamma & 1 - \gamma^2 \\ \gamma^2 - 1 & 2\gamma \end{pmatrix} \tag{6}$$

with $\gamma \in \mathbb{R}\backslash\{0\}$. The first case occurs if H does not admit a resonance at energy zero, it is referred to as the generic case (g.c.). The second, so-called exceptional case (e.c.), occurs when such a zero energy resonance exists. The contribution to the winding number coming from Γ_1 can be determined: $w(\Gamma_1) = -\frac{1}{2}$ in the generic case, and $w(\Gamma_1) = 0$ in the exceptional one. Thus, taking into account that $\Gamma_3 = \Gamma_4 = 1$ (the former because $S(\infty) = 1$) one obtains from (3)

$$\frac{1}{2\pi} \int_{\mathbb{R}_+} \mathrm{tr}_\lambda[\mathrm{i}S^*(\lambda)S'(\lambda)]\,\mathrm{d}\lambda = \begin{cases} N - \frac{1}{2}, & \text{g.c.} \\ N, & \text{e.c.} \end{cases} \tag{7}$$

where $N = \mathrm{Tr}(P_p)$ is the number of bound states of H. In particular, the correction term ν corresponds to w_1. This result is in accordance with the literature [4, 5, 8, 13].

If the potential is symmetric, a Levinson's theorem holds for each sector. In that situation, the exceptional case $\gamma = 1$ in (6) corresponds to an even zero energy resonance, and $\gamma = -1$ corresponds to an odd zero energy resonance. The results for the even and odd sectors are summarized in the following two tables.

Even sector	Γ_1^e	Γ_2^e	$S_e(0)$	w_1^e	w_2^e	$w(q(\Omega_e))$
g.c.	r_e	S_e	-1	$-\frac{1}{2}$	$-(N_e - \frac{1}{2})$	$-N_e$
e.c.	1	S_e	1	0	$-N_e$	$-N_e$

Odd sector	Γ_1^o	Γ_2^o	$S_o(0)$	w_1^o	w_2^o	$w(q(\Omega_o)$
g.c.	1	S_o	1	0	$-N_o$	$-N_o$
e.c.	r_o	S_o	-1	$\frac{1}{2}$	$-(N_o + \frac{1}{2})$	$-N_o$

Summing up the results of both sectors one obtains (7) as there is never an even and an odd zero energy resonance at the same time.

5. Conclusion

Levinson's theorem is an index theorem. We have elaborated the general framework supporting this statement, and corroborated it with one-dimensional scattering systems with point interaction or rapidly vanishing potentials. Our formulation reveals its topological nature and explains the corrections in a coherent and natural way. The proof is based on a new formula for the wave operator involving up to a compact operator the scattering operator and a universal function of the dilation operator. This formula is of independent interest and might be of use in other contexts as well.

References

[1] Aktosun T and Klaus M 2001 Small-energy asymptotics for the Schrödinger equation on the line *Inverse Problems* **17** 619–32
[2] Bellissard J 1986 *K-Theory of C^*-Algebras in Solid State Physics* (*Lecture Notes in Phys.* vol 257) (Berlin: Springer) pp 99–156
[3] Bollé D 1986 *Sum Rules in Scattering Theory and Applications to Statistical Mechanics* (*Mathematics + physics* vol 2) (Singapore: World Scientific) pp 84–153
[4] Bollé D, Gesztesy F and Klaus M 1987 Scattering theory for one-dimensional systems with $\int dx\, V(x) = 0$ *J. Math. Anal. Appl.* **122** 496–518
[5] Bollé D, Gesztesy F and Wilk S F J 1985 A complete treatment of low-energy scattering in one dimension *J. Operator Theory* **13** 3–31
[6] Booss B and Bleecker D D 1985 *Topology and Analysis. The Atiyah-Singer Index Formula and Gauge-Theoretic Physics (Universitext)* (New York: Springer)
[7] de Carvalho C A A and Nussenzveig H M 2002 Time delay *Phys. Rep.* **364** 83–174
[8] Dong S-H and Ma Z-Q 2000 Levinson's theorem for the Schrödinger equation in one dimension *Int. J. Theor. Phys.* **39** 469–81
[9] Kellendonk J and Richard S 2006 Levinson's theorem for Schrödinger operators with point interaction: a topological approach *J. Phys. A: Math. Gen.* **39** 14397–403
[10] Kellendonk J and Richard S 2008 Topological boundary maps in physics: general theory and applications *Perspectives in Operator Algebras and Mathematical Physcis* (Bucharest: Theta) pp 105–121
[11] Ma Z-Q 2006 The Levinson theorem *J. Phys. A: Math. Gen.* **39** R625–59
[12] Sakurai J J 1994 *Modern Quantum Mechanics* (New York: Addison-Wesley)
[13] Sassoli de Bianchi M 1994 Levinson's theorem, zero-energy resonances, and time delay in one-dimensional scattering systems *J. Math. Phys.* **35** 2719–33