New Formulae for the Wave Operators for a Rank One Interaction

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Abstract. We prove new formulae for the wave operators for a Friedrichs scattering system with a rank one perturbation, and we derive a topological version of Levinson's theorem for this model.

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1. Introduction and main results

Let us consider the Hilbert space $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, and let $H_0 \equiv X$ be the self-adjoint operator of multiplication by the variable, $i.e.$ $(H_0 f)(x) = x f(x)$ for any $f \in \mathcal{D}(H_0) \equiv \mathsf{L}^2(\mathbb{R}, (1+x^2)\,\mathrm{d}x)$. For $u \in \mathcal{H}$, we also consider the rank one perturbation of H_0 defined by

$$H_u f := H_0 f + \langle u, f \rangle u, \quad f \in \mathcal{D}(H_0).$$

It is well known that the wave operators $\Omega_\pm := \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{iH_u t}\,\mathrm{e}^{-iH_0 t}$ exist and are asymptotically complete, and that the scattering operator $S := \Omega_+^* \Omega_-$ is a unitary operator in $\mathcal{H}$. In fact, $S \equiv S(X)$ is simply an operator of multiplication by a function $S : \mathbb{R} \to \mathbb{T}$, with $\mathbb{T}$ the set of complex numbers of modulus 1.

   A rather explicit formula for the wave operators for this model was proposed in [8]. Its expression involves singular integrals that have to be manipulated with some care. In this Note, we propose a simpler formula for the wave operators, and put into light a straightforward consequence of it. However, we stress that contrary to [8], our formula and its corollary hold only if some additional (but weak) hypotheses on u are imposed.

   To state our results, and in particular to have an explicit formula of the Hilbert transform in terms of the generator of dilations (see Equation (2.3) below), we need to introduce the even / odd representation of $\mathcal{H}$. Given any function m on $\mathbb{R}$, we write m_{e} and m_{o} for the even part and the odd part of

m. We also set $\mathscr{H} := \mathsf{L}^2(\mathbb{R}_+; \mathbb{C}^2)$ and introduce the unitary map $\mathscr{U} : \mathcal{H} \to \mathscr{H}$ given on any $f \in \mathcal{H}$, $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in \mathscr{H}$, $x \in \mathbb{R}$ by

$$\mathscr{U} f := \sqrt{2} \begin{pmatrix} f_{\mathrm{e}} \\ f_{\mathrm{o}} \end{pmatrix} \quad \text{and} \quad \left[\mathscr{U}^* \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right](x) := \frac{1}{\sqrt{2}} \left[f_1(|x|) + \operatorname{sgn}(x) f_2(|x|) \right].$$

Now, observe that if m is a function on $\mathbb{R}$ and A the generator of dilations in $\mathcal{H}$, then we have

$$\mathscr{U} m(X) \mathscr{U}^* = \begin{pmatrix} m_{\mathrm{e}}(X_+) & m_{\mathrm{o}}(X_+) \\ m_{\mathrm{o}}(X_+) & m_{\mathrm{e}}(X_+) \end{pmatrix} \quad \text{and} \quad \mathscr{U} A \mathscr{U}^* = \begin{pmatrix} A_+ & 0 \\ 0 & A_+ \end{pmatrix},$$

where X_+ is the operator of multiplication by the variable in $\mathsf{L}^2(\mathbb{R}_+)$, and A_+ the generator of dilations in $\mathsf{L}^2(\mathbb{R}_+)$, namely $(\mathrm{e}^{itA_+} f)(x) := \mathrm{e}^{t/2} f(\mathrm{e}^t x)$ for $f \in \mathsf{L}^2(\mathbb{R}_+)$, $x \in \mathbb{R}_+$.

In the sequel we assume that the vector u satisfies the following assumption.

Assumption 1.1. *The function $u \in \mathcal{H}$ is Hölder continuous with exponent $\alpha > 0$. Furthermore, if $x_0 \in \mathbb{R}$ satisfies $u(x_0) = 0$ and $1 - \int_{\mathbb{R}} dy\, |u(y)|^2 (x_0 - y)^{-1} = 0$, then there exists an exponent $\alpha' > 1/2$ such that*

$$|u(x) - u(y)| \le \mathrm{Const.}\, |x - y|^{\alpha'}$$

for all x, y near x_0.

This assumption implies that u is bounded and satisfies $\lim_{|x| \to \infty} u(x) = 0$. Moreover, it is known that under Assumption 1.1, the operator H_u has at most a finite number of eigenvalues of multiplicity one [1, Sec. 2] (see also [6, 11]). Clearly, Assumption 1.1 is satisfied if $u \in \mathcal{H}$ is Hölder continuous with exponent $\alpha > 1/2$.

Our main result is the following representation of the wave operator Ω_- in $\mathscr{H}$.

Theorem 1.2. *Let u satisfy Assumption* 1.1. *Then, one has*

$$
\begin{aligned}
\mathscr{U} \Omega_- \mathscr{U}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &+ \tfrac{1}{2} \begin{pmatrix} 1 & -\overline{\phi}(A_+) \\ -\phi(A_+) & 1 \end{pmatrix} \\
&\times \begin{pmatrix} S_{\mathrm{e}}(X_+) - 1 & S_{\mathrm{o}}(X_+) \\ S_{\mathrm{o}}(X_+) & S_{\mathrm{e}}(X_+) - 1 \end{pmatrix} + K,
\end{aligned}
\tag{1.1}
$$

where $\phi(A_+) := \tanh(\pi A_+) + i[\cosh(\pi A_+)]^{-1}$ and K is a compact operator in $\mathscr{H}$.

Let us immediately mention that a similar formula holds for Ω_+. Indeed, by using $\Omega_+ = \Omega_- \overline{S}(X)$ one gets

$$
\begin{aligned}
\mathscr{U} \Omega_+ \mathscr{U}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &+ \tfrac{1}{2} \begin{pmatrix} 1 & \overline{\phi}(A_+) \\ \phi(A_+) & 1 \end{pmatrix} \\
&\times \begin{pmatrix} \overline{S}_{\mathrm{e}}(X_+) - 1 & \overline{S}_{\mathrm{o}}(X_+) \\ \overline{S}_{\mathrm{o}}(X_+) & \overline{S}_{\mathrm{e}}(X_+) - 1 \end{pmatrix} + K',
\end{aligned}
$$

where K' is a compact operator in $\mathcal{H}$. We also note that the proof of Theorem 1.2 will make clear why the minimal hypothesis $u \in \mathcal{H}$ is not sufficient in order to prove the claim.

We now state a corollary of the above theorem which corresponds to a topological version of Levinson's Theorem:

Corollary 1.3. *Let u satisfy Assumption* 1.1. *Then $S(\pm\infty) = 1$ and the following equality holds:*

$$\omega(S) = - \text{ number of eigenvalues of } H_u,$$

where $\omega(S)$ is the winding number of the continuous map $S : \mathbb{R} \to \mathbb{T}$.

Such a result was already known for more general perturbations but under stronger regularity conditions [2, 4] (see also [6, 14] for general information on the Friedrichs model). Our result does require neither the differentiability of the scattering matrix nor the differentiability of u. Nonetheless, if S is continuously differentiable, then the winding number can also be expressed in terms of an integral involving the (on-shell) time delay operator, which is the logarithmic derivative of the scattering matrix [13].

Remark 1.4. The authors emphasize that Corollary 1.3 is a straightforward consequence of formula (1.1), even if its proof requires the algebraic framework presented in Section 3. They do not doubt that for smooth u this result can also be obtained via more analytical technics, but one of their motivations was to show that Levinson's Theorem can directly be inferred from the explicit formula (1.1), without any further analysis (see [10] and references therein for similar proofs of Levinson's Theorem for other scattering systems).

The content of this Note is the following. In Section 2 we prove Formula (1.1) and derive some auxiliary results. In Section 3 we give a description of the algebraic framework involved in the proof of the Corollary 1.3, which is proved at the end of the section.

2. Derivation of the new formula

We start by recalling some notations and results borrowed from [1] and [8]. We shall always suppose that u satisfies Assumption 1.1.

For $x \in \mathbb{R}$ and $\varepsilon > 0$ we set

$$I_\pm^\varepsilon(x) := \int_{\mathbb{R}} \mathrm{d}y \, \frac{|u(y)|^2}{x - y \pm i\varepsilon} \, .$$

By Privalov's theorems, the limit $I_\pm(x) := \lim_{\varepsilon \searrow 0} I_\pm^\varepsilon(x)$ exists for all $x \in \mathbb{R}$. Furthermore, the set of x such that $I_\pm(x) = 1$ is finite [1, p. 396]. In consequence the expression $[1 - I_\pm(x)]^{-1}$ is well defined for almost every $x \in \mathbb{R}$, and the domain $\mathcal{D}_\pm$ of $[1 - I_\pm(X)]^{-1}$ in $\mathcal{H}$ is dense.

Let $\mathscr{F}$ denote the Fourier transform in $\mathcal{H}$, namely

$$(\mathscr{F}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}y \; \mathrm{e}^{-ixy} f(y), \quad f \in \mathcal{H} \cap \mathsf{L}^1(\mathbb{R}).$$

Given a Borel function m on $\mathbb{R}$, we set $m(D) := \mathscr{F}^* m(X)\mathscr{F}$. Finally, $\chi_{(-\infty,0)}$ stands for the characteristic function for the half-line $(-\infty, 0)$.

We are now in a position to recall the formula [8, eq. (56b)] for Ω_-. One has

$$\Omega_- = 1 - 2\pi i u(X)\chi_{(-\infty,0)}(D)\overline{u}(X)[1 - I_+(X)]^{-1}$$

on $\mathcal{D}_+$ (note that we use a convention for the sign $\pm$ of the wave operators $\Omega_{\pm}$ which differs from the one of [8]). So, if $K := -2\pi i[u(X), \chi_{(-\infty,0)}(D)]\overline{u}(X)[1 - I_+(X)]^{-1}$, then one gets on $\mathcal{D}_+$

$$\begin{aligned}
\Omega_- - 1 &= -2\pi i u(X)\chi_{(-\infty,0)}(D)\overline{u}(X)[1 - I_+(X)]^{-1} \\
&= \chi_{(-\infty,0)}(D)\big\{ -2\pi i|u(X)|^2[1 - I_+(X)]^{-1}\big\} + K \\
&= \chi_{(-\infty,0)}(D)\{S(X) - 1\} + K,
\end{aligned} \tag{2.1}$$

by using [8, Eq. (66b)] in the last equality.

In the next lemma, we determine the action of $\chi_{(-\infty,0)}(D)$ in $\mathscr{H}$. For that purpose, we define $\phi \in C([-\infty, \infty]; \mathbb{T})$ by $\phi(x) := \tanh(\pi x) + i[\cosh(\pi x)]^{-1}$ for all $x \in \mathbb{R}$.

Lemma 2.1. *One has* $\mathscr{U}\chi_{(-\infty,0)}(D)\mathscr{U}^* = \Phi(A_+)$, *where*

$$\Phi(A_+) := \tfrac{1}{2} \begin{pmatrix} 1 & -\overline{\phi}(A_+) \\ -\phi(A_+) & 1 \end{pmatrix}.$$

Proof. The usual Hilbert transform H on $\mathbb{R}$ satisfies $\mathrm{sgn}(D) = i\mathsf{H}$. Thus

$$\chi_{(-\infty,0)}(D) = \tfrac{1}{2}\big(1 - \mathrm{sgn}(D)\big) = \tfrac{1}{2}(1 - i\mathsf{H}). \tag{2.2}$$

Using the expression for $i\mathsf{H}$ in terms of the generator of dilations in $\mathcal{H}$ given in [10, Lemma 3], one gets

$$\begin{aligned}
&\mathscr{U} i\mathsf{H} \mathscr{U}^* \\
&= \begin{pmatrix} 0 & \tanh(\pi A_+) - i[\cosh(\pi A_+)]^{-1} \\ \tanh(\pi A_+) + i[\cosh(\pi A_+)]^{-1} & 0 \end{pmatrix}. \tag{2.3}
\end{aligned}$$

The claim follows then from (2.2) and (2.3). $\qquad\square$

We now recall some results on the scattering matrix.

Lemma 2.2. *Let u satisfy Assumption* 1.1. *Then the map S belongs to $C([-\infty, \infty]; \mathbb{T})$ and satisfies $S(\pm\infty) = 1$.*

Proof. The continuity of S follows from [1, Thm. 1.(i)]. The equalities $S(\pm\infty) = 1$ follow from the formula $S(x) - 1 = -2\pi i|u(x)|^2[1 - I_+(x)]^{-1}$ together with Lemma 1.(a) of [1] and the fact that $\lim_{|x|\to\infty} |u(x)|^2 = 0$. $\quad\square$

The last lemma deals with the remainder term K of Formula (1.1).

Lemma 2.3. *Let u satisfy Assumption* 1.1. *Then the operator*

$$[u(X), \chi_{(-\infty,0)}(D)]\overline{u}(X)[1 - I_+(X)]^{-1},$$

defined on $\mathcal{D}_+$, extends to a compact operator in $\mathcal{H}$.

Proof. (i) Define for all $x \in \mathbb{R}$ the function $\psi(x) := \overline{u}(x)[1 - I_+(x)]^{-1}$. We know that $\overline{u}$ is bounded and that $\lim_{|x|\to\infty} \overline{u}(x) = 0$. We also know from [1, Lemma 1.(a)] that I_+ is Hölder continuous and that $\lim_{|x|\to\infty} I_+(x) = 0$. So, outside any neighbourhood of the finite set of points where I_+ equals 1, the function ψ is bounded. Furthermore, Assumption 1.1 and [1, Lemma 1.(c)], imply that ψ is locally square integrable (see also [8, p. 2423]). Therefore, ψ can be written as $\psi = \psi_\infty + \psi_2$, with $\psi_\infty \in \mathsf{L}^\infty(\mathbb{R})$ and $\psi_2 \in \mathsf{L}^2(\mathbb{R})$ with support in a small neighbourhood of the points where I_+ equals 1.

We now show the compacity of the operator $[u(X), \chi_{(-\infty,0)}(D)]\psi_\infty(X)$ and of the operator $[u(X), \chi_{(-\infty,0)}(D)]\psi_2(X)$.

(ii) Choose a function $\varphi_1 \in C^\infty(\mathbb{R})$ and a function $\varphi_2 \in \mathsf{L}^\infty(\mathbb{R})$ with compact support such that $\varphi_1 + \varphi_2 = \chi_{(-\infty,0)}$. Then $[u(X), \varphi_1(D)]$ is compact due to [3, Thm. C], and $[u(X), \varphi_2(D)]$ is Hilbert-Schmidt due to [12, Thm. 4.1]. So

$$[u(X), \chi_{(-\infty,0)}(D)]\psi_\infty(X) = [u(X), \varphi_1(D)]\psi_\infty(X) + [u(X), \varphi_2(D)]\psi_\infty(X)$$

is a compact operator.

(iii) For each $f \in \mathcal{H}$ and almost every $x \in \mathbb{R}$, define the operator

$$(K_0 f)(x) := \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{d}y \, \frac{u(x) - u(y)}{y - x} \, \psi_2(y) f(y).$$

From the Assumption 1.1 we know that

$$|u(y + x) - u(y)| \leq \mathrm{Const.} \, |x|^{\alpha'}, \quad \alpha' > 1/2$$

for each $y \in \mathrm{supp}(\psi_2)$ and each $x \in \mathbb{R}$ with $|x|$ small enough. In particular, there exists $\delta > 0$ such that

$$4\pi^2 \int_{\mathbb{R}^2} \mathrm{d}x \, \mathrm{d}y \left| \frac{i}{2\pi} \frac{u(x) - u(y)}{y - x} \, \psi_2(y) \right|^2$$

$$= \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}x \, \frac{|u(y + x) - u(y)|^2}{x^2} \, |\psi_2(y)|^2$$

$$= \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R} \setminus [-\delta, \delta]} \mathrm{d}x \, \frac{|u(y + x) - u(y)|^2}{x^2} \, |\psi_2(y)|^2$$

$$+ \int_{\mathbb{R}} \mathrm{d}y \int_{-\delta}^{\delta} \mathrm{d}x \, \frac{|u(y + x) - u(y)|^2}{x^2} \, |\psi_2(y)|^2$$

$$\leq \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R} \setminus [-\delta, \delta]} \mathrm{d}x \, \frac{4\|u\|_\infty^2}{x^2} \, |\psi_2(y)|^2 + \mathrm{Const.} \int_{\mathbb{R}} \mathrm{d}y \int_{-\delta}^{\delta} \mathrm{d}x \, |x|^{2(\alpha' - 1)} \, |\psi_2(y)|^2$$

$$< \infty.$$

Thus, K_0 is a Hilbert-Schmidt operator. Furthermore, we have for $f \in \mathcal{D}_+$ and almost every $x \in \mathbb{R}$

$$\big\{[u(X), \chi_{(-\infty,0)}(D)]\psi_2(X)f\big\}(x) = -\frac{i}{2}\big\{[u(X), \mathsf{H}]\psi_2(X)f\big\}(x)$$
$$= -\frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{d}y \, \frac{u(x) - u(y)}{x - y} \, \psi_2(y) f(y)$$
$$= (K_0 f)(x).$$

Therefore, the operator $[u(X), \chi_{(-\infty,0)}(D)]\psi_2(X)$ extends to a Hilbert-Schmidt operator. $\qquad\square$

Proof of Theorem 1.2. The operator K extends to a compact operator due to Lemma 2.3. So Equation (2.1) holds on $\mathcal{H}$, and the claim follows from Lemma 2.1. $\qquad\square$

Remark 2.4. The proof of Corollary 1.3 relies on the fact that the range of the wave operators is the orthocomplement of the subspace spanned by the eigenvectors of H_u. Since the wave operators are complete, such a property holds if and only if H_u has no singularly continuous spectrum. Now, by using the characterization of the singular spectrum recalled in [5, p. 299] and by taking into account Lemmas 1 and 2 of [1] (which are valid since u satisfies Assumption 1.1), one easily gets that the singular spectrum of H_u only consists of a finite set. So Assumption 1.1 implies the absence of singularly continuous spectrum for H_u.

3. Algebraic framework

This section is dedicated to the presentation of the algebraic framework leading to Corollary 1.3. Since a similar construction already appears in [10] for the proof of Levinson's theorem in one dimensional potential scattering, we only sketch the construction and refer to this reference for more details.

We start by giving the definition of the Mellin transform associated with the generator of dilations A_+ in $\mathsf{L}^2(\mathbb{R}_+)$ (see [9, Sec. 2] for a general presentation when the operator acts in $\mathsf{L}^2(\mathbb{R}^n)$). Let $\mathscr{V} : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R})$ be defined by $(\mathscr{V}f)(x) := \mathrm{e}^{x/2} f(\mathrm{e}^x)$ for $x \in \mathbb{R}$, and remark that $\mathscr{V}$ is a unitary map with adjoint $\mathscr{V}^*$ given by $(\mathscr{V}^*g)(x) = x^{-1/2}g(\ln x)$ for $x \in \mathbb{R}_+$. Then, the Mellin transform $\mathscr{M} : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R})$ is defined by $\mathscr{M} := \mathscr{F}\mathscr{V}$. Its main property is that it diagonalizes the generator of dilations, namely, $\mathscr{M}A_+\mathscr{M}^* = X$. Formally, one also has $\mathscr{M}\ln(X_+)\mathscr{M}^* = -D$.

Let us now recall from Remark 2.4 that under the Assumption 1.1 the wave operators $\Omega_\pm$ are isometries with range projection $1 - P_\mathrm{p}$, where P_p is the projection onto the subspace spanned by the finite number N of eigenvectors of H_u. In particular, Ω_- is a Fredholm operator with $\mathrm{index}(\Omega_-) = -\mathrm{Tr}(P_\mathrm{p}) = -N$. Furthermore, we recall that any Fredholm operator F in $\mathcal{H}$ is invertible modulo a compact operator, that is, its image $q(F)$ in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible.

Now, assume that Ω_- belongs to a norm-closed subalgebra $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$ containing $\mathcal{K}(\mathcal{H})$. Moreover, assume that $\mathcal{E}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C\big(\mathbb{S}; M_2(\mathbb{C})\big)$, the algebra of continuous functions over the circle with values in the 2×2 matrices. Then, viewing $q(\Omega_-)$ as such a function, we can take pointwise its determinant to obtain a non-vanishing function over the circle. The winding number of that latter function can be related to the index of Ω_-; this is essentially the content of Corollary 1.3.

The choice of $\mathcal{E}$ is suggested by the formula obtained in Theorem 1.2. Indeed, we consider the closure $\mathscr{E}$ in $\mathcal{B}(\mathscr{H})$ of the algebra generated by elements of the form $\varphi(A_+)\psi(X_+)$, where φ is a continuous function on $\mathbb{R}$ with values in $M_2(\mathbb{C})$ which converges at $\pm\infty$, and ψ is a continuous function on $\mathbb{R}_+$ with values in $M_2(\mathbb{C})$ which converges at 0 and at $+\infty$. Stated differently, $\varphi \in C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big)$ with $\overline{\mathbb{R}} = [-\infty, \infty]$, and $\psi \in C\big(\overline{\mathbb{R}_+}; M_2(\mathbb{C})\big)$ with $\overline{\mathbb{R}_+} = [0, \infty]$. Let $\mathscr{J}$ be the norm closed algebra generated by $\varphi(A_+)\psi(X_+)$ with functions φ and ψ for which the above limits vanish. Then, $\mathscr{J}$ is an ideal in $\mathscr{E}$, and the same algebras are obtained if $\psi(X_+)$ is replaced by $\eta(\ln(X_+))$ with $\eta \in C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big)$ or $\eta \in C_0\big(\mathbb{R}; M_2(\mathbb{C})\big)$, respectively.

These algebras have already been studied in [7] in a different context. The authors constructed them in terms of the operators X and D on $\mathsf{L}^2(\mathbb{R}, E)$, with E an auxiliary Hilbert space, possibly of infinite dimension. In that situation, φ and η are norm continuous functions on $\overline{\mathbb{R}}$ with values in $\mathcal{K}(E)$. The isomorphism between our algebras and the algebras introduced in [7, Sec. 3.5], with $E = \mathbb{C}^2$, is given by the Mellin transform $\mathscr{M}$, or more precisely by $\mathscr{M} \otimes 1$, where 1 is identity operator in $M_2(\mathbb{C})$. For that reason, we shall freely use the results obtained in that reference, and refer to it for the proofs. For instance, it is proved that $\mathscr{J} = \mathcal{K}(\mathscr{H})$, and an explicit description of the quotient $\mathscr{E}/\mathscr{J}$ is given, which we specify now in our context.

To describe the quotient $\mathscr{E}/\mathscr{J}$, we consider the square $\square := \overline{\mathbb{R}_+} \times \overline{\mathbb{R}}$, whose boundary $\partial\square$ is the union of four parts: $\partial\square \equiv B_1 \cup B_2 \cup B_3 \cup B_4$, with $B_1 := \{0\} \times \overline{\mathbb{R}}$, $B_2 := \overline{\mathbb{R}_+} \times \{\infty\}$, $B_3 := \{\infty\} \times \overline{\mathbb{R}}$, and $B_4 := \overline{\mathbb{R}_+} \times \{-\infty\}$. It is proved in [7, Thm. 3.20] that $\mathscr{E}/\mathscr{J}$ is isomorphic to $C\big(\partial\square; M_2(\mathbb{C})\big)$. This algebra can be seen as the subalgebra of

$$C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big) \oplus C\big(\overline{\mathbb{R}_+}; M_2(\mathbb{C})\big) \oplus C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big) \oplus C\big(\overline{\mathbb{R}_+}; M_2(\mathbb{C})\big) \qquad (3.1)$$

given by elements $\gamma \equiv (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ which coincide at the corresponding end points, that is, $\gamma_1(\infty) = \gamma_2(0)$, $\gamma_2(\infty) = \gamma_3(\infty)$, $\gamma_3(-\infty) = \gamma_4(\infty)$, and $\gamma_4(0) = \gamma_1(-\infty)$. Furthermore, for any $\varphi \in C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big)$ and $\psi \in C\big(\overline{\mathbb{R}_+}; M_2(\mathbb{C})\big)$, the image of the operator $\varphi(A_+)\psi(X_+)$ through the quotient map
$q : \mathscr{E} \to C\big(\partial\square; M_2(\mathbb{C})\big)$ is given by $\gamma_1 = \varphi\psi(0)$, $\gamma_2 = \varphi(\infty)\psi$, $\gamma_3 = \varphi\psi(\infty)$ and $\gamma_4 = \varphi(-\infty)\psi$.

From what precedes we deduce that the subalgebra $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$, defined by $\mathcal{E} := \mathscr{U}^*\mathscr{E}\mathscr{U}$, contains the ideal of compact operators on $\mathcal{H}$ and that the quotient $\mathcal{E}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C\big(\partial\square; M_2(\mathbb{C})\big) \cong C\big(\mathbb{S}; M_2(\mathbb{C})\big)$. We are thus in the framework defined above, and the for any invertible element

γ of $C\big(\partial\square; M_2(\mathbb{C})\big)$, the winding number of its pointwise determinant is a well-defined quantity. So we are ready to give the proof of Corollary 1.3.

Proof of Corollary 1.3. We know from Theorem 1.2 and Lemma 2.2 that $\mathscr{U}\Omega_-\mathscr{U}^* \in \mathscr{E}$, or equivalently that $\Omega_- \in \mathcal{E}$. Due to Formula (1.1), the element γ belonging to (3.1) and associated with $q(\Omega_-)$ is given by suitable restrictions of the function $\Gamma : \overline{\mathbb{R}_+} \times \mathbb{R} \to M_2(\mathbb{C})$, where

$$\Gamma(x,y) := 1 + \Phi(y)\begin{pmatrix} s_{\mathrm{e}}(x) - 1 & s_{\mathrm{o}}(x) \\ s_{\mathrm{o}}(x) & s_{\mathrm{e}}(x) - 1 \end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix} s_{\mathrm{e}}(x) - \overline{\phi}(y)s_{\mathrm{o}}(x) + 1 & s_{\mathrm{o}}(x) - \overline{\phi}(y)[s_{\mathrm{e}}(x) - 1] \\ s_{\mathrm{o}}(x) - \phi(y)[s_{\mathrm{e}}(x) - 1] & s_{\mathrm{e}}(x) - \phi(y)s_{\mathrm{o}}(x) + 1 \end{pmatrix}.$$

Namely, $\gamma_1 = \Gamma(0, \cdot\,)$, $\gamma_2 = \Gamma(\cdot\,, +\infty)$, $\gamma_3 = \Gamma(+\infty, \cdot\,)$, and $\gamma_4 = \Gamma(\cdot\,, -\infty)$. The pointwise determinants of these functions are easily calculated by using the identity $\phi(\pm\infty) = \pm 1$: one gets $\det\gamma_1(y) = s_{\mathrm{e}}(0)$, $\det\gamma_2(x) = s(-x)$, $\det\gamma_3(y) = 1$ and $\det\gamma_4(x) = s(x)$.

The precise relation between the winding number of the map $\det\gamma : \partial\square \to \mathbb{T}$ and the index of Ω_- has been described in [10, Prop. 7]. However, the algebra corresponding to $\mathscr{E}$ in that reference was defined in terms of the operators A_+ and $B_+ = \frac{1}{2}\ln\big((D^2)_+\big)$ which satisfy the relation $[iA_+, B_+] = -1$. In our case, the algebra $\mathscr{E}$ has been constructed with the operators A_+ and $\ln(X_+)$ which satisfy the relation $[iA_+, \ln(X_+)] = 1$. Therefore, in order to apply [10, Prop. 7] in our setting, one previously needs to apply the automorphism of $C\big(\overline{\mathbb{R}}; M_2(\mathbb{C})\big)$ defined by $\widetilde{\eta}(x) := \eta(-x)$ for all $x \in \mathbb{R}$, or equivalently the automorphism of $C\big(\overline{\mathbb{R}_+}; M_2(\mathbb{C})\big)$ defined by $\widetilde{\psi}(x) := \psi(x^{-1})$ for all $x \in \mathbb{R}_+$. Therefore the pointwise determinants of the function $\widetilde{\gamma}_j$ associated with $q(\Omega_-)$ are $\det\widetilde{\gamma}_1(y) = 1$, $\det\widetilde{\gamma}_2(x) = s(-x^{-1})$, $\det\widetilde{\gamma}_3(y) = s_{\mathrm{e}}(0)$ and $\det\widetilde{\gamma}_4(x) = s(x^{-1})$.

Now, [10, Prop. 7] reads $\omega(\det\widetilde{\gamma}) = \mathrm{index}(\Omega_-) = -N$, where N is the number of eigenvalues of H_u. The convention used in that reference for the calculation of the winding number implies that the contribution of $x \mapsto \det\widetilde{\gamma}_2(x)$ is from $x = 0$ to $x = +\infty$ and the contribution of $x \mapsto \det\widetilde{\gamma}_4(x)$ is from $x = +\infty$ to $x = 0$. This corresponds to the calculation of the winding number of $x \mapsto \det\big(S(x)\big)$, from $x = -\infty$ to $x = +\infty$. Since the contributions of $\det\widetilde{\gamma}_1$ and $\det\widetilde{\gamma}_3$ are null because these terms are constant, the claim is proved. □

Acknowledgment

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