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Time delay is a common feature of quantum scattering theory

S. Richard [a,*,1], R. Tiedra de Aldecoa [b,2]

[a] Graduate School of Pure and Applied Sciences, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan
[b] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile

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ABSTRACT

We prove that the existence of time delay defined in terms of sojourn times, as well as its identity with Eisenbud–Wigner time delay, is a common feature of two-Hilbert spaces quantum scattering theory. All statements are model-independent.

1. Introduction

In quantum scattering theory, only a few results that are completely model-independent. The simplest one is certainly that the strong limit $\text{s-lim}_{t\to\pm\infty} K e^{-itH} P_{ac}(H)$ vanishes whenever H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, $P_{ac}(H)$ the projection onto the subspace of absolute continuity of H and K a compact operator in $\mathcal{H}$. Another famous result of this type is RAGE Theorem which establishes propagation estimates for the states in the continuous subspace of $\mathcal{H}$. At the same level of abstraction, one could also mention the role of H-smooth operators B which lead to estimates of the form $\int_{\mathbb{R}} dt\, \|Be^{-itH}\varphi\|^2 < \infty$ for $\varphi \in \mathcal{H}$.

Our aim in this paper is to add a new general result to this list. Originally, this result was presented as the existence of global time delay defined in terms of sojourn times and its identity with Eisenbud–Wigner time delay [10,31]. This identity was proved in different settings by various authors (see [2–5,8,11–14,16–18,23,24,27,29,30] and references therein), but a general and abstract statement has never been proposed. Furthermore, it had not been realised until very recently that its proof mainly relies on a general formula relating localisation operators to time operators [21]. Using this formula, we shall prove here that the existence and the identity of the two time delays is in fact a common feature of quantum scattering theory. On the way we shall need to consider a symmetrization procedure [3,6,11,15,17,26–29] which broadly extends the applicability of the theory but which also has the drawback of reducing the physical interpretation of the result.

Quantum scattering theory is mainly a theory of comparison: One fundamental question is whether, given a self-adjoint operator H in a Hilbert space $\mathcal{H}$ and a suitable state $\psi \in \mathcal{H}$, one can provide a rather simple description of the t-dependent state $e^{-itH}\psi$ of $\mathcal{H}$ as $t \to \pm\infty$? For that purpose, a possible approach is to look for a triple $(\mathcal{H}_0, H_0, J)$, with H_0 a self-adjoint operator in an auxiliary Hilbert space $\mathcal{H}_0$ and J a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$, such that the following strong limits exist

$$W_\pm := \text{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_0} P_{ac}(H_0).$$

* Corresponding author. On leave from Université de Lyon; Université Lyon 1; CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France.

E-mail addresses: richard@math.univ-lyon1.fr (S. Richard), rtiedra@mat.puc.cl (R. Tiedra de Aldecoa).

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In that case, for any ψ in the range of $W_\pm$ there exist $\varphi_\pm \in P_{ac}(H_0)\mathcal{H}_0$ such that the difference $e^{-itH}\psi - Je^{-itH_0}\varphi_\pm$ tends to zero in norm as $t \to \pm\infty$. If the operator H_0 is simpler than H (in a sense which highly depends on the context), then the wave operators $W_\pm$ provide, as expected, a simpler asymptotic description of the states $e^{-itH}\psi$. Furthermore, if the ranges of both operators $W_\pm$ are equal to $P_{ac}(H)\mathcal{H}$, then the study of the scattering operator $S := W_+^* W_-$ leads to further results on the scattering process. We recall that since S commutes with H_0, S decomposes into a family $\{S(\lambda)\}_{\lambda\in\sigma(H_0)}$ in the spectral representation $\int_{\sigma(H_0)}^{\oplus} d\lambda\, \mathcal{H}(\lambda)$ of H_0, with $S(\lambda)$ a unitary operator in $\mathcal{H}(\lambda)$ for almost every λ in the spectrum $\sigma(H_0)$ of H_0.

An important additional ingredient when dealing with time delay is a family of position-type operators which permits to define sojourn times, namely, a family of mutually commuting self-adjoint operators $\Phi \equiv (\Phi_1, \dots, \Phi_d)$ in $\mathcal{H}_0$ satisfying two appropriate commutation assumptions with respect to H_0. Roughly speaking, the first one requires that for some $z \in \mathbb{C} \setminus \mathbb{R}$ the map

$$\mathbb{R}^d \ni x \mapsto e^{-ix\cdot\Phi}(H_0 - z)^{-1}e^{ix\cdot\Phi} \in \mathscr{B}(\mathcal{H}_0)$$

is three times strongly differentiable. The second one requires that all the operators $e^{-ix\cdot\Phi}H_0e^{ix\cdot\Phi}$, $x \in \mathbb{R}^d$, mutually commute. Let also f be any non-negative Schwartz function on $\mathbb{R}^d$ with $f = 1$ in a neighbourhood of 0 and $f(-x) = f(x)$ for each $x \in \mathbb{R}^d$. Then, to define the time delay in terms of sojourn times one has to consider for any $r > 0$ the expectation values of the localisation operator $f(\Phi/r)$ on the freely evolving state $e^{-itH_0}\varphi$ as well as on the corresponding fully evolving state $e^{-itH}W_-\varphi$. However one immediately faces the problem that the evolution group $\{e^{-itH}\}_{t\in\mathbb{R}}$ acts in $\mathcal{H}$ whereas $f(\Phi/r)$ is an operator in $\mathcal{H}_0$. As explained in Section 4, a general solution for this problem consists in introducing a family $L(t)$ of (identification) operators from $\mathcal{H}$ to $\mathcal{H}_0$ which satisfies some natural requirements (in many examples, one can simply take $L(t) = J^*$ for all $t \in \mathbb{R}$). The sojourn time for the evolution group $\{e^{-itH}\}_{t\in\mathbb{R}}$ is then obtained by considering the expectation value of $f(\Phi/r)$ on the state $L(t)e^{-itH}W_-\varphi$. An additional sojourn time naturally appears in this general two-Hilbert spaces setting: the time spent by the scattering state $e^{-itH}W_-\varphi$ inside the time-dependent subset $(1 - L(t)^*L(t))\mathcal{H}$ of $\mathcal{H}$. Apparently, this sojourn time has never been discussed before in the literature. Finally, the total time delay is defined for fixed r as the integral over the time t of the expectations values involving the fully evolving state $L(t)e^{-itH}W_-\varphi$ minus the symmetrized sum of the expectations values involving the freely evolving state $e^{-itH_0}\varphi$ (see Eq. (4.4) for a precise definition). Our main result, properly stated in Theorem 4.3, is the existence of the limit as $r \to \infty$ of the total time delay and its identity with the Eisenbud–Wigner time delay (see (1.1) below) which we now define in this abstract setting.

Under the mentioned assumptions on Φ and H_0 it is shown in [21] how a time operator for H_0 can be defined: With the Schwartz function f introduced above, one defines a new function $R_f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and expresses the time operator in the (oversimplified) form

$$T_f := -\frac{1}{2}\big(\Phi \cdot R_f'\big(H_0'\big) + R_f'\big(H_0'\big) \cdot \Phi\big),$$

with $R_f' := \nabla R_f$ and $H_0' := (i[H_0, \Phi_1], \dots, i[H_0, \Phi_d])$ (see Section 3 for details). In suitable situations and in an appropriate sense, the operator T_f acts as $i\frac{d}{d\lambda}$ in the spectral representation of H_0 (for instance, when $H_0 = -\Delta$ in $\mathsf{L}^2(\mathbb{R}^d)$, this is verified with Φ the usual family of position operators, see [21, Section 7] for details and other examples). Accordingly, it is natural to define in this abstract framework the Eisenbud–Wigner time delay as the expectation value

$$-\langle\varphi, S^*[T_f, S]\varphi\rangle \tag{1.1}$$

for suitable $\varphi \in \mathcal{H}_0$.

The interest of the equality between both definitions of time delay is threefold. It generalises and unifies various results on time delay scattered in the literature. It provides a precise recipe for future investigations on the subject (for instance, for new models in two-Hilbert spaces scattering). And finally, it establishes a relation between the two formulations of scattering theory: Eisenbud–Wigner time delay is a product of the stationary formulation while expressions involving sojourn times are defined using the time dependent formulation. An equality relating these two formulations is always welcome.

In the last section (Section 5), we present a sufficient condition for the equality of the symmetrized time delay with the original (unsymmetrized) time delay. The physical interpretation of the latter was, a couple of decades ago, the motivation for the introduction of these concepts.

As a final remark, let us add a comment about the applicability of our abstract result. As already mentioned, most of the existing proofs, if not all, of the existence and the identity of both time delays can be recast in our framework. Furthermore, we are currently working on various new classes of scattering systems for which our approach leads to new results. Among others, we mention the case of scattering theory on manifolds which has recently attracted a lot of attention. Our framework is also general enough for a rigorous approach of time delay in the N-body problem (see [6,17,19,26] for earlier attempts in this direction). However, the verification of our abstract conditions for any non-trivial model always requires some careful analysis, in particular for the mapping properties of the scattering operator. As a consequence, we prefer to refer to [3,11, 27–29] for various incarnations of our approach and to present in this paper only the abstract framework for the time delay.

2. Operators H_0 and Φ

In this section, we recall the framework of [21] on a self-adjoint operator H_0 in a Hilbert space $\mathcal{H}_0$ and its relation with an abstract family $\Phi \equiv (\Phi_1, \dots, \Phi_d)$ of mutually commuting self-adjoint operators in $\mathcal{H}_0$ (we use the term "commute" for operators commuting in the sense of [20, Section VIII.5]). In comparison with the notations of [21], we add an index 0 to the space $\mathcal{H}$ and to the operators H, H' and H''; they are now denoted by $\mathcal{H}_0, H_0, H_0'$ and H_0'', respectively.

In order to express the regularity of H_0 with respect to Φ, we recall from [1] that a self-adjoint operator T with domain $\mathcal{D}(T) \subset \mathcal{H}_0$ is said to be of class $C^1(\Phi)$ if there exists $\omega \in \mathbb{C} \setminus \sigma(T)$ such that the map

$$\mathbb{R}^d \ni x \mapsto \mathrm{e}^{-ix\cdot\Phi}(T-\omega)^{-1}\mathrm{e}^{ix\cdot\Phi} \in \mathscr{B}(\mathcal{H}_0) \tag{2.1}$$

is strongly of class C^1 in $\mathcal{H}_0$. In such a case and for each $j \in \{1, \dots, d\}$, the set $\mathcal{D}(T) \cap \mathcal{D}(\Phi_j)$ is a core for T and the quadratic form $\mathcal{D}(T) \cap \mathcal{D}(\Phi_j) \ni \varphi \mapsto \langle T\varphi, \Phi_j\varphi\rangle - \langle \Phi_j\varphi, T\varphi\rangle$ is continuous in the topology of $\mathcal{D}(T)$. This form extends then uniquely to a continuous quadratic form $[T, \Phi_j]$ on $\mathcal{D}(T)$, which can be identified with a continuous operator from $\mathcal{D}(T)$ to its dual $\mathcal{D}(T)^*$. Finally, the following equality holds:

$$\left[\Phi_j, (T-\omega)^{-1}\right] = (T-\omega)^{-1}[T, \Phi_j](T-\omega)^{-1}.$$

In the sequel, we shall say that $i[T, \Phi_j]$ is essentially self-adjoint on $\mathcal{D}(T)$ if $[T, \Phi_j]\mathcal{D}(T) \subset \mathcal{H}_0$ and if $i[T, \Phi_j]$ is essentially self-adjoint on $\mathcal{D}(T)$ in the usual sense.

Our first main assumption concerns the regularity of H_0 with respect to Φ.

Assumption 2.1. The operator H_0 is of class $C^1(\Phi)$, and for each $j \in \{1, \dots, d\}$, $i[H_0, \Phi_j]$ is essentially self-adjoint on $\mathcal{D}(H_0)$, with its self-adjoint extension denoted by $\partial_j H_0$. The operator $\partial_j H_0$ is of class $C^1(\Phi)$, and for each $k \in \{1, \dots, d\}$, $i[\partial_j H_0, \Phi_k]$ is essentially self-adjoint on $\mathcal{D}(\partial_j H_0)$, with its self-adjoint extension denoted by $\partial_{jk} H_0$. The operator $\partial_{jk} H_0$ is of class $C^1(\Phi)$, and for each $\ell \in \{1, \dots, d\}$, $i[\partial_{jk} H_0, \Phi_\ell]$ is essentially self-adjoint on $\mathcal{D}(\partial_{jk} H_0)$, with its self-adjoint extension denoted by $\partial_{jk\ell} H_0$.

As shown in [21, Section 2], this assumption implies the invariance of $\mathcal{D}(H_0)$ under the action of the unitary group $\{\mathrm{e}^{ix\cdot\Phi}\}_{x\in\mathbb{R}^d}$. As a consequence, we obtain that each self-adjoint operator

$$H_0(x) := \mathrm{e}^{-ix\cdot\Phi} H_0 \mathrm{e}^{ix\cdot\Phi} \tag{2.2}$$

has domain $\mathcal{D}[H_0(x)] = \mathcal{D}(H_0)$. Similarly, the domains $\mathcal{D}(\partial_j H_0)$ and $\mathcal{D}(\partial_{jk} H_0)$ are left invariant by the action of the unitary group $\{\mathrm{e}^{ix\cdot\Phi}\}_{x\in\mathbb{R}^d}$, and the operators $(\partial_j H_0)(x) := \mathrm{e}^{-ix\cdot\Phi}(\partial_j H_0)\mathrm{e}^{ix\cdot\Phi}$ and $(\partial_{jk} H_0)(x) := \mathrm{e}^{-ix\cdot\Phi}(\partial_{jk} H_0)\mathrm{e}^{ix\cdot\Phi}$ are self-adjoint operators with domains $\mathcal{D}(\partial_j H_0)$ and $\mathcal{D}(\partial_{jk} H_0)$ respectively.

Our second main assumption concerns the family of operators $H_0(x)$.

Assumption 2.2. The operators $H_0(x)$, $x \in \mathbb{R}^d$, mutually commute.

This assumption is equivalent to the commutativity of each $H_0(x)$ with H_0. As shown in [21, Lemma 2.4], Assumptions 2.1 and 2.2 imply that the operators $H_0(x), (\partial_j H_0)(y)$ and $(\partial_{k\ell} H_0)(z)$ mutually commute for each $j, k, \ell \in \{1, \dots, d\}$ and each $x, y, z \in \mathbb{R}^d$. For simplicity, we write H_0' for the d-tuple $(\partial_1 H_0, \dots, \partial_d H_0)$, and define for each measurable function $g : \mathbb{R}^d \to \mathbb{C}$ the operator $g(H_0')$ by using the d-variables functional calculus. Similarly, we consider the family of operators $\{\partial_{jk} H_0\}$ as the components of a d-dimensional matrix which we denote by H_0''. The symbol $E^{H_0}(\cdot)$ denotes the spectral measure of H_0, and we use the notation $E^{H_0}(\lambda; \delta)$ for $E^{H_0}((\lambda-\delta, \lambda+\delta))$.

We now recall the definition of the critical values of H_0 and state some basic properties which have been established in [21, Lemma 2.6].

Definition 2.3. A number $\lambda \in \mathbb{R}$ is called a critical value of H_0 if

$$\lim_{\varepsilon \searrow 0} \left\| \left(H_0'^2 + \varepsilon\right)^{-1} E^{H_0}(\lambda; \delta) \right\| = +\infty \tag{2.3}$$

for each $\delta > 0$. We denote by $\kappa(H_0)$ the set of critical values of H_0.

Lemma 2.4. *Let H_0 satisfy Assumptions 2.1 and 2.2. Then the set $\kappa(H_0)$ possesses the following properties*:

(a) *$\kappa(H_0)$ is closed.*
(b) *$\kappa(H_0)$ contains the set of eigenvalues of H_0.*
(c) *The limit $\lim_{\varepsilon\searrow 0} \|(H_0'^2 + \varepsilon)^{-1} E^{H_0}(I)\|$ is finite for each compact set $I \subset \mathbb{R} \setminus \kappa(H_0)$.*
(d) *For each compact set $I \subset \mathbb{R} \setminus \kappa(H_0)$, there exists a compact set $U \subset (0, \infty)$ such that $E^{H_0}(I) = E^{|H_0'|}(U) E^{H_0}(I)$.*

In [21, Section 3] a Mourre estimate is also obtained under Assumptions 2.1 and 2.2. It implies spectral results for H_0 and the existence of locally H_0-smooth operators. We use the notation $\langle x\rangle := (1+x^2)^{1/2}$ for any $x \in \mathbb{R}^d$.

Theorem 2.5. *Let H_0 satisfy Assumptions 2.1 and 2.2. Then,*

(a) *the spectrum of H_0 in $\sigma(H_0) \setminus \kappa(H_0)$ is purely absolutely continuous,*
(b) *each operator $B \in \mathscr{B}(\mathcal{D}(\langle\Phi\rangle^{-s}), \mathcal{H}_0)$, with $s > 1/2$, is locally H_0-smooth on $\mathbb{R} \setminus \kappa(H_0)$.*

3. Integral formula for H_0

We recall in this section the main result of [21], which is expressed in terms of a function R_f appearing naturally when dealing with quantum scattering theory. The function R_f is a renormalised average of a function f of localisation around the origin $0 \in \mathbb{R}^d$. These functions were already used, in one form or another, in [11,21,28,29]. In these references, part of the results were obtained under the assumption that f belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^d)$. So, for simplicity, we shall assume from the very beginning that $f \in \mathscr{S}(\mathbb{R}^d)$ and also that f is even, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}^d$. Let us however mention that some of the following results easily extend to the larger class of functions introduced in [21, Section 4].

Assumption 3.1. The function $f \in \mathscr{S}(\mathbb{R}^d)$ is non-negative, even and equal to 1 on a neighbourhood of $0 \in \mathbb{R}^d$.

It is clear that $\text{s-lim}_{r\to\infty} f(\Phi/r) = 1$ if f satisfies Assumption 3.1. Furthermore, it also follows from this assumption that the function $R_f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ given by

$$R_f(x) := \int_0^\infty \frac{\mathrm{d}\mu}{\mu}\big(f(\mu x) - \chi_{[0,1]}(\mu)\big)$$

is well defined. The following properties of R_f are proved in [29, Section 2]: The function R_f belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and satisfies

$$R_f'(x) = \int_0^\infty \mathrm{d}\mu\, f'(\mu x)$$

as well as the homogeneity properties $x \cdot R_f'(x) = -1$ and $t^{|\alpha|}(\partial^\alpha R_f)(tx) = (\partial^\alpha R_f)(x)$ with $x \in \mathbb{R}^d \setminus \{0\}$, $\alpha \in \mathbb{N}^d$ and $t > 0$. Furthermore, if f is radial, then $R_f'(x) = -x^{-2}x$. We shall also need the function $F_f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ defined by

$$F_f(x) := \int_{\mathbb{R}} \mathrm{d}\mu\, f(\mu x). \tag{3.1}$$

The function F_f satisfies several properties as R_f such as $F_f(x) = tF_f(tx)$ for each $t > 0$ and each $x \in \mathbb{R}^d \setminus \{0\}$.

Now, we know from Lemma 2.4(a) that the set $\kappa(H_0)$ is closed. So we can define for each $t \geqslant 0$ the set

$$\mathscr{D}_t := \big\{\varphi \in \mathcal{D}\big(\langle\Phi\rangle^t\big) \;\big|\; \varphi = \eta(H_0)\varphi \text{ for some } \eta \in C_c^\infty\big(\mathbb{R} \setminus \kappa(H_0)\big)\big\}.$$

The set $\mathscr{D}_t$ is included in the subspace $\mathcal{H}_{\mathrm{ac}}(H_0)$ of absolute continuity of H_0, due to Theorem 2.5(a), and $\mathscr{D}_{t_1} \subset \mathscr{D}_{t_2}$ if $t_1 \geqslant t_2$. We refer the reader to [21, Section 6] for an account on density properties of the sets $\mathscr{D}_t$.

In the sequel, we sometimes write C^{-1} for an operator C a priori not invertible. In such a case, the operator C^{-1} will always be acting on a set where it is well defined. The next statement follows from [21, Proposition 5.2] and [21, Remark 5.4].

Proposition 3.2. *Let H_0 satisfy Assumptions 2.1 and 2.2, and let f satisfy Assumption 3.1. Then the map*

$$t_f : \mathscr{D}_1 \to \mathbb{C}, \quad \varphi \mapsto t_f(\varphi) := -\frac{1}{2}\sum_j \big\{\big\langle \Phi_j\varphi, (\partial_j R_f)\big(H_0'\big)\varphi\big\rangle + \big\langle(\partial_j R_f)\big(H_0'\big)\varphi, \Phi_j\varphi\big\rangle\big\},$$

is well defined. Moreover, the linear operator $T_f : \mathscr{D}_1 \to \mathcal{H}_0$ defined by

$$T_f\varphi := -\frac{1}{2}\left(\Phi \cdot R_f'\big(H_0'\big) + R_f'\left(\frac{H_0'}{|H_0'|}\right)\cdot \Phi\big|H_0'\big|^{-1} + iR_f'\left(\frac{H_0'}{|H_0'|}\right)\cdot\big(H_0''^{\mathsf{T}} H_0'\big)\big|H_0'\big|^{-3}\right)\varphi \tag{3.2}$$

satisfies $t_f(\varphi) = \langle\varphi, T_f\varphi\rangle$ for each $\varphi \in \mathscr{D}_1$. In particular, T_f is a symmetric operator if $\mathscr{D}_1$ is dense in $\mathcal{H}_0$.

Remark 3.3. The r.h.s. of formula (3.2) is a priori rather complicated and one could be tempted to define $T_f\varphi$ by the simpler expression $-\frac{1}{2}(\Phi \cdot R_f'(H_0') + R_f'(H_0') \cdot \Phi)\varphi$. Unfortunately, a precise meaning of this expression is not available in general, and one can only justify it in concrete examples. However, when f is radial, then $(\partial_j R_f)(x) = -x^{-2}x_j$, and T_f is equal on $\mathscr{D}_1$ to

$$T := \frac{1}{2}\left(\Phi \cdot \frac{H_0'}{(H_0')^2} + \frac{H_0'}{|H_0'|} \cdot \Phi \left|H_0'\right|^{-1} + \frac{iH_0'}{(H_0')^4} \cdot \left(H_0''^{\mathsf{T}} H_0'\right)\right). \tag{3.3}$$

The next theorem is the main result of [21]; it relates the evolution of the localisation operators $f(\Phi/r)$ to the operator T_f.

Theorem 3.4. *(See [21, Theorem 5.5].) Let H_0 satisfy Assumptions* 2.1 *and* 2.2*, and let f satisfy Assumption* 3.1*. Then we have for each $\varphi \in \mathscr{D}_2$*

$$\lim_{r\to\infty} \frac{1}{2} \int_0^\infty \mathrm{d}t \left\langle \varphi, \left(\mathrm{e}^{-itH_0} f(\Phi/r) \mathrm{e}^{itH_0} - \mathrm{e}^{itH_0} f(\Phi/r) \mathrm{e}^{-itH_0}\right)\varphi \right\rangle = \langle \varphi, T_f \varphi \rangle. \tag{3.4}$$

In particular, when the localisation function f is radial, the operator T_f in the r.h.s. of (3.4) is equal to the operator T, which is independent of f.

4. Symmetrized time delay

In this section we prove the existence of symmetrized time delay for a scattering system (H_0, H, J) with free operator H_0, full operator H, and identification operator J. The operator H_0 acts in the Hilbert space $\mathcal{H}_0$ and satisfies Assumptions 2.1 and 2.2 with respect to the family Φ. The operator H is a self-adjoint operator in a Hilbert space $\mathcal{H}$ satisfying Assumption 4.1 below. The operator $J : \mathcal{H}_0 \to \mathcal{H}$ is a bounded operator used to "identify" the Hilbert space $\mathcal{H}_0$ with a subset of $\mathcal{H}$.

The assumption on H concerns the existence, the isometry and the completeness of the generalised wave operators:

Assumption 4.1. The generalised wave operators

$$W_\pm := \text{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH} J \mathrm{e}^{-itH_0} P_{\mathrm{ac}}(H_0)$$

exist, are partial isometries with initial subspaces $\mathcal{H}_0^\pm$ and final subspaces $\mathcal{H}_{\mathrm{ac}}(H)$.

Sufficient conditions on $JH_0 - HJ$ ensuring the existence and the completeness of $W_\pm$ are given in [32, Chapter 5]. The main consequence of Assumption 4.1 is that the scattering operator

$$S := W_+^* W_- : \mathcal{H}_0^- \to \mathcal{H}_0^+$$

is a well-defined unitary operator commuting with H_0.

We now define the sojourn times for the quantum scattering system (H_0, H, J), starting with the sojourn time for the free evolution e^{-itH_0}. So, let $r > 0$ and let f be a non-negative element of $\mathscr{S}(\mathbb{R}^d)$ equal to 1 on a neighbourhood Σ of the origin $0 \in \mathbb{R}^d$. For $\varphi \in \mathscr{D}_0$, we set

$$T_r^0(\varphi) := \int_{\mathbb{R}} \mathrm{d}t \left\langle \mathrm{e}^{-itH_0}\varphi, f(\Phi/r)\mathrm{e}^{-itH_0}\varphi \right\rangle,$$

where the integral has to be understood as an improper Riemann integral. The operator $f(\Phi/r)$ is approximately the projection onto the subspace $E^\Phi(r\Sigma)\mathcal{H}_0$ of $\mathcal{H}_0$, with $r\Sigma := \{x \in \mathbb{R}^d \mid x/r \in \Sigma\}$. Therefore, if $\|\varphi\| = 1$, then $T_r^0(\varphi)$ can be approximately interpreted as the time spent by the evolving state $\mathrm{e}^{-itH_0}\varphi$ inside $E^\Phi(r\Sigma)\mathcal{H}_0$. Furthermore, the expression $T_r^0(\varphi)$ is finite for each $\varphi \in \mathscr{D}_0$, since we know from Lemma 2.5(b) that each operator $B \in \mathscr{B}(\mathcal{D}(\langle\Phi\rangle^{-s}), \mathcal{H}_0)$, with $s > \frac{1}{2}$, is locally H_0-smooth on $\mathbb{R} \setminus \kappa(H_0)$.

When defining the sojourn time for the full evolution e^{-itH}, one faces the problem that the localisation operator $f(\Phi/r)$ acts in $\mathcal{H}_0$ while the operator e^{-itH} acts in $\mathcal{H}$. The obvious modification would be to consider the operator $Jf(\Phi/r)J^* \in \mathscr{B}(\mathcal{H})$, but the resulting framework could be not general enough (see Remark 4.5 below). Sticking to the basic idea that the freely evolving state $\mathrm{e}^{-itH_0}\varphi$ should approximate, as $t \to \pm\infty$, the corresponding evolving state $\mathrm{e}^{-itH}W_\pm\varphi$, one should look for operators $L(t) : \mathcal{H} \to \mathcal{H}_0$, $t \in \mathbb{R}$, such that

$$\lim_{t\to\pm\infty} \left\| L(t)\mathrm{e}^{-itH} W_\pm \varphi - \mathrm{e}^{-itH_0}\varphi \right\| = 0. \tag{4.1}$$

Since we consider vectors $\varphi \in \mathscr{D}_0$, the operators $L(t)$ can be unbounded as long as $L(t)E^H(I)$ are bounded for any bounded subset $I \subset \mathbb{R}$. With such a family of operators $L(t)$, it is natural to define the sojourn time for the full evolution e^{-itH} by the expression

$$T_{r,1}(\varphi) := \int_{\mathbb{R}} \mathrm{d}t \left\langle L(t)\mathrm{e}^{-itH} W_-\varphi, f(\Phi/r)L(t)\mathrm{e}^{-itH} W_-\varphi\right\rangle. \tag{4.2}$$

Another sojourn time appearing naturally in this context is

$$T_2(\varphi) := \int_{\mathbb{R}} \mathrm{d}t \left\langle \mathrm{e}^{-itH} W_-\varphi, \big(1 - L(t)^*L(t)\big)\mathrm{e}^{-itH} W_-\varphi\right\rangle_{\mathcal{H}}. \tag{4.3}$$

The finiteness of $T_{r,1}(\varphi)$ and $T_2(\varphi)$ is proved under an additional assumption in Lemma 4.2 below.

The term $T_{r,1}(\varphi)$ can be approximatively interpreted as the time spent by the scattering state $\mathrm{e}^{-itH}W_-\varphi$, injected in $\mathcal{H}_0$ via $L(t)$, inside $E^{\Phi}(r\Sigma)\mathcal{H}_0$. The term $T_2(\varphi)$ can be seen as the time spent by the scattering state $\mathrm{e}^{-itH}W_-\varphi$ inside the time-dependent subset $(1 - L(t)^*L(t))\mathcal{H}$ of $\mathcal{H}$. If $L(t)$ is considered as a time-dependent quasi-inverse for the identification operator J (see [32, Section 2.3.2] for the related time-independent notion of quasi-inverse), then the subset $(1-L(t)^*L(t))\mathcal{H}$ can be seen as an approximate complement of $J\mathcal{H}_0$ in $\mathcal{H}$ at time t. Note that in concrete examples of two-Hilbert spaces quantum scattering systems, the necessity of the term $T_2(\varphi)$ can easily be illustrated (see for example [22]). Furthermore, when $\mathcal{H}_0 = \mathcal{H}$, one usually sets $L(t) = J^* = 1$, and then the term $T_2(\varphi)$ vanishes. Within this general framework, we say that

$$\tau_r(\varphi) := T_r(\varphi) - \frac{1}{2}\big\{T_r^0(\varphi) + T_r^0(S\varphi)\big\}, \tag{4.4}$$

with $T_r(\varphi) := T_{r,1}(\varphi) + T_2(\varphi)$, is the symmetrized time delay of the scattering system (H_0, H, J) with incoming state φ. This symmetrized version of the usual time delay

$$\tau_r^{\mathrm{in}}(\varphi) := T_r(\varphi) - T_r^0(\varphi)$$

is known to be the only time delay having a well-defined limit as $r \to \infty$ for complicated scattering systems (see for example [3,6,11,15,17,25–27]).

For the next lemma, we need the auxiliary quantity

$$\tau_r^{\mathrm{free}}(\varphi) := \frac{1}{2}\int_0^{\infty} \mathrm{d}t \left\langle \varphi, S^*\big[\mathrm{e}^{itH_0} f(\Phi/r)\mathrm{e}^{-itH_0} - \mathrm{e}^{-itH_0} f(\Phi/r)\mathrm{e}^{itH_0}, S\big]\varphi\right\rangle, \tag{4.5}$$

which is finite for all $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_0$. We refer the reader to [29, Eq. (4.1)] for a similar definition in the case of dispersive systems, and to [2, Eq. (3)], [14, Eq. (6.2)] and [16, Eq. (5)] for the original definition.

Lemma 4.2. *Let H_0, f and H satisfy Assumptions* 2.1, 2.2, 3.1 *and* 4.1. *For each $t \in \mathbb{R}$, let $L(t) : \mathcal{H} \to \mathcal{H}_0$ satisfy $L(t)E^H(I) \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ for any bounded subset $I \subset \mathbb{R}$. Finally, let $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_0$ be such that*

$$\big\|\big(L(t)W_- - 1\big)\mathrm{e}^{-itH_0}\varphi\big\| \in \mathsf{L}^1(\mathbb{R}_-, \mathrm{d}t) \quad \textit{and} \quad \big\|\big(L(t)W_+ - 1\big)\mathrm{e}^{-itH_0}S\varphi\big\| \in \mathsf{L}^1(\mathbb{R}_+, \mathrm{d}t). \tag{4.6}$$

Then $T_r(\varphi)$ is finite for each $r > 0$, and

$$\lim_{r\to\infty}\big\{\tau_r(\varphi) - \tau_r^{\mathrm{free}}(\varphi)\big\} = 0. \tag{4.7}$$

Proof. Direct computations with $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_0$ imply that

$$\begin{aligned}
I_r(\varphi) &:= T_{r,1}(\varphi) - \frac{1}{2}\big\{T_r^0(\varphi) + T_r^0(S\varphi)\big\} - \tau_r^{\mathrm{free}}(\varphi) \\
&= \int_{-\infty}^{0} \mathrm{d}t \left\{\left\langle L(t)\mathrm{e}^{-itH}W_-\varphi, f(\Phi/r)L(t)\mathrm{e}^{-itH}W_-\varphi\right\rangle - \left\langle \mathrm{e}^{-itH_0}\varphi, f(\Phi/r)\mathrm{e}^{-itH_0}\varphi\right\rangle\right\} \\
&\quad + \int_0^{\infty} \mathrm{d}t \left\{\left\langle L(t)\mathrm{e}^{-itH}W_-\varphi, f(\Phi/r)L(t)\mathrm{e}^{-itH}W_-\varphi\right\rangle - \left\langle \mathrm{e}^{-itH_0}S\varphi, f(\Phi/r)\mathrm{e}^{-itH_0}S\varphi\right\rangle\right\}.
\end{aligned}$$

Using the inequality

$$\big|\|\varphi\|^2 - \|\psi\|^2\big| \leqslant \|\varphi - \psi\| \cdot \big(\|\varphi\| + \|\psi\|\big), \quad \varphi, \psi \in \mathcal{H}_0,$$

the intertwining property of the wave operators and the identity $W_- = W_+S$, one gets the estimates

$$\begin{aligned}
&\big|\big\langle L(t)\mathrm{e}^{-itH}W_-\varphi, f(\Phi/r)L(t)\mathrm{e}^{-itH}W_-\varphi\big\rangle - \big\langle \mathrm{e}^{-itH_0}\varphi, f(\Phi/r)\mathrm{e}^{-itH_0}\varphi\big\rangle\big| \leqslant \mathrm{Const}\, g_-(t),\\
&\big|\big\langle L(t)\mathrm{e}^{-itH}W_-\varphi, f(\Phi/r)L(t)\mathrm{e}^{-itH}W_-\varphi\big\rangle - \big\langle \mathrm{e}^{-itH_0}S\varphi, f(\Phi/r)\mathrm{e}^{-itH_0}S\varphi\big\rangle\big| \leqslant \mathrm{Const}\, g_+(t),
\end{aligned}$$

where

$$g_-(t) := \big\|\big(L(t)W_- - 1\big)\mathrm{e}^{-itH_0}\varphi\big\| \quad \text{and} \quad g_+(t) := \big\|\big(L(t)W_+ - 1\big)\mathrm{e}^{-itH_0}S\varphi\big\|.$$

It follows by (4.6) that $|I_r(\varphi)|$ is bounded by a constant independent of r, and thus $T_{r,1}(\varphi)$ is finite for each $r > 0$. Then, using Lebesgue's dominated convergence theorem, the fact that $\text{s-lim}_{r\to\infty} f(\Phi/r) = 1$ and the isometry of W_- on $\mathcal{H}_0^-$, one obtains that

$$\begin{aligned}
\lim_{r\to\infty} I_r(\varphi) &= \int_{-\infty}^{0} \mathrm{d}t\, \big\{\big\langle L(t)\mathrm{e}^{-itH}W_-\varphi, L(t)\mathrm{e}^{-itH}W_-\varphi\big\rangle - \big\langle \mathrm{e}^{-itH_0}\varphi, \mathrm{e}^{-itH_0}\varphi\big\rangle\big\}\\
&\quad + \int_{0}^{\infty} \mathrm{d}t\, \big\{\big\langle L(t)\mathrm{e}^{-itH}W_-\varphi, L(t)\mathrm{e}^{-itH}W_-\varphi\big\rangle - \big\langle \mathrm{e}^{-itH_0}S\varphi, \mathrm{e}^{-itH_0}S\varphi\big\rangle\big\}\\
&= \int_{\mathbb{R}} \mathrm{d}t\, \big\langle \mathrm{e}^{-itH}W_-\varphi, \big(L(t)^*L(t) - 1\big)\mathrm{e}^{-itH}W_-\varphi\big\rangle_{\mathcal{H}}\\
&\equiv -T_2(\varphi).
\end{aligned}$$

Thus, $T_2(\varphi)$ is finite, and the equality (4.7) is verified. Since $T_r(\varphi) = T_{r,1}(\varphi) + T_2(\varphi)$, one also infers that $T_r(\varphi)$ is finite for each $r > 0$. □

The next theorem shows the existence of symmetrized time delay. It is a direct consequence of Lemma 4.2, Eq. (4.5) and Theorem 3.4. The apparently large number of assumptions reflects nothing more but the need of describing the very general scattering system (H_0, H, J); one needs hypotheses on the relation between H_0 and Φ, conditions on the localisation function f, a compatibility assumption between H_0 and H, a (time-dependent) quasi-inverse for the identification operator J, and conditions on the state φ on which the calculations are performed.

Theorem 4.3. *Let H_0, f and H satisfy Assumptions* 2.1, 2.2, 3.1 *and* 4.1. *For each $t \in \mathbb{R}$, let $L(t) : \mathcal{H} \to \mathcal{H}_0$ satisfy $L(t)E^H(I) \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ for any bounded subset $I \subset \mathbb{R}$. Finally, let $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_2$ verify $S\varphi \in \mathscr{D}_2$ and* (4.6)*. Then one has*

$$\lim_{r\to\infty} \tau_r(\varphi) = -\big\langle \varphi, S^*[T_f, S]\varphi\big\rangle, \tag{4.8}$$

with T_f defined by (3.2)*.*

Remark 4.4. Theorem 4.3 is the main result of the paper. It expresses the identity of the symmetrized time delay (defined in terms of sojourn times) and the Eisenbud–Wigner time delay for general scattering systems (H_0, H, J). The l.h.s. of (4.8) is equal to the global symmetrized time delay of the scattering system (H_0, H, J), with incoming state φ, in the dilated regions associated to the localisation operators $f(\Phi/r)$. The r.h.s. of (4.8) is the expectation value in φ of the generalised Eisenbud–Wigner time delay operator $-S^*[T_f, S]$. When T_f acts in the spectral representation of H_0 as the differential operator $i\frac{\mathrm{d}}{\mathrm{d}H_0}$, which occurs in most of the situations of interest (see for example [21, Section 7]), one recovers the usual Eisenbud–Wigner Formula:

$$\lim_{r\to\infty} \tau_r(\varphi) = -\Big\langle \varphi, iS^* \frac{\mathrm{d}S}{\mathrm{d}H_0}\varphi\Big\rangle.$$

Remark 4.5. Eq. (4.1) is equivalent to the existence of the limits

$$\widetilde{W}_\pm := \text{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH_0}L(t)\mathrm{e}^{-itH}P_{\mathrm{ac}}(H),$$

together with the equalities $\widetilde{W}_\pm W_\pm = P_0^\pm$, where $P_0^\pm$ are the orthogonal projections on the subspaces $\mathcal{H}_0^\pm$ of $\mathcal{H}_0$. In simple situations, namely, when $\mathcal{H}_0^\pm = \mathcal{H}_{\mathrm{ac}}(H_0)$ and $L(t) \equiv L$ is independent of t and bounded, sufficient conditions implying (4.1) are given in [32, Theorem 2.3.6]. In more complicated situations, namely, when $\mathcal{H}_0^\pm \neq \mathcal{H}_{\mathrm{ac}}(H_0)$ or $L(t)$ depends on t and is unbounded, the proof of (4.1) could be highly non-trivial. This occurs for instance in the case of the N-body systems. In such a situation, the operators $L(t)$ really depend on t and are unbounded (see for instance [9, Section 6.7]), and the proof of (4.1) is related to the problem of the asymptotic completeness of the N-body systems.

5. Usual time delay

We give in this section conditions under which the symmetrized time delay $\tau_r(\varphi)$ and the usual time delay $\tau_r^{\mathrm{in}}(\varphi)$ are equal in the limit $r \to \infty$. Heuristically, one cannot expect that this equality holds if the scattering is not elastic or is of multichannel type. However, for simple scattering systems, the equality of both time delays presents an interest. At the mathematical level, this equality reduces to giving conditions under which

$$\lim_{r\to\infty}\{T_r^0(S\varphi) - T_r^0(\varphi)\} = 0. \tag{5.1}$$

Eq. (5.1) means that the freely evolving states $e^{-itH_0}\varphi$ and $e^{-itH_0}S\varphi$ tend to spend the same time within the region defined by the localisation function $f(\Phi/r)$ as $r \to \infty$. Formally, the argument goes as follows. Suppose that $F_f(H_0')$, with F_f defined in (3.1), commutes with the scattering operator S. Then, using the change of variables $\mu := t/r$, $\nu := 1/r$, and the symmetry of f, one gets

$$\begin{aligned}
\lim_{r\to\infty}\{T_r^0(S\varphi) - T_r^0(\varphi)\} &= \lim_{r\to\infty}\int_{\mathbb{R}} \mathrm{d}t\,\langle\varphi, S^*[e^{itH_0} f(\Phi/r)e^{-itH_0}, S]\varphi\rangle - \langle\varphi, S^*[F_f(H_0'), S]\varphi\rangle \\
&= \lim_{\nu\searrow 0}\int_{\mathbb{R}} \mathrm{d}\mu\left\langle\varphi, S^*\left[\frac{1}{\nu}\{f(\mu H_0' + \nu\Phi) - f(\mu H_0')\}, S\right]\varphi\right\rangle \\
&= \int_{\mathbb{R}} \mathrm{d}\mu\,\langle\varphi, S^*[\Phi\cdot f'(\mu H_0'), S]\varphi\rangle = 0.
\end{aligned}$$

A rigorous proof of this argument is given in Theorem 5.3 below. Before this we introduce an assumption on the behavior of the C_0-group $\{e^{ix\cdot\Phi}\}_{x\in\mathbb{R}^d}$ in $\mathcal{D}(H_0)$, and then prove a technical lemma. We use the notation $\mathcal{G}$ for $\mathcal{D}(H_0)$ endowed with the graph topology, and $\mathcal{G}^*$ for its dual space. In the following proofs, we also freely use the notations of [1] for some regularity classes with respect to the group generated by Φ.

Assumption 5.1. The C_0-group $\{e^{ix\cdot\Phi}\}_{x\in\mathbb{R}^d}$ is of polynomial growth in $\mathcal{G}$, namely there exists $r > 0$ such that for all $x \in \mathbb{R}^d$

$$\|e^{ix\cdot\Phi}\|_{\mathscr{B}(\mathcal{G},\mathcal{G})} \leqslant \mathrm{Const}\langle x\rangle^r.$$

Lemma 5.2. *Let H_0 and Φ satisfy Assumptions 2.1, 2.2 and 5.1, and let $\eta \in C_c^\infty(\mathbb{R})$. Then there exists $\mathrm{c}, s > 0$ such that for all $\mu \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\nu \in (-1,1)\setminus\{0\}$*

$$\left\|\frac{1}{\nu}\{\eta(H_0(\nu x))e^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]} - \eta(H_0)e^{i\mu x\cdot H_0'}\}\right\| \leqslant \mathrm{c}(1+|\mu|)\langle x\rangle^s.$$

Proof. Let us first observe that $H_0 \in C_{\mathrm{u}}^1(\Phi;\mathcal{G},\mathcal{H}_0)$. Indeed, since $H_0 \in \mathscr{B}(\mathcal{G},\mathcal{H}_0)$, $\partial_j H_0 \in \mathscr{B}(\mathcal{D}(\partial_j H_0),\mathcal{H}_0) \subset \mathscr{B}(\mathcal{G},\mathcal{H}_0)$ and $\partial_{jk}H_0 \in \mathscr{B}(\mathcal{D}(\partial_{jk}H_0),\mathcal{H}_0) \subset \mathscr{B}(\mathcal{D}(\partial_j H_0),\mathcal{H}_0) \subset \mathscr{B}(\mathcal{G},\mathcal{H}_0)$ for any $j,k \in \{1,\dots,d\}$, it follows that $H_0 \in C^2(\Phi;\mathcal{G},\mathcal{H}_0)$. So, one has in particular that $H_0 \in C_{\mathrm{u}}^1(\Phi;\mathcal{G},\mathcal{H}_0)$. Now, for $x \in \mathbb{R}^d$ and $\mu \in \mathbb{R}$, we define the function

$$g_{x,\mu} : (-1,1)\setminus\{0\} \to \mathscr{B}(\mathcal{H}_0), \qquad \nu \mapsto e^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}\eta(H_0)$$

and observe that $g_{x,\mu}$ is continuous in norm with

$$g_{x,\mu}(0) := \lim_{\nu\to 0} g_{x,\mu}(\nu) = e^{i\mu x\cdot H_0'}\eta(H_0).$$

On another hand, since $\eta(H_0)$ belongs to $C_{\mathrm{u}}^1(\Phi)$, one has in $\mathscr{B}(\mathcal{H}_0)$ the equalities

$$\frac{1}{\nu}\{\eta(H_0(\nu x)) - \eta(H_0)\} = \frac{1}{\nu}\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\eta(H_0(t\nu x)) = i\sum_j x_j\int_0^1 \mathrm{d}t\,e^{-it\nu x\cdot\Phi}[\eta(H_0),\Phi_j]e^{it\nu x\cdot\Phi}.$$

So, combining the two equations, one obtains that

$$\begin{aligned}
&\frac{1}{\nu}\{\eta(H_0(\nu x))e^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]} - \eta(H_0)e^{i\mu x\cdot H_0'}\} \\
&\quad= \frac{1}{\nu}\{\eta(H_0(\nu x)) - \eta(H_0)\}e^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]} + \frac{1}{\nu}\{g_{x,\mu}(\nu) - g_{x,\mu}(0)\} \\
&\quad= i\sum_j x_j\int_0^1 \mathrm{d}t\,e^{-it\nu x\cdot\Phi}[\eta(H_0),\Phi_j]e^{it\nu x\cdot\Phi}e^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]} + \frac{1}{\nu}\{g_{x,\mu}(\nu) - g_{x,\mu}(0)\}.
\end{aligned} \tag{5.2}$$

In order to estimate the difference $g_{x,\mu}(\nu) - g_{x,\mu}(0)$, observe first that one has in $\mathscr{B}(\mathcal{H}_0)$ for any bounded set $I \subset \mathbb{R}$

$$\frac{1}{\nu}\big[H_0(\nu x) - H_0\big]E^{H_0}(I) = \frac{1}{\nu}\int_0^1 \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} H_0(t\nu x)E^{H_0}(I) = \int_0^1 \mathrm{d}t\, x \cdot H_0'(t\nu x)E^{H_0}(I).$$

So, if $\varepsilon \in \mathbb{R}$ is small enough and if the bounded set $I \subset \mathbb{R}$ is chosen such that $\eta(H_0) = E^{H_0}(I)\eta(H_0)$, one obtains in $\mathscr{B}(\mathcal{H}_0)$

$$\begin{aligned} g_{x,\mu}(\nu+\varepsilon) - g_{x,\mu}(\nu) &= \big\{ \mathrm{e}^{i\mu \int_0^1 \mathrm{d}t\, x\cdot H_0'(t(\nu+\varepsilon)x)E^{H_0}(I)} - \mathrm{e}^{i\mu \int_0^1 \mathrm{d}t\, x\cdot H_0'(t\nu x)E^{H_0}(I)} \big\}\eta(H_0) \\ &= \mathrm{e}^{i\mu \int_0^1 \mathrm{d}u\, x\cdot H_0'(u\nu x)E^{H_0}(I)} \big\{ \mathrm{e}^{i\mu \int_0^1 \mathrm{d}t\, x\cdot [H_0'(t(\nu+\varepsilon)x) - H_0'(t\nu x)]E^{H_0}(I)} - 1 \big\}\eta(H_0) \\ &= \mathrm{e}^{i\mu \int_0^1 \mathrm{d}u\, x\cdot H_0'(u\nu x)E^{H_0}(I)} \big\{ \mathrm{e}^{i\mu \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s\, t\varepsilon \sum_{j,k} x_j x_k (\partial_{jk}H_0)(t(\nu+s\varepsilon)x)E^{H_0}(I)} - 1 \big\}\eta(H_0). \end{aligned}$$

Note that the property $\partial_j H_0 \in C_{\mathrm{u}}^1(\Phi; \mathcal{G}, \mathcal{H}_0)$ (which follows from our assumptions and from an argument similar to the one presented at the beginning of this proof) has been taken into account for the last equality. Then, multiplying the above expression by ε^{-1} and taking the limit $\varepsilon \to 0$ in $\mathscr{B}(\mathcal{H}_0)$ leads to

$$g_{x,\mu}'(\nu) = i\mu \mathrm{e}^{i\mu \int_0^1 \mathrm{d}u\, x\cdot H_0'(u\nu x)} \int_0^1 \mathrm{d}t\, t \sum_{j,k} x_j x_k (\partial_{jk}H_0)(t\nu x)\eta(H_0). \tag{5.3}$$

This formula, together with Eq. (5.2) and the mean value theorem, implies that

$$\begin{aligned} \left\| \frac{1}{\nu}\big\{ \eta\big(H_0(\nu x)\big)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x) - H_0]} - \eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'} \big\} \right\| &\leqslant \mathrm{Const}|x| + \sup_{\xi\in[0,1]} \big\| g_{x,\mu}'(\xi\nu) \big\| \\ &\leqslant \mathrm{Const}|x| + \mathrm{Const}\, x^2|\mu| \sup_{\xi\in[0,1]} \sum_{j,k} \big\| (\partial_{jk}H_0)(\xi\nu x)\eta(H_0) \big\|. \end{aligned} \tag{5.4}$$

But one has

$$(\partial_{jk}H_0)(\xi\nu x)\eta(H_0) = \mathrm{e}^{-i\xi\nu x\cdot\Phi}(\partial_{jk}H_0)\mathrm{e}^{i\xi\nu x\cdot\Phi}\eta(H_0)$$

with $\eta(H_0) \in \mathscr{B}(\mathcal{H}_0, \mathcal{G})$ and $(\partial_{jk}H_0) \in \mathscr{B}(\mathcal{G}, \mathcal{H}_0)$. So, it follows from Assumption 5.1 that there exists $r > 0$ such that

$$\big\| (\partial_{jk}H_0)(\xi\nu x)\eta(H_0) \big\| \leqslant \mathrm{Const}\langle \xi\nu x\rangle^r.$$

Hence, one finally gets from (5.4) that for each $\nu \in (-1,1) \setminus \{0\}$

$$\left\| \frac{1}{\nu}\big\{ \eta\big(H_0(\nu x)\big)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x) - H_0]} - \eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'} \big\} \right\| \leqslant \mathrm{Const}\big(1 + |\mu|\big)\langle x\rangle^{r+2},$$

which proves the claim with $s := r + 2$. □

In the sequel, the symbol $\mathscr{F}$ stands for the Fourier transformation, and the measure $\underline{\mathrm{d}}x$ on $\mathbb{R}^d$ is chosen so that $\mathscr{F}$ extends to a unitary operator in $\mathsf{L}^2(\mathbb{R}^d)$.

Theorem 5.3. *Let H_0, f, H and Φ satisfy Assumptions* 2.1, 2.2, 3.1, 4.1 *and* 5.1, *and let $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_2$ satisfy $S\varphi \in \mathscr{D}_2$ and*

$$\big[F_f\big(H_0'\big), S\big]\varphi = 0. \tag{5.5}$$

Then the following equality holds:

$$\lim_{r\to\infty}\big\{ T_r^0(S\varphi) - T_r^0(\varphi) \big\} = 0.$$

Note that the l.h.s. of (5.5) is well defined due to the homogeneity property of F_f. Indeed, one has

$$\big[F_f\big(H_0'\big), S\big]\varphi = \left[\big|H_0'\big|^{-1}\eta(H_0)F_f\left(\frac{H_0'}{|H_0'|}\right), S \right]\varphi$$

for some $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R} \setminus \kappa(H_0))$, and thus $[F_f(H_0'), S]\varphi \in \mathcal{H}$ due to Lemma 2.4(d) and the compacticity of $F_f(\mathbb{S}^{d-1})$.

Proof. Let $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_2$ satisfies $S\varphi \in \mathscr{D}_2$, take a real $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R} \setminus \kappa(H_0))$ such that $\varphi = \eta(H_0)\varphi$, and set $\eta_t(s) := \mathrm{e}^{its}\eta(s)$ for each $s, t \in \mathbb{R}$. Using (5.5), the definition of F_f and the change of variables $\mu := t/r$, $\nu := 1/r$, one gets

$$\begin{aligned}
T^0_{1/\nu}(S\varphi)-T^0_{1/\nu}(\varphi) &= \int_{\mathbb{R}} \mathrm{d}\mu \left\langle \varphi, S^*\left[\frac{1}{\nu}\left\{\eta_{\frac{\mu}{\nu}}(H_0)f(\nu\Phi)\eta_{-\frac{\mu}{\nu}}(H_0)-f\left(\mu H_0'\right)\right\}, S\right]\varphi\right\rangle \\
&= \int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \varphi, S^*\left[\frac{1}{\nu}\left\{\mathrm{e}^{i\nu x\cdot\Phi}\eta_{\frac{\mu}{\nu}}\left(H_0(\nu x)\right)\eta_{-\frac{\mu}{\nu}}(H_0)-\mathrm{e}^{i\mu x\cdot H_0'}\right\}, S\right]\varphi\right\rangle \\
&= \int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \varphi, S^*\left[\frac{1}{\nu}\left(\mathrm{e}^{i\nu x\cdot\Phi}-1\right)\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}, S\right]\varphi\right\rangle \\
&\quad + \int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \varphi, S^*\left[\frac{1}{\nu}\left\{\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}-\eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'}\right\}, S\right]\varphi\right\rangle. \qquad (5.6)
\end{aligned}$$

To prove the statement, it is sufficient to show that the limit as $\nu \searrow 0$ of each of these two terms is equal to zero. This is done in points (i) and (ii) below.

(i) For the first term, one can easily adapt the method [21, Theorem 5.5] (points (ii) and (iii) of the proof) in order to apply Lebesgue's dominated convergence theorem to (5.6). So, one gets

$$\begin{aligned}
&\lim_{\nu\searrow 0}\int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \varphi, S^*\left[\frac{1}{\nu}\left(\mathrm{e}^{i\nu x\cdot\Phi}-1\right)\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}, S\right]\varphi\right\rangle \\
&\quad = i\int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\{\left\langle (x\cdot\Phi)S\varphi, \mathrm{e}^{i\mu x\cdot H_0'}S\varphi\right\rangle - \left\langle (x\cdot\Phi)\varphi, \mathrm{e}^{i\mu x\cdot H_0'}\varphi\right\rangle\right\},
\end{aligned}$$

and the change of variables $\mu' := -\mu$, $x' := -x$, together with the symmetry of f, implies that this expression is equal to zero.

(ii) For the second term, it is sufficient to prove that

$$\lim_{\nu\searrow 0}\int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \psi, \frac{1}{\nu}\left\{\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}-\eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'}\right\}\psi\right\rangle \qquad (5.7)$$

is equal to zero for any $\psi \in \mathscr{D}_2$ satisfying $\eta(H_0)\psi = \psi$. For the moment, let us assume that we can interchange the limit and the integrals in (5.7) by invoking Lebesgue's dominated convergence theorem. Then, taking Eqs. (5.2) and (5.3) into account, one obtains

$$\begin{aligned}
&\lim_{\nu\searrow 0}\int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \psi, \frac{1}{\nu}\left\{\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}-\eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'}\right\}\psi\right\rangle \\
&\quad = \int_{\mathbb{R}} \mathrm{d}\mu \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \psi, \left\{i\left[\eta(H_0), x\cdot\Phi\right]\mathrm{e}^{i\mu x\cdot H_0'} + \frac{i\mu}{2}\mathrm{e}^{i\mu x\cdot H_0'}\sum_{j,k}x_jx_k(\partial_{jk}H_0)\eta(H_0)\right\}\psi\right\rangle,
\end{aligned}$$

and the change of variables $\mu' := -\mu$, $x' := -x$, together with the symmetry of f, implies that this expression is equal to zero. So, it only remains to show that one can really apply Lebesgue's dominated convergence theorem in order to interchange the limit and the integrals in (5.7). For this, let us set for $\nu \in (-1,1)\setminus\{0\}$ and $\mu \in \mathbb{R}$

$$L(\nu,\mu) := \int_{\mathbb{R}^d} \underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle \psi, \frac{1}{\nu}\left\{\eta\left(H_0(\nu x)\right)\mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}-\eta(H_0)\mathrm{e}^{i\mu x\cdot H_0'}\right\}\psi\right\rangle.$$

By using Lemma 5.2 and the fact that $\mathscr{F}f \in \mathscr{S}(\mathbb{R}^d)$, one gets that $|L(\nu,\mu)| \leqslant \mathrm{Const}(1+|\mu|)$ with a constant independent of ν. Therefore $|L(\nu,\mu)|$ is bounded uniformly in $\nu \in (-1,1)\setminus\{0\}$ by a function in $\mathsf{L}^1([-1,1],\mathrm{d}\mu)$.

For the case $|\mu| > 1$, we first remark that there exists a compact set $I \subset \mathbb{R}\setminus\kappa(H_0)$ such that $\eta(H_0) = E^{H_0}(I)\eta(H_0)$. Due to Lemma 2.4(d), there also exists $\zeta \in C_c^\infty((0,\infty))$ such that

$$\eta\left(H_0(\nu x)\right) = \eta\left(H_0(\nu x)\right)\zeta\left(H_0'(\nu x)^2\right)$$

for all $x \in \mathbb{R}^d$ and $\nu \in \mathbb{R}$. So, using the notations

$$A^I_{\nu,\mu}(x) := \mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]}E^{H_0}(I) \equiv \mathrm{e}^{i\frac{\mu}{\nu}[H_0(\nu x)-H_0]E^{H_0}(I)}E^{H_0}(I)$$

and

$$B^I_{\mu}(x) := \mathrm{e}^{i\mu x\cdot H_0'}E^{H_0}(I) \equiv \mathrm{e}^{i\mu x\cdot H_0'E^{H_0}(I)}E^{H_0}(I),$$

one can rewrite $L(\nu,\mu)$ as

$$L(\nu,\mu)=\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle\psi,\frac{1}{\nu}\left\{\eta\big(H_0(\nu x)\big)\zeta\big(H_0'(\nu x)^2\big)A^I_{\nu,\mu}(x)-\eta(H_0)\zeta\big(H_0'^2\big)B^I_\mu(x)\right\}\psi\right\rangle.$$

Now, using the same technics as in the proof of Lemma 5.2, one shows that the maps $A^I_{\nu,\mu}:\mathbb{R}^d\to\mathscr{B}(\mathcal{H}_0)$ and $B^I_\mu:\mathbb{R}^d\to\mathscr{B}(\mathcal{H}_0)$ are differentiable, with derivatives

$$\big(\partial_j A^I_{\nu,\mu}\big)(x)=i\mu(\partial_j H_0)(\nu x)A^I_{\nu,\mu}(x)\quad\text{and}\quad\big(\partial_j B^I_\mu\big)(x)=i\mu(\partial_j H_0)B^I_\mu(x).$$

Thus, setting

$$C_j:=\big(H_0'\big)^{-2}\zeta\big(H_0'^2\big)(\partial_j H_0)\eta(H_0)\in\mathscr{B}(\mathcal{H}_0)\quad\text{and}\quad V_x:=\mathrm{e}^{-ix\cdot\Phi},$$

one can even rewrite $L(\nu,\mu)$ as

$$L(\nu,\mu)=(i\mu)^{-1}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}C_jV^*_{\nu x}\big(\partial_jA^I_{\nu,\mu}\big)(x)-C_j\big(\partial_jB^I_\mu\big)(x)\right\}\psi\right\rangle.$$

We shall now use repeatedly the following argument: Let $g\in\mathscr{S}(\mathbb{R}^n)$ and let $X:=(X_1,\dots,X_n)$ be a family of self-adjoint and mutually commuting operators in $\mathcal{H}_0$. If all X_j are of class $C^2(\Phi)$, then the operator $g(X)$ belongs to $C^2(\Phi)$, and $[[g(X),\Phi_j],\Phi_k]\in\mathscr{B}(\mathcal{H}_0)$ for all j,k. Such a statement has been proved in [21, Proposition 5.1] in a greater generality. Here, the operator C_j is of the type $g(X)$, since all the operators $H_0,\partial_jH_0,\dots,\partial_dH_0$ are of class $C^2(\Phi)$. Thus, we can perform a first integration by parts (with vanishing boundary contributions) with respect to x_j to obtain

$$\begin{aligned}L(\nu,\mu)=&-(i\mu)^{-1}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_j(\mathscr{F}f)\big](x)\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}C_jV^*_{\nu x}A^I_{\nu,\mu}(x)-C_jB^I_\mu(x)\right\}\psi\right\rangle\\&-\mu^{-1}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\big\langle\psi,V_{\nu x}[C_j,\Phi_j]V^*_{\nu x}A^I_{\nu,\mu}(x)\psi\big\rangle.\end{aligned}$$

Now, the scalar product in the first term can be written as

$$(i\mu)^{-1}\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}DV^*_{\nu x}\big(\partial_jA^I_{\nu,\mu}\big)(x)-D\big(\partial_jB^I_\mu\big)(x)\right\}\psi\right\rangle$$

with $D:=(H_0')^{-2}\zeta(H_0'^2)\eta(H_0)\in\mathscr{B}(\mathcal{H}_0)$. Thus, a further integration by parts leads to

$$L(\nu,\mu)=-\mu^{-2}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_j^2(\mathscr{F}f)\big](x)\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}DV^*_{\nu x}A^I_{\nu,\mu}(x)-DB^I_\mu(x)\right\}\psi\right\rangle\tag{5.8}$$

$$-i\mu^{-2}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_j(\mathscr{F}f)\big](x)\big\langle\psi,V_{\nu x}[D,\Phi_j]V^*_{\nu x}A^I_{\nu,\mu}(x)\psi\big\rangle\tag{5.9}$$

$$-\mu^{-1}\sum_j\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\,(\mathscr{F}f)(x)\big\langle\psi,V_{\nu x}[C_j,\Phi_j]V^*_{\nu x}A^I_{\nu,\mu}(x)\psi\big\rangle.\tag{5.10}$$

By setting $E_k:=(H_0')^{-4}\zeta(H_0'^2)(\partial_kH_0)\eta(H_0)\in\mathscr{B}(\mathcal{H}_0)$ and by performing a further integration by parts, one obtains that (5.8) is equal to

$$\begin{aligned}&i\mu^{-3}\sum_{j,k}\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_j^2(\mathscr{F}f)\big](x)\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}E_kV^*_{\nu x}\big(\partial_kA^I_{\nu,\mu}\big)(x)-E_k\big(\partial_kB^I_\mu\big)(x)\right\}\psi\right\rangle\\&\quad=-i\mu^{-3}\sum_{j,k}\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_k\partial_j^2(\mathscr{F}f)\big](x)\left\langle\psi,\frac{1}{\nu}\left\{V_{\nu x}E_kV^*_{\nu x}A^I_{\nu,\mu}(x)-E_kB^I_\mu(x)\right\}\psi\right\rangle\\&\qquad+\mu^{-3}\sum_{j,k}\int_{\mathbb{R}^d}\underline{\mathrm{d}}x\big[\partial_j^2(\mathscr{F}f)\big](x)\big\langle\psi,V_{\nu x}[E_k,\Phi_k]V^*_{\nu x}A^I_{\nu,\mu}(x)\psi\big\rangle.\end{aligned}$$

By mimicking the proof of Lemma 5.2, with $\eta(H_0)$ replaced by E_k, one obtains that there exist $\mathrm{c}, s > 0$ such that for all $|\mu| > 1$, $x \in \mathbb{R}^d$ and $\nu \in (-1, 1) \setminus \{0\}$

$$\left\| \frac{1}{\nu}\left\{V_{\nu x} E_k V_{\nu x}^* A_{\nu,\mu}^I(x) - E_k B_\mu^I(x)\right\}\right\| \leqslant \mathrm{c}\big(1+|\mu|\big)\langle x\rangle^s.$$

So, the terms (5.8) and (5.9) can be bounded uniformly in $\nu \in (-1,1)\setminus\{0\}$ by a function in $\mathsf{L}^1(\mathbb{R}\setminus[-1,1], \mathrm{d}\mu)$. For the term (5.10), a direct calculation shows that it can be written as

$$-i\mu^{-2}\sum_{j,k}\int_{\mathbb{R}} \underline{\mathrm{d}}x\,(\mathscr{F} f)(x)\big\langle V_{\nu x}^*\psi, [C_j, \Phi_j] V_{\nu x}^* C_k V_{\nu x}\big(\partial_k A_{\nu,-\mu}^I\big)(-x) V_{\nu x}^*\psi\big\rangle.$$

So, doing once more an integration by parts with respect to x_k, one also obtains that this term is bounded uniformly in $\nu \in (-1,1)\setminus\{0\}$ by a function in $\mathsf{L}^1(\mathbb{R}\setminus[-1,1], \mathrm{d}\mu)$.

The last estimates, together with our previous estimate for $|\mu| \leqslant 1$, show that $|L(\nu,\mu)|$ is bounded uniformly in $|\nu| < 1$ by a function in $\mathsf{L}^1(\mathbb{R}, \mathrm{d}\mu)$. So, one can interchange the limit $\nu \searrow 0$ and the integration over μ in (5.7). The interchange of the limit $\nu \searrow 0$ and the integration over x in (5.7) is justified by the bound obtained in Lemma 5.2. □

The existence of the usual time delay is now a direct consequence of Theorems 4.3 and 5.3:

Theorem 5.4. *Let H_0, f, H and Φ satisfy Assumptions 2.1, 2.2, 3.1, 4.1 and 5.1. For each $t \in \mathbb{R}$, let $L(t): \mathcal{H} \to \mathcal{H}_0$ satisfy $L(t)E^H(I) \in \mathscr{B}(\mathcal{H}, \mathcal{H}_0)$ for any bounded subset $I \subset \mathbb{R}$. Finally, let $\varphi \in \mathcal{H}_0^- \cap \mathscr{D}_2$ verify $S\varphi \in \mathscr{D}_2$, (4.6) and (5.5). Then one has*

$$\lim_{r\to\infty}\tau_r^{\mathrm{in}}(\varphi) = \lim_{r\to\infty}\tau_r(\varphi) = -\big\langle\varphi, S^*[T_f, S]\varphi\big\rangle,$$

with T_f defined by (3.2).

Remark 5.5. In $\mathsf{L}^2(\mathbb{R}^d)$, the position operators Q_j and the momentum operators P_j are related to the free Schrödinger operator by the commutation formula $P_j = i[-\Delta/2, Q_j]$. Therefore, if one interprets the collection $\{\Phi_1,\dots,\Phi_d\}$ as a family of position operators, then it is natural (by analogy to the Schrödinger case) to think of $H_0' \equiv (i[H_0,\Phi_1],\dots,i[H_0,\Phi_d])$ as a velocity operator for H_0. As a consequence, one can interpret the commutation assumption (5.5) as the conservation of (a function of) the velocity operator H_0' by the scattering process, and the meaning of Theorem 5.4 reduces to the following: If the scattering process conserves the velocity operator H_0', then the usual and the symmetrized time delays are equal.

There are several situations where the commutation assumption (5.5) is satisfied. Here we present three of them:

(i) Suppose that H_0 is of class $C^1(\Phi)$, and assume that there exists $v \in \mathbb{R}^d\setminus\{0\}$ such that $H_0' = v$. Then the operator $F_f(H_0')$ reduces to the scalar $F_f(v)$, and $[F_f(H_0'), S] = 0$ in $\mathscr{B}(\mathcal{H}_0)$. This occurs for instance in the case of Friedrichs-type and Stark operators (see [21, Section 7.1]).

(ii) Suppose that Φ has only one component and that $H_0' = H_0$. Then the operator $F_f(H_0') \equiv F_f(H_0)$ is diagonalizable in the spectral representation of H_0. We also know that S is decomposable in the spectral representation of H_0. Thus (5.5) is satisfied for each $\varphi \in \mathscr{D}_0$, since diagonalizable operators commute with decomposable operators. This occurs in the case of Φ-homogeneous operators H_0 such as the free Schrödinger operator (see [21, Section 7.2] and also [7, Sections 10 and 11]).

(iii) More generally, suppose that $F_f(H_0')$ is diagonalizable in the spectral representation of H_0. Then (5.5) is once more satisfied for each $\varphi \in \mathscr{D}_0$, since diagonalizable operators commute with decomposable operators. For instance, in the case of the Dirac operator and of dispersive systems with a radial symbol, we have neither $H_0' = v \in \mathbb{R}^d\setminus\{0\}$, nor $H_0' = H_0$. But if we suppose f radial, then $F_f(H_0')$ is nevertheless diagonalizable in the spectral representation of H_0 (see [21, Section 7.3] and [29, Remark 4.9]).

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