Twisted Crossed Products and Magnetic Pseudodifferential Operators

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Abstract

There is a connection between the Weyl pseudodifferential calculus and crossed product C^*-algebras associated with certain dynamical systems. And in fact both topics are involved in the quantization of a non-relativistic particle moving in $\mathbb{R}^N$. Our paper studies the situation in which a variable magnetic field is also present. The Weyl calculus has to be modified, giving a functional calculus for a family of operators (positions and magnetic momenta) with highly non-trivial commutation relations. On the algebraic side, the dynamical system is twisted by a cocycle defined by the flux of the magnetic field, leading thus to twisted crossed products. Following mainly [MP1] and [MP2], we outline the interplay between the modified pseudodifferential setting and the C^*-algebraic formalism at an abstract level as well as in connection with magnetic fields.

Key words and phrases: Magnetic field, gauge invariance, quantization, pseudodifferential operator, Weyl calculus, canonical commutation relations, Moyal product, dynamical system, twisted crossed products.

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Introduction

One of the purposes of the present article is to study the mathematical objects involved in the quantization of physical systems placed in a magnetic field. Two structures emerge naturally and in a correlated manner: (1) a modified form of the pseudodifferential Weyl calculus and (2) twisted crossed product C^*-algebras. In fact, the connection between (1) and (2) is present in a more general setting and showing this is also one of our aims. Under favorable circumstances, both (1) and (2) involve algebras of symbols defined on variables having a physical interpretation and the representations of these algebras may be seen as functional calculi associated with families of non-commuting observables.

Let us first explain roughly what we mean by “quantization”. We assume that a given physical system admits *a phase space*, at least in a weak sense. We do not give a precise definition of this concept, but approximatively it is a space containing the canonical coordinates, often interpreted as “positions” and “momenta”. These coordinates enter in the equations that describe in Classical Mechanics the time evolution of the system. The phase space is usually modelled mathematically by a symplectic manifold and its points are called *the states of the system*. The *observables*, the quantities that can be measured, are described by (smooth) functions on the phase space. On the other hand, the quantum description of the same system requires a Hilbert space and the observables are represented by self-adjoint or normal operators acting in the Hilbert space. Then a *quantization* would be a systematic procedure to assign quantum observables (operators) to classical ones (functions). This procedure should obey a set of principles, which are difficult to state once for all in a universal way.

We stress that - for the moment - our approach is only a first (important) step towards a deformation quantization à la Rieffel (cf. [Ri3] or [La]). All is done for a fixed value of Planck’s constant; in fact we take $\hbar = 1$. We hope to be able to let $\hbar$ vary in a subsequent publication.

We also note that the word “phase space” should not be taken too strictly; in particular it can differ from a cotangent bundle. If the configuration space is taken to be $\mathbb{Z}^N$ for instance, there is no cotangent bundle, but aside multiplication operators, one is also interested in finite-difference operators. Actually our “phase space” Ξ will be the direct product of an abelian, locally compact group X and its dual $X^\sharp$. X serves as a configuration space and $X^\sharp$ is somehow connected with a momentum observable, but this may be rather vague if not enough structure is present. Anyhow, the main interest of our work lies in the fact that this momentum observable can be “twisted”, the main example being the magnetic momentum. This means that it may correspond not to a usual action θ in the configuration space, but to a twisted action (θ, ω), where ω is a 2-cocycle of the group X with values in some general functions space.

Let us comment on the most important case, $X = \mathbb{R}^N$. A common point of view is that the Weyl form of the usual pseudodifferential calculus in $\mathbb{R}^N$ (cf. [Fo], [Hor], [Sh1]) is a functional calculus for the family of operators $(Q_1, \dots, Q_N, P_1, \dots, P_N)$, where Q_j is the operator of multiplication by the j’th coordinate and $P_j = -i\partial_j$. This calculus is suited to the description of a non-relativistic quantum particle moving in $\mathbb{R}^N$ and having no internal structure. Its complexity is due to the fact that the operators Q_j and P_j do not commute. If the particle is placed in a magnetic field B, the momentum P_j has to be replaced by the magnetic momentum $\Pi_j^A = P_j - A_j(Q)$, where A is a vector potential assigned to the magnetic field: $B = dA$. The problem of constructing a functional calculus for the more complicated family $(Q_1, \dots, Q_N, \Pi_1^A, \dots, \Pi_N^A)$ was tackled in [KO1], [KO2] and [MP2]; the result was called *the magnetic Weyl calculus*. Now there is a higher (but still manageable) degree of non-commutativity, in terms of phase factors defined by the flux of the magnetic field through suitable triangles and the circulation of the vector potential through suitable segments.

Crossed product C^*-algebras (for which we refer to [Pe] or to [RW]) originated in physics (cf. [DKR]) and had an exceptional career, very often independent of the initial motivation. There also exists a more involved version, but still very natural, called *twisted crossed product*. It was initiated in [BS] and found its full strength in [PR1] and [PR2]; a useful review is [Pa]. The motivations were of a pure mathematical nature. It was shown in [MP1] that the concept of twisted crossed product C^*-algebra is extremely natural in connection with the quantization of physical systems in the presence of magnetic fields. The point is that a reformulation of the magnetic commutation relations puts into evidence *a twisted dynamical system*, the twist being related to the flux of the magnetic field. This reveals automatically a formalism which is in a certain sense isomorphic to the magnetic Weyl calculus. But the C^*-algebraic version is more precise and flexible. It also gives a more transparent view on cohomological matters related to gauge covariance.

Essentially, this is an expository article, summarizing results of [MP1] and [MP2] and underlining the connection between them. But a large part of the approach is rephrased and several new results are included.

Let us describe the content. First of all, in an introductory section, we review briefly some basic facts about the quantization of a non-relativistic, spinless particle moving in the flat N-dimensional configuration space (in the absence of any magnetic field). This should be a motivation for the main body of the text, since the key notions (canonical commutation relations, Weyl calculus, dynamical systems and their crossed products) appear already in a simple form.

In Section 2 we review twisted dynamical systems, covariant representations and twisted crossed products. We are placed in a general setting, but with some simplifying assumptions which should make this topic more popular to non-specialists. Some general facts in group cohomology are also presented briefly, leading to a general form of gauge covariance. The assumption that the algebra on which the group acts is actually composed of bounded, uniformly continuous functions defined on the group leads to several specific properties, as the existence of a general class of Schrödinger-type representations. This also eases the way to the next section.

In Section 3, by a simple reformulation, we get objects generalizing in some sense the pseudodifferential calculus. It might be instructive to note that parts of the pseudodifferential theory depend only on a rather general setting and still work for commutation relations more involved than those satisfied by Q and P.

In Section 4 we show how all these particularize to the twisted dynamical system associated with a particle in a magnetic field. This physical case is the main motivation of our work.

Our interest in twisted crossed products and in the magnetic Weyl calculus lies beyond this general setting. In a forthcoming publication we will show that the twisted crossed product C^*-algebras contain the functional calculus of suitable classes of Schrödinger operators with magnetic fields. This opens the way towards extending spectral or propagation results as those of [ABG], [GI1], [Ma1], [AMP] to magnetic Schrödinger operators. This will be done by C^*-algebraic techniques, involving constructions and results of the present paper.

Our hope is that our approach could be interesting for at least three groups of people: C^*-algebraists (especially those involved in C^*-dynamical systems), people working in pseudodifferential theory and mathematical physicists interested in quantum systems with magnetic fields. Since none of the three needs to be an expert in the other two topics, we shall try to be rather elementary. We also defer more deep or more technical developments as well as applications to future works. The present purpose is just to show that a new pseudodifferential calculus is well justified physically, has a life of its own and can be recast in the language of twisted crossed products. A really critical point is C^*-algebras. Although they are no longer considered to be a close friend of the mathematical physicists working in quantum theory, our opinion is that they provide very useful techniques and insights and we intend to continue to use them in connection with magnetic fields. If one is interested in pseudodifferential operators with magnetic fields but without C^*-algebras, (s)he could consult [KO1], [KO2] and [MP2].

1 Zero magnetic field - a heuristic presentation

The class of physical systems for which a reliable quantization is already achieved is rather poor. The outstanding example is certainly the non-relativistic quantum particle moving in $\mathbb{R}^N$ and having no additional (internal) degree of freedom. We describe briefly this situation before introducing any magnetic field, since this serves both as a motivation and as a comparison theory.

1.1 The Weyl calculus - a paradigm of quantization

We consider a non-relativistic particle without internal structure, moving in the configuration space $X \equiv \mathbb{R}^N$. The phase space of this system is the cotangent bundle $T^*X = X \times X^\star \equiv \mathbb{R}^{2N}$, denoted from now on by Ξ, on which we have the canonical symplectic form

$$\sigma(\xi, \eta) = \sigma\left[(x,p),(y,k)\right] := y \cdot p - x \cdot k,$$

where $y \cdot p$ is the action of the linear functional $p \in X^\star \equiv \mathbb{R}^N$ on the vector $y \in X$. Thus a classical observable is just a (smooth real or complex) function defined on T^*X. The symplectic form serves among others in defining the Poisson bracket

$$\{f, g\} := \sigma(\nabla f, \nabla g) = \sum_{j=1}^{N} \left(\partial_{p_j} f \; \partial_{q_j} g - \partial_{p_j} g \; \partial_{q_j} f\right),$$

endowing the set of observables with the structure of a Lie algebra. Here we have made the usual abuse of identifying the tangent space at a point $\xi \in \Xi$ with the linear space Ξ itself and thus extending the canonical symplectic form σ to the tangent bundle $T\Xi$. This Poisson bracket plays an important role in the Hamiltonian formulation of the equations of motion.

Let us now describe briefly and rather formally the quantization of this system. One first deals with the coordinate functions $q_1, \dots q_N$ and $p_1, \dots, p_N$ satisfying the relations

$$\{q_i, q_j\} = 0,\ \{p_i, p_j\} = 0,\ \{p_i, q_j\} = \delta_{ij}, \qquad i, j = 1, \dots, N.$$

A dogma of the physical community is to assign to them the self-adjoint operators $Q_1, \dots, Q_N, P_1, \dots, P_N$ satisfying the same relations but with the Poisson bracket $\{\cdot, \cdot\}$ replaced by $i[\cdot, \cdot]$ (here $[S, T] := ST - TS$ is the commutator). Thus we should have

$$i[Q_i, Q_j] = 0,\ i[P_i, P_j] = 0,\ i[P_i, Q_j] = \delta_{ij}, \qquad i, j = 1, \dots, N. \tag{1.1}$$

It is known that if one also asks the irreducibility of the family $Q_1, \dots, Q_N, P_1, \dots, P_N$ and identifies unitarily equivalent families, then there is only one possible choice for Q and P : the Hilbert space is $\mathcal{H} := L^2(X)$, Q_j is the operator of multiplication by x_j and P_j is the differential operator $-i\partial_j$. For the exact form of this statement, involving the unitary groups generated by Q and P, called *the Stone-von Neumann Theorem*, see [Fo].

The natural next step is the quantization of more general functions. This can be seen as the problem of constructing a functional calculus $f \mapsto f(Q, P)$ for the family $Q_1, \dots, Q_N, P_1, \dots, P_N$ of $2N$ self-adjoint, non-commuting operators. One would also like to define a "quantum" multiplication $(f, g) \mapsto f \circ g$ satisfying $(f \circ g)(Q, P) = f(Q, P)g(Q, P)$ as well as an involution $f \to f^\circ$ leading to $f^\circ(Q, P) = f(Q, P)^*$. The deviation of $\circ$ from pointwise multiplication is imputable to the fact that Q and P do not commute. The solution of these problems is called *the Weyl calculus.* The prescription is $f(Q, P) = \mathfrak{Op}(f)$, with

$$[\mathfrak{Op}(f)u](x) := \int_{\mathbb{R}^{2N}} dy\, dp\, e^{i(x-y)\cdot p} f\left(\frac{x+y}{2}, p\right) u(y), \qquad u \in \mathcal{H}, \tag{1.2}$$

the involution is $f^\circ(\xi) := \overline{f(\xi)}$ and the multiplication (called *the Moyal product*) is

$$(f \circ g)(\xi) := 4^N \int_\Xi d\eta \int_\Xi d\zeta\, e^{-2i\sigma(\xi-\eta, \xi-\zeta)} f(\eta)\, g(\zeta), \qquad \xi \in \Xi. \tag{1.3}$$

The two formulae must be taken with some care: for many symbols f and g they need a suitable reinterpretation.

Let us try to show where all these come from. We consider the strongly continuous unitary maps $X \ni x \mapsto U(x) := e^{-ix\cdot P} \in \mathcal{U}(\mathcal{H})$ and $X^\star \ni p \mapsto V(p) := e^{-iQ\cdot p} \in \mathcal{U}(\mathcal{H})$, acting on $\mathcal{H}$ as

$$[U(x)u](y) = u(y-x) \quad \text{and} \quad [V(p)u](y) = e^{-iy\cdot p}\, u(y), \qquad u \in \mathcal{H},\ y \in X.$$

These operators satisfy the *Weyl form of the canonical commutation relations*

$$U(x)V(p) = e^{ix\cdot p}\, V(p)U(x), \qquad x \in X,\ p \in X^\star, \tag{1.4}$$

as well as the identities $U(x)U(x') = U(x')U(x)$ and $V(p)V(p') = V(p')V(p)$ for $x, x' \in X$ and $p, p' \in X^\star$. These can be considered as a reformulation of (1.1) in terms of bounded operators.

A convenient way to condense the maps U and V in a single one is to define *the Schrödinger Weyl system* $\{W(x, p) \mid x \in X,\ p \in X^\star\}$ by

$$W(x, p) := e^{\frac{i}{2}x\cdot p}\, U(-x)V(p) = e^{-\frac{i}{2}x\cdot p}\, V(p)U(-x), \tag{1.5}$$

which satisfies the relation $W(\xi)W(\eta) = e^{\frac{i}{2}\sigma(\xi,\eta)}\, W(\xi+\eta)$ for any $\xi, \eta \in \Xi$. This equality encodes all the commutation relations between the basic operators Q and P. Explicitly, the action of W on $u \in \mathcal{H}$ is given by

$$[W(x, p)u](y) = e^{-i\left(\frac{1}{2}x+y\right)\cdot p}\, u(y+x), \qquad x, y \in X,\ p \in X^\star. \tag{1.6}$$

For a family of m commuting self-adjoint operators $S_1, \dots, S_m$ one usually defines a functional calculus by the formula $f(S) := \int_{\mathbb{R}^m} dt\, \check{f}(t)e^{-it\cdot S}$, where $t \cdot S = t_1S_1 + \ldots + t_mS_m$ and $\check{f}$ is the inverse Fourier transform of f, conveniently normalized. The formula (1.2) can be obtained by a similar computation. For that purpose, let us define the *symplectic Fourier transformation* $\mathcal{F}_\Xi :\ \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ by

$$(\mathcal{F}_\Xi f)(\xi) := \int_\Xi d\eta\, e^{i\sigma(\xi,\eta)} f(\eta).$$

Now, for any function $f : \Xi \to \mathbb{C}$ belonging to the Schwartz space $\mathcal{S}(\Xi)$, we set

$$\mathfrak{Op}(f) := \int_\Xi d\xi\, (\mathcal{F}_\Xi^{-1} f)(\xi)\, W(\xi). \tag{1.7}$$

By using (1.6), one gets formula (1.2). Then it is easy to verify that the relation $\mathfrak{Op}(f)\mathfrak{Op}(g) = \mathfrak{Op}(f \circ g)$ holds for $f, g \in \mathcal{S}(\Xi)$ if one uses the Moyal product introduced in (1.3).

1.2 Connection with C^*-dynamical systems and crossed products

One can turn to the crossed product formalism just by examining the composition law $\circ$ in conjunction with a partial Fourier transformation. It is more instructive to take the natural path and recover it by reformulating the canonical commutation relations, so as to get a dynamical system.

The family $\{V(p)\}_{p\in X^\star}$ is just part of the functional calculus of the position operator Q, defined by the Spectral Theorem. For any Borel function $a : X \to \mathbb{C}$ one denotes by $a(Q)$ the normal (unbounded) operator in $L^2(X)$ of multiplication by the function a. The formula (1.4) is just a particular case of the more general one

$$U(-x)\,a(Q)\,U(x) = a(Q + x), \qquad x \in X.$$

Usually one works with a's belonging to some C^*-algebra $\mathcal{A}$ of continuous functions on X and $a \to a(Q)$ becomes a representation of $\mathcal{A}$. In its turn, U is a unitary, strongly continuous group representation. Let us now point out a general framework encompassing this situation.

Definition 1.1. A *C^*-dynamical system* is a triple $(\mathcal{A}, \theta, X)$ formed by a locally compact group X, a C^*-algebra $\mathcal{A}$ and a group morphism $\theta : X \to \mathfrak{Aut}(\mathcal{A})$ of X into the group of automorphisms of $\mathcal{A}$ which is continuous in the sense that for any $a \in \mathcal{A}$, the map $X \ni x \mapsto \theta_x(a) \in \mathcal{A}$ is continuous.

Definition 1.2. A *covariant representation* of the C^*-dynamical system $(\mathcal{A}, \theta, X)$ is a triple $(\mathcal{H}, r, T)$, where $\mathcal{H}$ is a (separable) Hilbert space, $r : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a non-degenerate $(^*)$-representation of $\mathcal{A}$ in $\mathcal{H}$ and $T : X \to \mathcal{U}(\mathcal{H})$ a strongly continuous unitary representation of X in $\mathcal{H}$, such that for all $x \in X$ and $a \in \mathcal{A}$ one has $T(x)r(a)T(-x) = r[\theta_x(a)]$.

It is quite obvious how to recover our Quantum Mechanical framework from these general definitions. X is the configuration space $\mathbb{R}^N$ and θ is the action of X by translations on some suitable C^*-algebra $\mathcal{A}$ of functions on X: $[\theta_x(a)](y) := a(y + x)$. To be "suitable" $\mathcal{A}$ has to be stable under translations ($a \in \mathcal{A} \Rightarrow \theta_x(a) \in \mathcal{A}$, $\forall x \in X$) and composed of bounded, uniformly continuous functions, because this ensures the desired continuity of θ. A covariant representation is recovered by taking $\mathcal{H} = L^2(X)$, $T(x) = U(-x) = e^{+ix\cdot P}$ and $r(a) = a(Q)$. This natural representation is called *the Schrödinger representation.* In a certain sense, the kinematics of our quantum particle is described by two composed objects. First a classical setting, consisting of a configuration space and a group acting in a particular way on this configuration space, the action being then raised to functions. Second, a quantum implementation of the classical system in a Hilbert space by a suitable algebra of position observables and a unitary representation of the group, the two satisfying some natural commutation relations. We note that we are quite close of the concept of *imprimitivity system*, often used in the foundational theory of Quantum Mechanics; we refer to [Va].

A digression: With any locally compact group X (we shall assume it abelian) one associates naturally *the group C^*-algebra* $C^*(X)$. It is the completion in some suitable norm of the space $L^1(X)$ (with respect to the Haar measure), which is a Banach *-algebra with the involution $\alpha^*(x) := \overline{\alpha(-x)}$ and the convolution product $(\alpha \star \beta)(x) = \int_X dy\, \alpha(y)\, \beta(x - y)$. The virtue of $C^*(X)$ consists in the fact that its non-degenerate representations are in one-to-one correspondence with the unitary representations of the group X. In one direction, if $T : X \to \mathcal{U}(\mathcal{H})$ is such a group representation, then $s_{\mathrm{T}}(\alpha) := \int_X dx\, \alpha(x) T(x)$ is a representation of the Banach *-algebra $L^1(X)$, that extends to a representation of the group C^*-algebra. Since the group X was taken abelian, a Fourier transformation realizes an isomorphism between $C^*(X)$ and the C^*-algebra $\mathcal{C}_0(X^\star)$ (with the pointwise operations and the sup-norm) of all continuous functions vanishing at infinity on the dual $X^\star$. By composing s_{T} with the isomorphism one gets a representation σ_{T} of $\mathcal{C}_0(X^\star)$. In the particular case in which $X = \mathbb{R}^N$ is represented in $L^2(X)$ by translations, $T(x) = e^{ix\cdot P}$, this representation is exactly the functional calculus of the momentum operator: $\sigma_{\mathrm{T}}(b) = b(P)$, $\forall b \in \mathcal{C}_0(X^\star)$. We see now why the L^1-norm has to be replaced: not only a C^*-norm has better technical properties, but, very concretely, the norm of the operator $b(P)$ equals the sup-norm of b and not the L^1-norm of its Fourier transform.

After this digression, let us come back to the above C^*-dynamical system $(\mathcal{A}, \theta, X)$, covariantly represented by some arbitrary $(\mathcal{H}, r, T)$. We would like to define a single C^*-algebra containing in some subtle sense both $\mathcal{A}$ and $C^*(X) \cong \mathcal{C}_0(X^\star)$ and taking also into account the action of X by automorphisms of $\mathcal{A}$. And, hopefully, the two maps composing the covariant representation should be condensed into a single representation of this large C^*-algebra. Very roughly, for our Quantum Mechanical problem, one tries to put together position and momentum observables such that their commutation rules are respected. This was also our point of view on the pseudodifferential calculus in Subsection **1.1**, but with a different presentation of the commutation relations. It comes out that the right construction is as follows:

(1) On the Banach space $L^1(X;\mathcal{A})$ (with the natural norm) one defines an involution $\phi^\diamond(x) := [\phi(-x)]^*$, * being the involution in $\mathcal{A}$, and a composition law

$$(\phi \diamond \psi)(x) := \int_X dy\, \theta_{(y-x)/2}\, [\phi(y)]\, \theta_{y/2}\, [\psi(x-y)]\,. \tag{1.8}$$

Then $L^1(X;\mathcal{A})$ becomes a Banach *-algebra.

(2) One introduces the C^*-norm $\|\phi\| := \sup\|\pi(\phi)\|$, the "sup" being taken over all the non-degenerate representations of $L^1(X;\mathcal{A})$. The completion of $L^1(X;\mathcal{A})$ with respect to $\|\cdot\|$ is denoted by $\mathcal{A} \rtimes_\theta X$ and is called *the crossed product of* $\mathcal{A}$ *by the action* θ *of the group* X. Obviously this generalizes the concept of group C^*-algebra, that we recover for $\mathcal{A} = \mathbb{C}$.

Let us point out that in order to recover the Weyl calculus and not the Kohn-Nirenberg pseudodifferential calculus we have a slightly unusual form for the product operation in the crossed product algebra. In fact this is isomorphic with the usual one (see for example **3.1**).

Now, for a given covariant representation $(\mathcal{H}, r, T)$, one sets $r \rtimes T : \mathcal{A} \rtimes_\theta X \to \mathcal{B}(\mathcal{H})$, uniquely defined by the action on $L^1(X;\mathcal{A})$:

$$(r \rtimes T)(\phi) := \int_X dx\, r\left[\theta_{x/2}\big(\phi(x)\big)\right] T(x).$$

This is, indeed, a non-degenerate representation of the C^*-algebra $\mathcal{A} \rtimes_\theta X$. In fact there is also a converse construction, hence there are no other non-degenerate representations.

In the special case ($X = \mathbb{R}^N$, action by translations on an invariant C^*-algebra of bounded, uniformly continuous functions on X) *the composition law* $\diamond$ *is isomorphic to the Moyal product* (1.3). The isomorphism is just a partial Fourier transformation, also transporting the involutions $^\diamond$ and $^\circ$ one into the other. If $(\mathcal{H}, r, T)$ is the Schrödinger representation, then by composing $r \rtimes T$ with the isomorphism one gets the mapping $\mathfrak{Op}$. So, in this particular case, the crossed product is just another form of the Weyl pseudodifferential calculus. Remark that the possibility of choosing the algebra $\mathcal{A}$ will offers a flexibility which was not evident at the pseudodifferential level. In the sequel we shall extend all these facts to a much larger setting.

2 Twisted crossed products

This section is mainly dedicated to a brief summary of twisted C^*-dynamical systems, twisted crossed products and of their representations. We follow the standard references [BS], [PR1], [PR2] and [Pa]. To simplify, we undertake various hypotheses which are not needed for part of the arguments. Primarily, we assume that an *abelian* locally compact group acts upon an *abelian* C^*-algebra. This will favour later on the use of Fourier transforms and of Gelfand theory. Instead of working in terms of universal properties, we treat the twisted crossed product as the envelopping C^*-algebra of a L^1-type Banach *-algebra. To make the transition towards pseudodifferential operators and the magnetic case, we introduce at the end of the present section a special type of twisted crossed products, in which the algebra is composed of continuous functions defined on the group. It is preceded and prepared by some considerations in group cohomology. Our setting is that of Polish modules, as in [Mo], but we use continuous and not Borel cochains, so [Gui] is also relevant for our framework.

2.1 Twisted C^*-dynamical systems and their covariant representations

Let us start abruptly with the relevant definition and explain afterwards its ingredients. Some of the explanations will be used only later on.

Definition 2.1. We call *abelian twisted* C^**-dynamical system* a quadruplet $(\mathcal{A}, \theta, \omega, X)$, where

(i) X is an abelian, second countable locally compact group,

(ii) $\mathcal{A}$ is an abelian, separable C^*-algebra,

(iii) $\theta : X \to \mathfrak{Aut}(\mathcal{A})$ is a group morphism from X to the group of automorphisms of $\mathcal{A}$, such that $x \mapsto \theta_x(a)$ is continuous for all $a \in \mathcal{A}$,

(iv) ω is a strictly continuous normalized 2-cocycle on X with values in the unitary group of the multiplier algebra of $\mathcal{A}$.

The couple (θ, ω) is called *a twisted action of X on $\mathcal{A}$*. Very often, we shall use the shorter expression *twisted dynamical system* for the quadruplet $(\mathcal{A}, \theta, \omega, X)$.

Remarks:

(A) Almost everything in this section would be true, with only some slight modifications, without assuming $\mathcal{A}$ and X to be abelian. However, our main interest lies in the connection between twisted dynamical systems and pseudodifferential theories. And for this purpose commutativity is extremely useful, almost essential. Therefore we do assume it from the very beginning. Separability conditions are needed to remain as close as possible to standard references ([BS], [PR1], [PR2]), but also to assign to $(\mathcal{A}, \theta, \omega, X)$ a "perfect" group cohomology in the sense of Moore (see [Mo] and **2.3**). We intend to weaken the separability condition on $\mathcal{A}$ in a forthcoming publication.

(B) The 2-cocycle, introduced at (iv) and explained at (D), has to be unitary-valued. In a non-unital C^*-algebra unitarity makes no sense, which requires the introduction of the multiplier algebra. We refer to [Pe] or to [RW] for this useful concept and recall only some simple facts. It is known that any non-unital C^*-algebra $\mathcal{A}$ can be embedded into several larger unital C^*-algebras as an essential ideal ($\mathcal{A}$ is an essential ideal of $\mathfrak{M}$ if $\mathcal{A} \cap \mathcal{I} \neq 0$ for all non-zero closed ideals $\mathcal{I}$ in $\mathfrak{M}$). Among these algebras there is a largest one, unique up to isomorphisms, called *the multiplier algebra* and denoted by $\mathfrak{M}(\mathcal{A})$. It can be introduced nicely as the class of all double centralizers of $\mathcal{A}$, but a concrete definition might be easier to grasp. Since $\mathcal{A}$ can be represented faithfully in some Hilbert space $\mathcal{H}$, we shall just imagine that $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then one sets $\mathfrak{M}(\mathcal{A}) := \{T \in \mathcal{B}(\mathcal{H}) \mid Ta,\ aT \in \mathcal{A},\ \forall a \in \mathcal{A}\}$. It can be shown that $\mathfrak{M}(\mathcal{A})$ is also commutative and that a different faithful representation would lead to some isomorphic copy of $\mathfrak{M}(\mathcal{A})$. Let us notice that if the C^*-algebra $\mathcal{A}$ is already unital, then one has naturally $\mathcal{A} = \mathfrak{M}(\mathcal{A})$. The C^*-algebra $\mathfrak{M}(\mathcal{A})$ is endowed with a second natural topology, the topology generated by the family of seminorms $m \mapsto \|ma\|$ for all $a \in \mathcal{A}$, $\|\cdot\|$ being the norm of $\mathcal{B}(\mathcal{H})$. This topology, with respect to which $\mathfrak{M}(\mathcal{A})$ is complete and contains $\mathcal{A}$ densely, is called *the strict topology.*

(C) Since $\mathfrak{M}(\mathcal{A})$ is unital, we can consider the unitary group $\mathcal{U}(\mathcal{A}) \equiv \mathcal{U}\mathfrak{M}(\mathcal{A}) := \{m \in \mathfrak{M}(\mathcal{A}) \mid m^* m = 1\}$, simply called *the unitary group* of $\mathcal{A}$. By restricting the strict topology to $\mathcal{U}(A)$ we get a topological group. Since $\mathcal{A}$ was supposed separable, its topology is Polish (metrizable, separable and complete).

(D) If one forgets about ω, the remaining data $(\mathcal{A}, \theta, X)$ form a *C^*-dynamical system* (with plenty of extra assumptions) according to [DKR], [Pe] or [RW]; see also Definition 1.1. Let us now explain the point (iv) of Definition 2.1. A 2-cocycle is a function $\omega : X \times X \to \mathcal{U}(\mathcal{A})$, continuous with respect to the strict topology on $\mathcal{U}(\mathcal{A})$, such that for all $x, y, z \in X$:

$$\omega(x + y, z)\omega(x, y) = \theta_x[\omega(y, z)]\omega(x, y + z). \tag{2.1}$$

We shall also assume it to be normalized:

$$\omega(x, 0) = \omega(0, x) = 1, \qquad \text{for all } x \in X. \tag{2.2}$$

It is known that any automorphism of $\mathcal{A}$ extends uniquely to an automorphism of $\mathfrak{M}(\mathcal{A})$ and, obviously, leaves $\mathcal{U}(\mathcal{A})$ invariant. By applying this fact to θ_x and by denoting the extension with the same symbol, one gives a sense to (2.1). Actually, by suitable particularizations in (2.1), we get $\theta_{-x}[\omega(x, 0)] = \omega(0, 0) = \omega(0, x)$, $\forall x \in X$, hence for normalization it suffices to ask $\omega(0, 0) = 1$. The required continuity can be rephrased by saying that for any $a \in \mathcal{A}$, the map $X \times X \ni (x, y) \mapsto a\omega(x, y) \in \mathcal{A}$ is continuous. In fact Borel conditions could be imposed instead of continuity for most of the constructions and results; we do not pursue this here. "2-cocycle" is a concept belonging to group cohomology. We shall give further details in Subsection **2.3**.

Let us now discuss some special features due to the fact that we assume $\mathcal{A}$ abelian.

(E) Gelfand theory describes completely the structure of *abelian* C^*-algebras. We first note that if S is a locally compact space, then $\mathcal{C}_0(S) := \{a : S \to \mathbb{C} \mid a \text{ continuous}, a \to 0 \text{ when } x \to \infty\}$ is an abelian C^*-algebra with the operations defined pointwise and the sup-norm. $\mathcal{C}_0(S)$ has a unit if and only if S is compact. Actually *all abelian C^*-algebras are of this form.* We define *the Gelfand spectrum $S_\mathcal{A}$* of $\mathcal{A}$ to be the family of all characters of $\mathcal{A}$ (*a character* is just a morphism $\nu : \mathcal{A} \to \mathbb{C}$). With the topology of simple convergence $S_\mathcal{A}$ is a locally compact space, which is compact exactly when $\mathcal{A}$ is unital. And the mapping $\mathfrak{G} : \mathcal{A} \to \mathcal{C}_0(S_\mathcal{A})$ given by $[\mathfrak{G}(a)](\nu) := \nu(a)$, for $a \in \mathcal{A}$ and $\nu \in S_\mathcal{A}$ is an isomorphism.

(F) If the C^*-algebra $\mathcal{A} \cong \mathcal{C}_0(S_\mathcal{A})$ is not unital, then $\mathcal{BC}(S_\mathcal{A})$, the C^*-algebra of all bounded and continuous complex functions on $S_\mathcal{A}$, surely is. It contains $\mathcal{C}_0(S_\mathcal{A})$ as an essential ideal. In fact $\mathcal{BC}(S_\mathcal{A})$ can be identified with the multiplier algebra $\mathfrak{M}(\mathcal{A})$ of $\mathcal{A}$ (see [RW]). Thus the unitary group of $\mathcal{A}$ is identified with $\mathcal{C}(S_\mathcal{A}, \mathbb{T})$, the family of all continuous functions on $S_\mathcal{A}$ taking values in the group $\mathbb{T}$ of complex numbers of absolute value 1. Moreover, the strict topology on $\mathcal{C}(S_\mathcal{A}, \mathbb{T})$ coincides with the topology of uniform convergence on compact subsets of $S_\mathcal{A}$.

(G) Abelian C^*-algebras being so special $\big($cf. (E)$\big)$, the corresponding C^*-dynamical systems are also special. They are in fact given by topological dynamical systems and this explains the terminology. The central remark is that the only automorphisms of $\mathcal{C}_0(S_\mathcal{A})$ are those implemented by homeomorphisms of the underlying locally compact space $S_\mathcal{A}$. Here "implementation" means just composition of the elements of $\mathcal{C}_0(S_\mathcal{A})$ with the homeomorphism. Thus θ induces an action of X through homeomorphisms of $S_\mathcal{A}$.

The coherent way to represent a twisted dynamical system in a Hilbert space is given by

Definition 2.2. Given a twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$, we call *covariant representation* a Hilbert space $\mathcal{H}$ together with two maps $r : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $T : X \to \mathcal{U}(\mathcal{H})$ satisfying

(i) r is a non-degenerate representation,

(ii) T is strongly continuous and $T(x)T(y) = r[\omega(x, y)]T(x + y), \quad \forall x, y \in X,$

(iii) $T(x)r(a)T(x)^* = r[\theta_x(a)], \quad \forall x \in X,\ a \in \mathcal{A}.$

One observes that T is a sort of generalized projective representation of X. The usual notion of projective representation corresponds to the case in which for all $x, y \in X$, $\omega(x, y) \in \mathbb{T}$, i.e. "$\omega(x, y)$ is a constant function on the spectrum $S_\mathcal{A}$ of $\mathcal{A}$". Let us already mention that constant magnetic fields lead to such a situation. Condition (iii) says that at a represented level the automorphism θ_x is implemented by the unitary equivalence associated to $T(x)$. It could also be interpreted as an a priori prescribed commutation rule between the elements of the group and the elements of the algebra when they are put together in $\mathcal{B}(\mathcal{H})$ by representations.

2.2 Twisted crossed products and their representations

Let $(\mathcal{A}, \theta, \omega, X)$ be a twisted dynamical system. We start by mixing together the algebra $\mathcal{A}$ and the space $L^1(X)$ in a way to form a Banach *-algebra. We define $L^1(X; \mathcal{A})$, the Bochner integrable equivalence classes of $\mathcal{A}$-valued functions (with respect to the Haar measure), endowed with the norm $\|\phi\| := \int_X dx\, \|\phi(x)\|_\mathcal{A}$. Let us also fix an element τ of the set $\mathfrak{End}(X)$ of continuous endomorphisms of X. Particular cases are $\mathfrak{o}, \mathfrak{1} \in \mathfrak{End}(X)$, $\mathfrak{o}(x) := 0$ and $\mathfrak{1}(x) := x$, for all $x \in X$. Addition and substraction of endomorphisms are well-defined. For elements ϕ, ψ of $L^1(X; \mathcal{A})$ and for any point $x \in X$ we set

$$(\phi \diamond^\omega_\tau \psi)(x) := \int_X dy\, \theta_{\tau(y-x)}\,[\phi(y)]\; \theta_{(\mathfrak{1}-\tau)y}\,[\psi(x - y)]\; \theta_{-\tau x}\,[\omega(y, x - y)] \tag{2.3}$$

and (a^* is the adjoint of a in $\mathcal{A}$)

$$\phi^{\diamond^\omega_\tau}(x) := \theta_{-\tau x}[\omega(x, -x)^{-1}]\,\theta_{(\mathfrak{1}-2\tau)x}\,[\phi(-x)^*]\,. \tag{2.4}$$

The expression (2.4) becomes much simpler if $\omega(x, -x) = 1$, which will be the case in most of the applications.

Lemma 2.3. *For two functions ϕ and ψ in $L^1(X; \mathcal{A})$ and for $\tau \in \mathfrak{End}(X)$, the function $\phi \diamond^\omega_\tau \psi$ belongs to $L^1(X; \mathcal{A})$. With the composition law $\diamond^\omega_\tau$ and the involution $^{\diamond^\omega_\tau}$, $L^1(X; \mathcal{A})$ is a Banach *-algebra. These Banach *-algebras are isomorphic for different τ's.*

Proof. The fact that $L^1(X; \mathcal{A})$ is stable under $\diamond^\omega_\tau$ follows from the relations

$$\|\theta_{\tau(y-x)}\,[\phi(y)]\,\theta_{(\mathfrak{1}-\tau)y}\,[\psi(x - y)]\,\theta_{-\tau x}\,[\omega(y, x - y)]\,\|_\mathcal{A} \le \|\phi(y)\|_\mathcal{A}\|\psi(x - y)\|_\mathcal{A},$$

$$\textstyle\int_X dx\; \|(\phi \diamond^\omega_\tau \psi)(x)\|_\mathcal{A} \le \int_X dx \int_X dy\; \|\phi(y)\|_\mathcal{A}\|\psi(x - y)\|_\mathcal{A} = \|\phi\|\|\psi\|.$$

The associativity of this composition law is easily deduced from the 2-cocycle property of ω. All the other requirements also follow by routine calculations.

The isomorphisms are the mappings

$$m_{\tau,\tau'} : L^1(X; \mathcal{A}) \to L^1(X; \mathcal{A}), \quad (m_{\tau,\tau'}\phi)\,(x) := \theta_{(\tau'-\tau)x}[\phi(x)], \qquad x \in X.$$

On the first copy of L^1 one considers the structure defined by τ' and on the second that defined by τ. Note the obvious relations $m_{\tau,\tau'}m_{\tau',\tau''} = m_{\tau,\tau''}$ and $[m_{\tau,\tau'}]^{-1} = m_{\tau',\tau}$ for all $\tau, \tau', \tau'' \in \mathfrak{End}(X)$. ☐

One finds in the literature only the case $\tau = \mathfrak{o}$. We introduced all these isomorphic structures because they help in understanding τ-quantizations in pseudodifferential theory.

We recall that *a C^*-norm* on a *-algebra has to satisfy $\|a^*a\| = \|a\|^2$. Since C^*-norms have many technical advantages and since $\|\!|\cdot|\!\|$ has not this C^*-property, we shall make now some adjustments, valid in an abstract setting. A Banach *-algebra $\mathcal{C}$ with norm $\|\!|\cdot|\!\|$ is called *an A^*-algebra* when it admits a C^*-norm or, equivalently, when it has an injective representation in a Hilbert space [Ta]. In this case we can consider *the standard C^*-norm* on it, defined as the supremum of all the C^*-norms, that we shall denote by $\|\cdot\|$. A rather explicit formula for $\|\cdot\|$ is $\|b\| = \sup\{\|\pi(b)\|_{\mathcal{B}(\mathcal{H})} \mid (\pi, \mathcal{H}) \text{ is a representation}\}$. One has $\|\phi\| \le \|\!|\phi|\!\|$ for all ϕ. The completion with respect to this norm will be a C^*-algebra containing $\mathcal{C}$ as a dense *-subalgebra. We call it *the envelopping C^*-algebra of $\mathcal{C}$*. It is known that $\big(L^1(X;\mathcal{A}), \diamond^\omega_\tau, ^{\diamond^\omega_\tau}, \|\!|\cdot|\!\|\big)$ is indeed an A^*-algebra.

Definition 2.4. The envelopping C^*-algebra of $\big(L^1(X;\mathcal{A}), \diamond^\omega_\tau, ^{\diamond^\omega_\tau}, \|\!|\cdot|\!\|\big)$ will be called *the twisted crossed product of $\mathcal{A}$ by X associated with the twisted action (θ, ω) and the endomorphism τ*. It will be denoted by $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$.

The C^*-algebra $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ has a rather abstract nature. But most of the time one uses efficiently the fact that $L^1(X;\mathcal{A})$ is a dense *-subalgebra, on which everything is very explicitly defined. Let us even observe that the algebraic tensor product $\mathcal{A} \odot L^1(X)$ may be identified with the dense *-subspace of $L^1(X;\mathcal{A})$ (hence of $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ also) formed of functions with finite-dimensional range. The isomorphism $m_{\tau,\tau'}$ extends nicely to an isomorphism from $\mathcal{A} \rtimes^\omega_{\theta,\tau'} X$ to $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$.

The next lemma shows clearly the importance of twisted crossed products as a way to bring together the informations contained in a twisted dynamical system.

Lemma 2.5. *If $(\mathcal{H}, r, T)$ is a covariant representation of $(\mathcal{A}, \theta, \omega, X)$ and $\tau \in \mathfrak{End}(X)$, then $r \rtimes_\tau T$ defined on $L^1(X;\mathcal{A})$ by*

$$(r \rtimes_\tau T)\phi := \int_X dx\, r\left[\theta_{\tau x}\big(\phi(x)\big)\right] T(x)$$

extends to a representation of $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$, called the integrated form of (r, T). *One has $r \rtimes_{\tau'} T = (r \rtimes_\tau T) \circ m_{\tau,\tau'}$ if $\tau, \tau' \in \mathfrak{End}(X)$.*

Proof. Some easy computations show that $r \rtimes_\tau T$ is a representation of the Banach *-algebra $\big(L^1(X;\mathcal{A}), \diamond^\omega_\tau, ^{\diamond^\omega_\tau}\big)$. Then, by taking into account the formula for $\|\cdot\|$, one gets $\|(r \rtimes_\tau T)\phi\|_{\mathcal{B}(\mathcal{H})} \le \|\phi\|$, $\forall \phi \in L^1(X;\mathcal{A})$. Thus $r \rtimes_\tau T$ extends to $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ by density and, by approximation, this extension has all the required algebraic properties.

The relation $r \rtimes_{\tau'} T = (r \rtimes_\tau T) \circ m_{\tau,\tau'}$ is checked readily on $L^1(X;\mathcal{A})$ and obviously extends to the full twisted crossed product. $\square$

We note that one can recover the covariant representation from $r \rtimes_\tau T$. Actually, there is a one-to-one correspondence between covariant representations of a twisted dynamical system and non-degenerate representations of the twisted crossed product. This correspondence preserves equivalence, irreducibility and direct sums. We do not give explicit formulae, since we do not use them.

Finally, we would like to show that $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ is generated in some sense (by using representations) by $\mathcal{A}$ and the L^1-space of the group X. The next result was included without proof in [MP1]. For completeness, we prove it here by using the approach of [GI1] for the untwisted case.

Proposition 2.6. *Let $(\mathcal{H}, r, T)$ be a covariant representation of the twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$.*

(a) *The mapping $s_T : L^1(X) \to \mathcal{B}(\mathcal{H})$ given by $s_T(b) := \int_X dx\, b(x)T(x)$ is a linear contraction,*

(b) *The norm closure of the vector space generated by $\{r(a)s_T(b) \mid a \in \mathcal{A},\, b \in L^1(X)\}$ is equal to the norm closure of the vector space generated by $\{s_T(b)r(a) \mid a \in \mathcal{A},\, b \in L^1(X)\}$. Furthermore, both coincide with the C^*-algebra $(r \rtimes_\tau T)\Big(\mathcal{A} \rtimes^\omega_{\theta,\tau} X\Big)$ for any $\tau \in \mathfrak{End}(X)$.*

Proof. The proof of statement (a) is obvious. The first part of statement (b) is a simple corollary of the following claim: For any $b \in L^1(X)$, any $a \in \mathcal{A}$ and any $\varepsilon > 0$ there exist $\{x_k\}_{k=1}^L$ with $x_k \in X$ and $\{b_k\}_{k=1}^L$ with $b_k \in L^1(X)$ such that

$$\left\| s_T(b)r(a) - \sum_{k=1}^L r[\theta_{x_k}(a)]s_T(b_k) \right\| \le \varepsilon. \tag{2.5}$$

To prove this, consider a finite set $\{J_k\}_{k=0}^L$ of continuous functions on X such that $0 \le J_k \le 1$ and $\sum_{k=0}^L J_k = 1$. The functions J_k will have a compact support for $k \ge 1$. Let also $\{x_k\}_{k=1}^L$ be a finite set of elements of X. Then one has

$$\begin{aligned} s_{\mathrm{T}}(b)r(a) &= \sum_{k=1}^L r[\theta_{x_k}(a)] \int_X dx J_k(x)b(x)T(x) + \sum_{k=1}^L \int_X dx\big(r[\theta_x(a)] - r[\theta_{x_k}(a)]\big) J_k(x)b(x)T(x) \\ &\quad + \int_X dx\; r[\theta_x(a)]J_0(x)b(x)T(x). \end{aligned}$$

By setting $b_k := bJ_k \in L^1(X)$, the l.h.s. term of (2.5) is less or equal to

$$\max_{k=1,\dots,L} \sup_{x\in \operatorname{supp} J_k} \|\theta_x(a) - \theta_{x_k}(a)\| \, \|b\|_{L^1} + \|a\| \int_{\operatorname{supp} J_0} dx|b(x)|, \tag{2.6}$$

Let K be a compact subset of X such that $\int_{X\setminus K} dx|b(x)| \le \frac{\varepsilon}{2\|a\|}$, and let V be a neighbourhood of 0 in X such that $\sup_{x\in V} \|\theta_x(a) - a\| \le \frac{\varepsilon}{2\|b\|_{L^1}}$. We now choose the set $\{x_k\}_{k=1}^L$ such that $K \subset \cup_{k=1}^L (x_k + V)$, and the collection $\{J_k\}_{k=0}^L$ such that $\operatorname{supp} J_k \subset x_k + V$ for $k \in \{1,\dots,L\}$ and $\operatorname{supp} J_0 \subset X \setminus K$. The inequality (2.5) is then easily obtained from (2.6).

For the final statement of (b), let us observe that $(r \rtimes_\tau T)\,[m_{\tau,\mathbf{o}}(a \otimes b)] = r(a)s_{\mathrm{T}}(b)$ for any $a \in \mathcal{A}$ and any $b \in L^1(X)$. The conclusion follows from the density of $\mathcal{A} \odot L^1(X)$ in $L^1(X;\mathcal{A})$. □

2.3 Group cohomology

We recall some definitions in group cohomology. They will be used in the next sections to show that standard matters as gauge invariance and τ-quantizations have a cohomological flavour. Now they will serve to isolate twisted dynamical systems for which a generalization of the Schrödinger representation exists.

Let X be an abelian, locally compact group and $\mathcal{U}$ a Polish abelian group. Recall that a Polish group is a group with a compatible metrizable, separable and complete topology. In our applications $\mathcal{U}$ will usually not be locally compact, being the unitary group of an abelian C^*-algebra, as in Subsection **2.1**. We also assume that $\mathcal{U}$ is an *X-module*, i.e. that there exists a continuous action θ of X by automorphisms of $\mathcal{U}$. We shall use for X and $\mathcal{U}$ additive and multiplicative notations, respectively.

The class of all continuous functions : $X^n \to \mathcal{U}$ is denoted by $C^n(X;\mathcal{U})$; it is obviously an abelian group (we use once again multiplicative notations). Elements of $C^n(X;\mathcal{U})$ are called *(continuous) n-cochains*. For any $n \in \mathbb{N}$, we define *the coboundary map* $\delta^n : C^n(X;\mathcal{U}) \to C^{n+1}(X;\mathcal{U})$ by

$$[\delta^n(\rho)]\,(x_1,\dots,x_n,x_{n+1}) := \theta_{x_1}[\rho(x_2,\dots,x_{n+1})] \prod_{j=1}^n \rho(x_1,\dots,x_j + x_{j+1},\dots,x_{n+1})^{(-1)^j} \rho(x_1,\dots,x_n)^{(-1)^{n+1}}.$$

It is easily shown that δ^n is a group morphism and that $\delta^{n+1} \circ \delta^n = 1$ for any $n \in \mathbb{N}$. It follows that $\operatorname{im}(\delta^n) \subset \ker(\delta^{n+1})$.

Definition 2.7. (a) $Z^n(X;\mathcal{U}) := \ker(\delta^n)$ is called *the set of n-cocycles (on X, with coefficients in $\mathcal{U}$).*

(b) $B^n(X;\mathcal{U}) := \operatorname{im}(\delta^{n-1})$ is called *the set of n-coboundaries.*

One notices that $Z^n(X;\mathcal{U})$ and $B^n(X;\mathcal{U})$ are subgroups of $C^n(X;\mathcal{U})$, and that $B^n(X;\mathcal{U}) \subset Z^n(X;\mathcal{U})$.

Definition 2.8. The quotient $H^n(X;\mathcal{U}) := Z^n(X;\mathcal{U})/B^n(X;\mathcal{U})$ is called *the n'th group of cohomology (of X with coefficients in $\mathcal{U}$).* Its elements are called *classes of cohomology.*

We shall need only the cases $n = 0,1,2$, which we outline now for the convenience of the reader. For $n = 0$, parts of the definitions are simple conventions. We set $C^0(X;\mathcal{U}) := \mathcal{U}$. One has $\left[\delta^0(a)\right](x) = \frac{\theta_x(a)}{a}$, $\forall a \in \mathcal{U}, x \in X$. This implies that $Z^0(X;\mathcal{U}) = \{a \in \mathcal{U} \mid a \text{ is a fixed point}\}$. By convention, $B^0(X;\mathcal{U}) = \{1\}$.

The mapping $\delta^1 : C^1(X;\mathcal{U}) \to C^2(X;\mathcal{U})$ is given by $\left[\delta^1(\lambda)\right](x,y) = \frac{\lambda(x)\theta_x[\lambda(y)]}{\lambda(x+y)}$. Thus a 1-cochain λ is in $Z^1(X;\mathcal{U})$ if it is *a crossed morphism*, i.e. if it satisfies $\lambda(x)\theta_x[\lambda(y)] = \lambda(x+y)$ for any $x,y \in X$. Particular cases are the elements of $B^1(X;\mathcal{U})$ (called *principal morphisms*), those of the form $\lambda(x) = \frac{\theta_x(a)}{a}$ for some $a \in \mathcal{U}$.

For $n = 2$ one encounters a situation which was already taken into account in the definition of twisted dynamical systems. The formula for the coboundary map is

$$\left[\delta^2(\omega)\right](x,y,z) = \theta_x[\omega(y,z)]\omega(x+y,z)^{-1}\omega(x,y+z)\omega(x,y)^{-1}.$$

Thus a 2-cocycle is just a function satisfying the relation (2.1). $B^2(X;\mathcal{U})$ is composed of 2-cocycles of the form $\omega(x,y) = \frac{\lambda(x)\theta_x[\lambda(y)]}{\lambda(x+y)}$ for some 1-cochain λ.

We are mainly interested in the case of X-modules coming from C^*-dynamical systems, as in Subsection **2.1**, the group $\mathcal{U}$ being the unitary group of some abelian C^*-algebra. Our developments will need especially the case of algebras $\mathcal{A}$ of continuous functions defined on the group itself. The next result will be extremely significant for our formalism. For $n = 2$, it is a continuous version of Lemma 5.1 of [GI2]. Recall that $\mathcal{U} := \mathcal{C}(X;\mathbb{T})$, endowed with the strict topology, can be interpreted as the unitary group associated with the C^*-algebra $\mathcal{C}_0(X)$.

Lemma 2.9. *For $n \geq 1$, $H^n\big(X;\mathcal{C}(X;\mathbb{T})\big) = \{1\}$.*

Proof. Let $\rho^n \in Z^n\big(X;\mathcal{C}(X;\mathbb{T})\big)$, i.e. ρ^n is a continuous n-cochain satisfying for any $y_1,\dots,y_{n+1} \in X$

$$\theta_{y_1}\left[\rho^n(y_2,\dots,y_{n+1})\right]\prod_{j=1}^{n}\rho^n(y_1,\dots,y_j+y_{j+1},\dots,y_{n+1})^{(-1)^j}\rho^n(y_1,\dots,y_n)^{(-1)^{n+1}} = 1.$$

We set in this relation $y_1 = q$, $y_j = x_{j-1}$ for $j \geq 2$ and rephrase it as

$$\theta_q\left[\rho^n(x_1,\dots,x_n)\right] =$$
$$\rho^n(q+x_1,x_2,\dots,x_n)\prod_{j=1}^{n-1}\rho^n(q,x_1,\dots,x_j+x_{j+1},\dots,x_n)^{(-1)^j}\rho^n(q,x_1,\dots,x_{n-1})^{(-1)^n},$$

which is an identity in $\mathcal{C}(X;\mathbb{T})$. One calculates both sides at the point $x = 0$ and obtain

$$\left[\rho^n(x_1,\dots,x_n)\right](q) =$$
$$\left[\rho^n(q+x_1,x_2,\dots,x_n)\right](0)\prod_{j=1}^{n-1}\left[\rho^n(q,x_1,\dots,x_j+x_{j+1},\dots,x_n)^{(-1)^j}\right](0)\left[\rho^n(q,x_1,\dots,x_{n-1})^{(-1)^n}\right](0).$$

This means exactly $\rho^n = \delta^{n-1}(\rho^{n-1})$ for

$$\left[\rho^{n-1}(z_1,\dots,z_{n-1})\right](q) := \left[\rho^n(q,z_1,\dots,z_{n-1})\right](0) \tag{2.7}$$

and thus any n-cocyle is at least formally an n-coboundary.

We show now that ρ^{n-1} has the right continuity properties. Let us recall that if $\mathcal{C}(X;\mathbb{T})$ is endowed with the topology of uniform convergence on compact sets of X and if Y is a locally compact space, then $\mathcal{C}\big(Y;\mathcal{C}(X;\mathbb{T})\big)$ can naturally be identified with $\mathcal{C}(X\times Y;\mathbb{T})$ (the proof of this statement is an easy exercice). So ρ^n can be interpreted as an element of $\mathcal{C}(X\times X^n;\mathbb{T})$. Being obtained from ρ^n by a restriction ρ^{n-1} belongs to $\mathcal{C}(X^n;\mathbb{T})$, and thus can be interpreted as an element of $\mathcal{C}\big(X^{n-1};\mathcal{C}(X;\mathbb{T})\big) \equiv C^{n-1}\big(X;\mathcal{C}(X;\mathbb{T})\big)$, which finishes the proof. ☐

Definition 2.10. Let $\mathcal{U}$ be an abelian Polish X-module with action θ and $\omega \in Z^2(X;\mathcal{U})$. We say that ω is *pseudo-trivial* if there exists another abelian Polish X-module $\mathcal{U}'$ with action θ' such that $\mathcal{U}$ is a subgroup of $\mathcal{U}'$, for each $x \in X$ one has $\theta_x = \theta'_x|_{\mathcal{U}}$ and $\omega \in B^2(X;\mathcal{U}')$.

Thus, to produce pseudo-trivial 2-cocycles, one has to find some $\omega \in B^2(X;\mathcal{U}')$ such that $\omega(x,y) \in \mathcal{U} \subset \mathcal{U}'$ for any $x,y \in X$ and such that $(x,y) \mapsto \omega(x,y) \in \mathcal{U}$ is continuous with respect to the topology of $\mathcal{U}$. This is possible in principle because the product $\lambda(x)\theta_x[\lambda(y)][\lambda(x+y)]^{-1}$ can be better-behaved than any of its factors. The particular choice $[\lambda(z)](q) = [\omega(q,z)](0)$ we made in the proof of Lemma 2.9 will lead in **4.1** to the physicists' familiar transversal gauge.

Let us emphasize that most of the time pseudo-triviality cannot be improved to a bona fide triviality. Very often, all the functions λ for which one has $\omega = \delta^1(\lambda)$ do not take all their values in $\mathcal{U}$ or miss the right continuity. We shall outline such a situation in the next subsection.

2.4 Standard twisted crossed products

When trying to transform the formalism of twisted crossed products into a pseudodifferential theory, one has to face the possible absence of an analogue of the Schrödinger representation and this would lead us too far from the initial motivation. The existence of a generalized Schrödinger representation is assured by the pseudo-triviality of the 2-cocycle, and thus we restrict ourselves to a specific class of twisted dynamical systems. In the same time we also restrict to algebras $\mathcal{A}$ of complex continuous functions on X. This also is not quite compulsory for a pseudodifferential theory, but it leads to a simple implementation of pseudo-triviality (by Lemma 2.9) and covers easily the important magnetic case.

Definition 2.11. Let X be an abelian, locally compact group. We call *X-algebra* a C^*-algebra of bounded, uniformly continuous functions on X, stable by translations: $\theta_x(a) := a(\cdot + x) \in \mathcal{A}$ for all $a \in \mathcal{A}$ and $x \in X$.

The C^*-algebra $\mathcal{BC}_u(X) := \{a : X \to \mathbb{C} \mid a \text{ is bounded and uniformly continuous}\}$ is the largest one on which the action θ of translations with elements of X is norm-continuous. But we shall denote by θ_x even the x-translation on $\mathcal{C}(X)$, the *-algebra of all continuous complex functions on X (which is not a normed algebra if X is not compact). The restriction of θ_x on $\mathcal{BC}(X)$ is only strictly continuous.

Let us fix an X-algebra $\mathcal{A}$ with spectrum $S_\mathcal{A}$. It will be technically useful to note the existence of a continuous map $\delta^\mathcal{A} : X \to S_\mathcal{A}$ with a dense range . One sets $\delta^\mathcal{A}_x : \mathcal{A} \to \mathbb{C}$, $\delta^\mathcal{A}_x(a) := a(x)$ and the continuity is obvious. The fact that it has a dense range follows from the identification of the multiplier algebra $\mathfrak{M}(\mathcal{A})$ with a C^*-subalgebra of $\mathcal{BC}(X)$ (cf. the proof of Proposition 2.14) and a simple argument with Stone-Čech compactifications. Remark that $\delta^\mathcal{A}$ is injective exactly when $\mathcal{C}_0(X) \subset \mathcal{A}$. If in addition $\mathcal{A}$ is unital, $S_\mathcal{A}$ is a compactification of X. Now the Gelfand isomorphism $\mathcal{A} \cong \mathcal{C}_0(S_\mathcal{A})$ can be put in very concrete terms: a bounded continuous function $a : X \to \mathbb{C}$ belongs to $\mathcal{A}$ if and only if there exists a (necessarily unique) function $\tilde{a} \in \mathcal{C}_0(S_\mathcal{A})$ such that $\tilde{a} \circ \delta^\mathcal{A} = a$. One has a similar criterion for $\mathfrak{M}(\mathcal{A})$ with $\tilde{a} \in \mathcal{BC}(S_\mathcal{A})$.

Definition 2.12. A function $b : X \to \mathbb{C}$ is *of type* $\mathcal{A}$ if there exists $\tilde{b} \in \mathcal{C}(S_\mathcal{A})$ such that $\tilde{b} \circ \delta^\mathcal{A} = b$.

If $\mathcal{A}$ is an X-algebra, then $(\mathcal{A}, \theta, X)$ is a C^*-dynamical system. If we twist it, we get

Definition 2.13. *A standard twisted dynamical system* is a twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$ for which $\mathcal{A}$ is an X-algebra. The C^*-algebra $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ is called *a standard twisted crossed product.*

Proposition 2.14. *If $(\mathcal{A}, \theta, \omega, X)$ is a standard twisted dynamical system, then ω is pseudo-trivial.*

Proof. We shall prove the triviality of the 2-cocyle in the unitary group of the C^*-algebra $\mathcal{C}_0(X)$, i.e. in $\mathcal{C}(X;\mathbb{T})$ endowed with the strict topology. We first show that the multiplier algebra of $\mathcal{A}$ can be identified with a C^*-subalgebra of $\mathcal{BC}(X)$, the multiplier algebra of $\mathcal{C}_0(X)$. One remarks that, the trivial case $\mathcal{A} = \{0\}$ excluded, the invariance of $\mathcal{A}$ under translations implies the non-degeneracy of the natural faithful representation of $\mathcal{A}$ in $L^2(X)$. We keep the same notation for $\mathcal{A}$ and for its image in $\mathcal{B}\big(L^2(X)\big)$. It follows that $\mathfrak{M}(\mathcal{A})$ is contained in the double commutant of $\mathcal{A}$ (cf. [FD]), which itself is contained in $L^\infty(X)$ (also represented in $L^2(X)$ by multiplication operators). Moreover, any element m of $\mathfrak{M}(\mathcal{A})$ is also a continuous map on X: If m would not be continuous in x_0, one could find by translational invariance some element a of $\mathcal{A}$ which is not vanishing in a neighbourhood of x_0 and this obviously makes $ma \in \mathcal{A}$ impossible. Thus $\mathfrak{M}(\mathcal{A}) \subset \mathcal{BC}(X)$.

Let us now observe that $\mathcal{U}(\mathcal{A})$ is a bounded subset of $\mathcal{BC}(X)$, and on bounded subsets of $\mathcal{BC}(X)$ the strict topology coincides with the topology of uniform convergence on compact subsets of X. Similarly the strict topology on bounded subsets of $\mathfrak{M}(\mathcal{A})$ coincides with the topology of uniform convergence on compact subsets of $S_\mathcal{A}$. But $\delta^\mathcal{A} : X \to S_\mathcal{A}$ being continuous we deduce that on $\mathcal{U}(\mathcal{A})$ the strict topology of $\mathfrak{M}(\mathcal{A})$ induces a finer topology than $\mathcal{BC}(X)$.

Thus $\mathcal{U}(\mathcal{A})$ is naturally identified with a subgroup of $\mathcal{C}(X;\mathbb{T})$, and the strict topology on $\mathcal{U}(\mathcal{A})$ is finer than the strict topology of $\mathcal{C}(X;\mathbb{T})$. ω can hence be considered as an element of $Z^2\big(X;\mathcal{C}(X;\mathbb{T})\big)$, which coincides with $B^2\big(X;\mathcal{C}(X;\mathbb{T})\big)$ by Lemma 2.9, and this finishes the proof. ☐

Example. The simplest X-algebra is composed only of constant functions. If one takes $\mathcal{A} = \mathbb{C}$ then θ_x is the identity for all x and $\omega : X \times X \to \mathbb{T}$ is sometimes called *a multiplier of the group* X. Then $\mathbb{C} \rtimes^\omega_{\mathrm{id},\tau} X$ does no longer depend on τ; it is denoted by $C^*_\omega(X)$ and called *the twisted C^*-algebra of the group X associated with* ω. Its non-degenerate representations are in one-to-one correspondence with the ω-projective representations of X. We know from Proposition 2.14 that ω will be trivial if we enlarge $\mathbb{T}$ to $\mathcal{C}(X;\mathbb{T})$. But $\omega \in B^2(X;\mathbb{T})$ if and only if it is symmetric ($\omega(x,y) = \omega(y,x)$, $\forall x,y \in X$); this is proved in [Kl1]. Since for many groups X other non-symmetric multipliers are available, we see that very often triviality cannot be achieved within $\mathbb{T}$. We shall encounter examples later on; in our setting they correspond roughly to constant magnetic fields for $X = \mathbb{R}^n$ and give "non-commutative tori" for $X = \mathbb{Z}^N$.

Remark 2.15. If ω, ω' are two cohomologous elements of $Z^2\big(X;\mathcal{U}(\mathcal{A})\big)$, i.e. $\omega = \delta^1(\lambda)\,\omega'$ for some $\lambda \in C^1\big(X;\mathcal{U}(\mathcal{A})\big)$, then the C^*-algebras $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ and $\mathcal{A} \rtimes^{\omega'}_{\theta,\tau} X$ are naturally isomorphic: on $L^1(X;\mathcal{A})$ the isomorphism is given by $\left[i^\lambda_\tau(\phi)\right](x) := \theta_{-\tau x}[\lambda(x)]\phi(x)$. Thus $\mathcal{C}_0(X) \rtimes^\omega_{\theta,\tau} X$ does not depend on ω; this will be strengthened in Proposition 2.17 (b). However this does not work if λ only belongs to $C^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ and $\mathcal{A}$ is not $\mathcal{C}_0(X)$; in general $\theta_{-\tau x}[\lambda(x)]\phi(x)$ gets out of $\mathcal{A}$ and i^λ_τ is no longer well-defined. For ω and ω' defining different classes of cohomology, $\mathcal{A}\rtimes^\omega_{\theta,\tau}X$ and $\mathcal{A}\rtimes^{\omega'}_{\theta,\tau}X$ are in general different C^*-algebras.

In the sequel we fix a standard twisted dynamical system $(\mathcal{A},\theta,\omega,X)$. One observes that the untwisted system $(\mathcal{A},\theta,X)$ always has an obvious covariant representation $(\mathcal{H},r,T)$, with $\mathcal{H}:=L^2(X)$ (with the Haar measure), $r(a)\equiv a(Q)=$ multiplication with a and $[T(y)u](x):=[U(-y)u](x)=u(x+y)$. For $X=\mathbb{R}^N$ it coincides with the "untwisted" Schrödinger representation introduced in Subsection **1.2**. Let us now choose $\lambda\in C^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ such that $\delta^1(\lambda)=\omega$ (identity in $Z^2\big(X;\mathcal{C}(X;\mathbb{T})\big)$). We set $T^\lambda(y):=r\big(\lambda(y)\big)T(y)$. Explicitly, for any $x\in X$ and $u\in\mathcal{H}$, $\big[T^\lambda(y)u\big](x)=[\lambda(y)](x)u(x+y)\equiv\lambda(x;y)u(x+y)$. Let us already mention that the point (b) in the next proposition is at the root of gauge invariance for magnetic pseudodifferential operators.

Proposition 2.16. (a) *$(\mathcal{H},r,T^\lambda)$ is a covariant representation of $(\mathcal{A},\theta,\omega,X)$,*

(b) *If μ is another element of $C^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ such that $\delta^1(\mu)=\omega$, then there exists $c\in\mathcal{C}(X;\mathbb{T})$ such that $\mu(x)=\frac{\theta_x(c)}{c}\lambda(x)$, $\forall x\in X$. Moreover, $T^\mu(x)=r(c^{-1})T^\lambda(x)r(c)$ for all $x\in X$.*

Proof. The proof of the first statement consists in trivial verifications. For the second statement, one first notes that $\frac{\mu}{\lambda}$ belongs to $\ker(\delta^1)=Z^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$. Since this set is equal to $B^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ by Lemma 2.9, there exists $c\in C^0\big(X;\mathcal{C}(X;\mathbb{T})\big)\equiv\mathcal{C}(X;\mathbb{T})$ satisfying $\mu(x)=\frac{\theta_x(c)}{c}\lambda(x)$, $\forall x\in X$. The last claim of the proposition follows from $r[\theta_x(c)]T(x)=T(x)r(c)$. □

We call $(\mathcal{H},r,T^\lambda)$ *the Schrödinger covariant representation associated with the* 1*-cochain* λ. Let us now recall the detailed form of the composition laws on $L^1(X;\mathcal{A})$. For simplicity we shall use notations as $\phi(x;y)$ for $[\phi(y)](x)$ and $\omega(x;y,z)$ for $[\omega(y,z)](x)$. With these notations and for any $\phi,\psi\in L^1(X;\mathcal{A})$, the relations (2.3) and (2.4) read respectively

$$(\phi\diamond^\omega_\tau\psi)(q;x)=\int_X dy\,\phi\big(q+\tau(y-x);y\big)\,\psi\big(q+(1-\tau)y;x-y\big)\,\omega\big(q-\tau x;y,x-y\big)$$

and

$$(\phi^{\diamond^\omega_\tau})(q;x)=\omega\big(q-\tau x;x,-x\big)^{-1}\overline{\phi\big(q+(1-2\tau)x;-x\big)},$$

where x,y,q are elements of X.

Let us also denote for convenience by $\mathfrak{Rep}^\lambda_\tau$ the representation $r\rtimes_\tau T^\lambda$ in $L^2(X)$ of the twisted crossed product $\mathcal{A}\rtimes^\omega_{\theta,\tau}X$. Its explicit action on $L^1(X;\mathcal{A})$ is given by

$$\Big[\Big(\mathfrak{Rep}^\lambda_\tau(\phi)\Big)\,u\Big](x)=\int_X dy\,\phi(x+\tau y;y)\,\lambda(x;y)\,u(x+y)=\int_X dy\,\phi\big((1-\tau)x+\tau y;y-x\big)\,\lambda(x;y-x)\,u(y).$$

We gather some important properties of $\mathfrak{Rep}^\lambda_\tau$ in

Proposition 2.17. (a) *In the setting of Proposition 2.16 (b), one has $\mathfrak{Rep}^\mu_\tau(\phi)=r(c^{-1})\mathfrak{Rep}^\lambda_\tau(\phi)r(c)$.*

(b) *$\mathfrak{Rep}^\lambda_\tau[\mathcal{C}_0(X)\rtimes^\omega_{\theta,\tau}X]=\mathcal{K}\big(L^2(X)\big)$, the C^*-algebra of all compact operators in $L^2(X)$.*

(c) *If $\mathcal{C}_0(X)\subset\mathcal{A}$, then $\mathfrak{Rep}^\lambda_\tau$ is irreducible.*

(d) *$\mathfrak{Rep}^\lambda_\tau$ is faithful.*

Proof. (a) The proof of this statement consists in a simple verification.

(b) By Lemma 2.9, ω belongs to $B^2\big(X;\mathcal{C}(X;\mathbb{T})\big)$, i.e. there exists $\lambda\in C^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ such that $\delta^1(\lambda)=\omega$. We may then consider the following isomorphism

$$i^{\lambda^{-1}}_\tau:\Big(L^1\big(X;\mathcal{C}_0(X)\big),\diamond^1_0,{}^{\diamond^1_0}\Big)\to\Big(L^1\big(X;\mathcal{C}_0(X)\big),\diamond^\omega_\tau,{}^{\diamond^\omega_\tau}\Big),\quad\Big[i^{\lambda^{-1}}_\tau(\phi)\Big](x)=\theta_{-\tau x}\left[\lambda^{-1}(x)\,\phi(x)\right],\tag{2.8}$$

that extends to an isomorphism between the non-twisted crossed product $\mathcal{C}_0(X)\rtimes^1_{\theta,0}X$ and our $\mathcal{C}_0(X)\rtimes^\omega_{\theta,\tau}X$ (this is consistent with Remark 2.15). One easily checks that $\mathfrak{Rep}^\lambda_\tau\big[i^{\lambda^{-1}}_\tau(\phi)\big]=\int_X dx\,r[\phi(x)]T(x)$ for all ϕ in $\Big(L^1(X;\mathcal{A}),\diamond^1_0,{}^{\diamond^1_0}\Big)$. But it is known that the image of $\mathcal{C}_0(X)\rtimes^1_{\theta,0}X$ through the representation $r\rtimes T\equiv\mathfrak{Rep}^1_0$ is equal to the algebra $\mathcal{K}\big(L^2(X)\big)$ of compact operators in $L^2(X)$, cf. for example [GI1].

(c) If $\mathcal{C}_0(X) \subset \mathcal{A}$ then $\mathcal{C}_0(X) \rtimes^{\omega}_{\theta,\tau} X$ can be identified to a C^*-subalgebra of $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ and the irreducibility of $\mathfrak{Rep}^{\lambda}_{\tau}\left(\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X\right)$ follows from the irreducibility of $\mathcal{K}\big(L^2(X)\big)$, by (b).

(d) Let us recall from [PR1] the regular representation of the twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$. It consists in the triple $(\mathcal{H}', r', T')$, where $\mathcal{H}'$ is the Hilbert space $L^2\big(X; L^2(X)\big)$, and where the two maps act on $\xi \in \mathcal{H}'$ as

$$[r'(a)\xi](x) = \theta_x(a)\xi(x) \quad \text{and} \quad [T'(y)\xi](x) = \omega(x, y)\xi(x + y) \qquad \text{for all } x, y \in X \text{ and } a \in \mathcal{A}.$$

It follows by straightforward verifications that $(\mathcal{H}', r', T')$ is a covariant representation.

Since $\mathcal{H}'$ is canonically isomorphic to $L^2(X \times X)$, let us set $\xi(\cdot; x) := \xi(x)$ and introduce the unitary operator $W^{\lambda} : L^2(X \times X) \to L^2(X \times X)$, $[W^{\lambda}\xi](x; y) := \lambda(x; y)\, \xi(x; x + y)$. Its adjoint is given by $[(W^{\lambda})^*\xi](x; y) = \lambda^{-1}(x; y - x)\; \xi(x; y - x)$. Some easy calculations show then that $\left[(W^{\lambda})^* r'(a) W^{\lambda} \xi\right](x; y) = a(y)\; \xi(x; y)$. Moreover, one has

$$\left[(W^{\lambda})^* T'(z) W^{\lambda} \xi\right](x; y) = \lambda^{-1}(x; y - x)\omega(x; y - x, z)\lambda(x; y - x + z)\xi(x; y + z) = \lambda(y; z)\xi(x; y + z),$$

where we have used that $\omega = \delta^1(\lambda)$. Equivalently, one has $(W^{\lambda})^* r'(a) W^{\lambda} = \mathbf{1} \otimes a(Q)$ and $(W^{\lambda})^* T'(z) W^{\lambda} = \mathbf{1} \otimes \lambda(Q; z)T(z) \equiv \mathbf{1} \otimes T^{\lambda}(z)$ in $L^2(X) \otimes L^2(X)$. Thus the regular representation is unitarily equivalent to the representation $(\mathcal{H} \otimes \mathcal{H}, \mathbf{1} \otimes r, \mathbf{1} \otimes T^{\lambda})$. Since the regular representation induces a faithful representation $r' \rtimes T'$ of $\mathcal{A} \rtimes^{\omega}_{\theta} X$ in $\mathcal{H}'$, cf. Theorem 3.11 of [PR1] (X is amenable, being abelian), the Schrödinger representation induces faithful representations of $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ in $\mathcal{H}$ for any $\tau \in \mathfrak{End}(X)$. $\square$

Remark 2.18. If $\mathcal{C}_0(X)$ is not contained in $\mathcal{A}$, then the conclusion in (c) may fail. If, for example, $\mathcal{A} = \mathbb{C}$ (with the trivial action) and $\omega = 1$ then any translation $T(x)$ commutes with $\mathfrak{Rep}(\mathbb{C} \rtimes X)$, thus $\mathfrak{Rep}(\mathbb{C} \rtimes X)$ is reducible by Schur's Lemma.

Remark 2.19. It is well-known that $\mathcal{K}\big(L^2(X)\big)$ admits a single class of irreducible representations. Thus, by (c) and (d), all the irreducible representations of $\mathcal{C}_0(X) \rtimes^{\omega}_{\theta,\tau} X$ are unitarily equivalent to the Schrödinger representations $\mathfrak{Rep}^{\lambda}_{\tau}$ (which form *always* a single class, by Proposition 2.17 (a)). This is by no means a general property. Assume that $\mathcal{A}$ contains $\mathcal{C}_0(X)$ and is unital. Then its spectrum $S_{\mathcal{A}}$ is a compactification of X. Since X sits in $S_{\mathcal{A}}$ as a dense, X-invariant open set, there exist closed invariant subsets of $S_{\mathcal{A}} \setminus X$. Let F be one of them; it will be called *an asymptotic set*. For instance, one can ask it to be minimal with respect to the properties above, i.e. it will be a quasi-orbit disjoint of X; this will be assumed in the sequel. $\mathcal{A}$ being identified with $\mathcal{C}(S_{\mathcal{A}})$, $\mathcal{C}^F(S_{\mathcal{A}}) := \{a \in \mathcal{A} \mid a|_F = 0\}$ is obviously an invariant ideal that we call $\mathcal{A}^F$. It is easy to see that the multiplier algebra of $\mathcal{A}^F$ contains the multiplier algebra of $\mathcal{A}$, so that, by restriction, the twisted dynamical system $(\mathcal{A}^F, \theta, \omega, X)$ makes sense. The twisted crossed product $\mathcal{A}^F \rtimes^{\omega}_{\theta,\tau} X$ may be identified with an ideal of $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ and the quotient to $\big(\mathcal{A}/\mathcal{A}^F\big) \rtimes^{\omega^F}_{\theta,\tau} X$. To understand ω^F, remark that $\mathcal{A}/\mathcal{A}^F$ is canonically isomorphic to the C^*-algebra $\mathcal{C}(F)$ of all continuous functions on F. In this interpretation, for each $x, y \in X$, $\omega(x, y) \in \mathcal{U}(\mathcal{A})$ first extends to $S_{\mathcal{A}}$ and then is restricted to F, giving thus a 2-cocycle $\omega^F : X \times X \to \mathcal{U}[\mathcal{C}(F)] = \mathcal{C}(F; \mathbb{T})$. If we choose now an irreducible representation R^F of the quotient, one gets an irreducible representation of the initial twisted crossed product just by composing with the quotient map. This representation is no longer faithful, hence it cannot be unitarily equivalent to the initial one. We can choose for R^F once again a Schrödinger-type representation, but associated with the asymptotic twisted dynamical system $(\mathcal{C}(F), \theta, \omega^F, X)$. To do this, we need to show that it is standard. This follows if we interpret $\mathcal{C}(F)$ as an X-algebra. This is done simply by choosing a point ν_0 on F whose quasi-orbit is the entire F. This will lead to an embedding of $\mathcal{C}(F)$ into $\mathcal{BC}_u(X)$. Thus the quotient is once again a standard twisted dynamical system, but with a "simpler" C^*-algebra and a "simpler" 2-cocycle ω^F. Then one can fix a pseudo-trivialization λ^F of ω^F and thus we may take $R^F = \mathfrak{Rep}^{\lambda^F}_{\tau}$.

Remark 2.20. Let us summary some facts obtained above, that will be relevant in the next sections. We fix a standard twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$. The C^*-algebras $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ are isomorphic to each other for different elements $\tau \in \mathfrak{End}(X)$. This also has a cohomological nature, but in Subsection **3.2** we shall be in a better position to discuss this. So let us fix τ. For some choices of $\mathcal{A}$, $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ may be isomorphic to the untwisted crossed product $\mathcal{A} \rtimes^{1}_{\theta,\tau} X$. This happens when ω is a 2-coboundary with respect to the unitary group of $\mathcal{A}$; the typical case is $\mathcal{A} = \mathcal{C}_0(X)$. Definitely, this does not happen too often. In general ω can be trivialized only by using a larger unitary group and this is not enough to conclude that it can be removed from $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ by an isomorphism. On the other hand, for fixed ω, one constructs a family of Schrödinger representations of $\mathcal{A} \rtimes^{\omega}_{\theta,\tau} X$ indexed by all the 1-cochains which define the pseudo-trivializable ω with respect to the larger X-module $\mathcal{C}(X;\mathbb{T})$. This is possible since the representation r of the algebra $\mathcal{A}$ is a restriction of a much larger representations. The Schrödinger representations assigned to two 1-cochains giving he same ω are unitary equivalent. This happens since the 1-cochains are cohomologous, once again with respect to the big X-module $\mathcal{C}(X;\mathbb{T})$. This phenomenon is a general instance of what is called "gauge covariance" for magnetic algebras and representations.

3 The pseudodifferential calculus associated with a twisted dynamical system

We have introduced a class of C^*-algebras, called standard twisted crossed products, as well as their family of Schrödinger representations, defined by pseudo-trivializations of the 2-cocycle ω. In the first subsection, by a partial Fourier transformation, we shall get from these data a sort of pseudodifferential calculus. Let us denote by $X^\sharp$ the dual group of X. Then certain classes of functions on $X \times X^\sharp$ will be organised in C^*-algebras with some natural involution and a product involving ω and generalizing the well-known Moyal product appearing in Quantum Mechanics. The composition between the partial Fourier transformation and the Schrödinger representation will lead to a rule of assigning operators to symbols belonging to these C^*-algebras. This will be a generalization of the pseudodifferential (in particular of the Weyl) rule valid for $X = \mathbb{R}^N$ in the absence of any 2-cocycle.

The axioms of a covariant representation are a sort of a priori commutation relations. We can reinterpret them in the form of a Weyl system - a family of unitary operators satisfying a relation which generalizes that of a projective representation. This Weyl system will be introduced and studied in the second subsection. We also show that the pseudodifferential prescription may be considered as a functional calculus associated with this Weyl system, by mimicking an approach that is standard in the commutative case.

3.1 Generalized pseudodifferential algebras and operators

Let $X^\sharp$ be the dual group of X, i.e. the set of all continuous morphisms (characters) $\chi : X \to \mathbb{T}$. Endowed with the composition law $(\chi \cdot \kappa)(x) := \chi(x)\kappa(x)$, $x \in X$ and with the topology of uniform convergence on compact subsets of X, $X^\sharp$ is a second-countable locally compact abelian group. The Haar measures on X and $X^\sharp$ will be normalized in such a way that the Fourier transformations

$$\mathcal{F}_X : L^1(X) \to \mathcal{C}_0(X^\sharp), \quad (\mathcal{F}_X\, b)(\chi) = \int_X dx\, \overline{\chi(x)}\, b(x)$$

and

$$\overline{\mathcal{F}}_X : L^1(X) \to \mathcal{C}_0(X^\sharp), \quad \left(\overline{\mathcal{F}}_X\, b\right)(\chi) = \int_X dx\, \chi(x)\, b(x)$$

induce unitary maps from $L^2(X)$ to $L^2(X^\sharp)$. The inverses of these maps act on $L^2(X^\sharp) \cap L^1(X^\sharp)$ as $\left(\overline{\mathcal{F}}_{X^\sharp}\, c\right)(x) = \int_{X^\sharp} d\chi\, \chi(x)\, c(\chi)$ and $\left(\mathcal{F}_{X^\sharp}\, c\right)(x) = \int_{X^\sharp} d\chi\, \overline{\chi(x)}\, c(\chi)$.

Let us now consider the twisted dynamical system $(\mathcal{A}, \theta, \omega, X)$. We define the mapping $\mathbf{1} \otimes \overline{\mathcal{F}}_X : L^1(X;\mathcal{A}) \to \mathcal{C}_0(X^\sharp;\mathcal{A})$ by $\left[\left(\mathbf{1} \otimes \overline{\mathcal{F}}_X\right)(\phi)\right](\chi) = \int_X dx\, \chi(x)\, \phi(x)$ (equality in $\mathcal{A}$). We recall that $\mathcal{A} \odot L^1(X)$ is a dense subspace of $L^1(X;\mathcal{A})$ and observe that $\left(\mathbf{1} \otimes \overline{\mathcal{F}}_X\right)(a \otimes b) = a \otimes \left(\overline{\mathcal{F}}_X b\right)$. Let us now also fix an element $\tau \in \mathfrak{End}(X)$. We transport all the structure of the Banach *-algebra $\left(L^1(X;\mathcal{A}), \diamond^\omega_\tau, ^{\diamond^\omega_\tau}, \|\!|\cdot\|\!|\right)$ to the corresponding subset of $\mathcal{C}_0(X^\sharp;\mathcal{A})$ via $\mathbf{1} \otimes \overline{\mathcal{F}}_X$. The space $\left(\mathbf{1} \otimes \overline{\mathcal{F}}_X\right) L^1(X;\mathcal{A})$ will also be a Banach *-algebra with a composition law $\circ^\omega_\tau$, an involution $^{\circ^\omega_\tau}$ and the norm $\|\!|(\mathbf{1} \otimes \overline{\mathcal{F}}_X^{-1}) \cdot \|\!|$. Its envelopping C^*-algebra will be denoted by $\mathfrak{C}^\omega_{\mathcal{A},\tau}$. The map $\mathbf{1} \otimes \overline{\mathcal{F}}_X$ extends canonically to an isomorphism between $\mathcal{A} \rtimes^\omega_{\theta,\tau} X$ and $\mathfrak{C}^\omega_{\mathcal{A},\tau}$. We remark that $\left(\mathbf{1} \otimes \overline{\mathcal{F}}_X\right)\left[\mathcal{A} \odot L^1(X)\right]$ is already not very explicit, since one has no direct characterization of the space $\overline{\mathcal{F}}_X\left[L^1(X)\right]$. Concerning $\mathfrak{C}^\omega_{\mathcal{A},\tau}$, we do not even know if it consists entirely of $\mathcal{A}$-valued distributions on $X^\sharp$ (whenever this makes sense). However, usually one can work efficiently on suitable dense subsets.

We deduce now the explicit form of the composition law and of the involution. Let us denote simply $\mathbf{1} \otimes \overline{\mathcal{F}}_X$ by $\mathfrak{F}$. One gets for any $f, g \in \mathfrak{F} L^1(X;\mathcal{A})$

$$(f \circ^\omega_\tau g)(q,\chi) := \left(\mathfrak{F}\left[(\mathfrak{F}^{-1} f) \diamond^\omega_\tau (\mathfrak{F}^{-1} g)\right]\right)(q,\chi) =$$

$$\int_X dx \int_X dy \int_{X^\sharp} d\kappa \int_{X^\sharp} d\gamma\, \chi(x)\, \overline{\kappa(y)}\, \overline{\gamma(x-y)}\, f(q + \tau(y-x), \kappa)\, g(q + (\mathbf{1} - \tau)y, \gamma)\, \omega(q - \tau x; y, x-y)$$

and

$$\left(f^{\circ^\omega_\tau}\right)(q,\chi) := \left(\mathfrak{F}\left[\left(\mathfrak{F}^{-1}f\right)^{\diamond^\omega_\tau}\right]\right)(q,\chi) = \int_X dx \int_{X^\sharp} d\kappa\, \left(\chi\cdot\kappa^{-1}\right)(x)\,\omega(q-\tau x; x, -x)^{-1}\,\overline{f(q+(1-2\tau)x,\kappa)}.$$

Both expressions make sense as iterated integrals; under more stringent conditions on f and g, the integrals will be absolutely convergent.

The constructions and formulae above can be given (with some slight adaptations) for general abelian twisted dynamical system. We ask now that our twisted dynamical system be standard, which makes ω pseudo-trivial. For any continuous function $\lambda : X \to \mathcal{C}(X;\mathbb{T})$ such that $\delta^1(\lambda) = \omega$, the corresponding Schrödinger covariant representation $(\mathcal{H}, r, T^\lambda)$ gives rise to the Schrödinger representation of $\mathcal{A}\rtimes^\omega_{\theta,\tau} X$ that we denoted by $\mathfrak{Rep}^\lambda_\tau$. We get a representation of $\mathfrak{C}^\omega_{\mathcal{A},\tau}$ just by composing with $\mathfrak{F}^{-1}$; it will be denoted by $\mathfrak{Op}^\lambda_\tau$. By simple calculations one obtains

Proposition 3.1. (a) *The representation* $\mathfrak{Op}^\lambda_\tau := \mathfrak{Rep}^\lambda_\tau \circ \mathfrak{F}^{-1} : \mathfrak{C}^\omega_{\mathcal{A},\tau} \to \mathcal{B}(\mathcal{H})$ *is faithful and acts on* $f \in \mathfrak{F}L^1(X;\mathcal{A})$ *by the formula*

$$\left[\mathfrak{Op}^\lambda_\tau(f)u\right](x) = \int_X dy \int_{X^\sharp} d\chi\, \chi(x-y)\,\lambda(x; y-x)\, f[(\mathtt{1}-\tau)x + \tau y, \chi]\, u(y), \quad u\in\mathcal{H}, \quad x\in X, \tag{3.1}$$

where the right-hand side is viewed as an iterated integral.

(b) *If* $\mu \in C^1\big(X;\mathcal{C}(X;\mathbb{T})\big)$ *is another* 1*-cochain, giving a second pseudo-trivialization of the* 2*-cocycle* ω*, then* $\mu = \delta^0(c)\lambda$ *for some* $c\in\mathcal{C}(X;\mathbb{T})$ *and* $\mathfrak{Op}^\lambda_\tau$*,* $\mathfrak{Op}^\mu_\tau$ *are unitarily equivalent:*

$$r(c^{-1})\,\mathfrak{Op}^\lambda_\tau(f)\,r(c) = \mathfrak{Op}^\mu_\tau(f), \quad \forall f\in\mathfrak{C}^\omega_{\mathcal{A},\tau}. \tag{3.2}$$

In (3.1) the integral is absolutely convergent under various assumptions on f and u, for instance if $u \in L^1(X)\cap L^2(X)$ and f is of class L^1 in χ. One could dare and call (3.2) *the gauge-covariance of the generalized pseudodifferential calculus.*

For different τ's, the C^*-algebras $\mathfrak{C}^\omega_{\theta,\tau}$ are isomorphic. If $\tau,\tau'\in\mathfrak{End}(X)$, then $\mathfrak{m}_{\tau,\tau'} := \mathfrak{F}\circ m_{\tau,\tau'}\mathfrak{F}^{-1}$ defines an isomorphism $\mathfrak{C}^\omega_{\theta,\tau'} \cong \mathfrak{C}^\omega_{\theta,\tau}$. We recall that $(m_{\tau,\tau'}\phi)(q;x) = \phi(q+(\tau'-\tau)x; x)$, $\forall x,q\in X$, $\forall \phi\in L^1(X;\mathcal{A})$. This isomorphism is constructed in order to satisfy $\mathfrak{Op}^\lambda_{\tau'} = \mathfrak{Op}^\lambda_\tau \circ \mathfrak{m}_{\tau,\tau'}$ and thus gives the transformation of the τ-symbol of a generalized pseudodifferential operator into its τ'-symbol.

We support the assertion that the choice of the parameter τ is a matter of ordering only by a weak example. Let us assume that the X-algebra $\mathcal{A}$ is unital. Then the element $f = 1\otimes b$ is in $\mathfrak{C}^\omega_{\theta,\tau}$ for any $b : X^\sharp \to \mathbb{C}$ with $\mathcal{F}_{X^\sharp}\, b \in L^1(X)$. The operator $\mathfrak{Op}^\lambda_\tau(1\otimes b)$ does not depend on τ. We denote it by $\mathfrak{op}^\lambda(b)$; its action on $u\in\mathcal{H}$ is given by

$$\left[\mathfrak{op}^\lambda(b)u\right](x) = \int_X dy\ \lambda(x; y-x)\,(\mathcal{F}_{X^\sharp}\, b)\,(y-x)u(y).$$

Let us now consider an arbitrary element $a\in\mathcal{A}$. Simple calculations for $\tau = \mathtt{0}$ and $\tau = \mathtt{1}$ show that $\mathfrak{Op}^\lambda_{\mathtt{0}}(a\otimes b) = r(a)\mathfrak{op}^\lambda(b)$ and $\mathfrak{Op}^\lambda_{\mathtt{1}}(a\otimes b) = \mathfrak{op}^\lambda(b)r(a)$. We point out that $b \to \mathfrak{op}^\lambda(b)$ is not a closed functional calculus: $(1\otimes b_1)\circ^\omega_\tau(1\otimes b_2)$ is in general a function depending on both variables, hence one cannot write $\mathfrak{op}^\lambda(b_1)\mathfrak{op}^\lambda(b_2) = \mathfrak{op}^\lambda(b)$ for some function b defined on $X^\sharp$. It is not difficult to extend the morphism $f\to\mathfrak{Op}^\lambda_\tau(f)$ to include elements $f = a\otimes b$ with $a\in\mathcal{A}$ and $b : X^\sharp\to\mathbb{C}$ being the Fourier transform of some bounded measure on X. Then we see that $\mathfrak{Op}^\lambda_\tau(a\otimes 1) = r(a)$ for all λ and τ.

3.2 Generalized Weyl systems and the functional calculus

From now on the product group $X\times X^\sharp$ will be denoted simply by Ξ; it is locally compact, second-countable and abelian. We shall rephrase the relations verified by the Schrödinger covariant representation $(r,\mathcal{H},T^\lambda)$ of the standard twisted dynamical system $(\mathcal{A},\theta,\omega,X)$, insisting on the role played by Ξ. In this way we generalize the Weyl system of **1.1**.

Remark first that r is the restriction to $\mathcal{A}$ of a representation (also denoted by r) of $\mathcal{BC}(X)$. On the other hand, the dual group $X^\sharp$ is naturally a subset of $\mathcal{BC}(X)$ (any character is a continuous function on X with values in the bounded subset $\mathbb{T}$ of $\mathbb{C}$). Thus we can consider the map $V : X^\sharp\to\mathcal{U}(\mathcal{H})$, $V(\chi) := r(\chi)^*$ =operator of multiplication by $\overline{\chi}$ in $\mathcal{H} = L^2(X)$. Obviously V is a unitary group representation, the same appearing in

Subsection **1.1** for the particular case $X = \mathbb{R}^N$. We set also $U^\lambda(x) := T^\lambda(x)^*$ for any $x \in X$. By putting $a = \overline{\chi}$ in the covariance relation $T^\lambda(x)r(a)T^\lambda(x)^* = r[\theta_x(a)]$ and using $\theta_x(\overline{\chi}) = \overline{\chi(x)}\,\overline{\chi}$, one gets readily

$$U^\lambda(x)V(\chi) = \chi(x)V(\chi)U^\lambda(x), \quad \forall (x,\chi) \in \Xi, \tag{3.3}$$

which is a generalization of the Weyl form (1.4) of the canonical commutation relations.

Let us also fix an endomorphism τ of the group X. By generalizing (1.5) one sets for all $\xi = (y,\chi) \in \Xi$

$$W_\tau^\lambda(y,\chi) := \chi[(1-\tau)y]\,U^\lambda(y)^*\,V(\chi) = \chi[-\tau y]\,V(\chi)\,U^\lambda(y)^*.$$

Explicitly, one has for $u \in \mathcal{H}$

$$\left[W_\tau^\lambda(y,\chi)u\right](x) = \chi[-x-\tau y]\,\lambda(x;y)\,u(x+y). \tag{3.4}$$

Definition 3.2. The family of unitary operators $\{W_\tau^\lambda(\xi)\}_{\xi\in\Xi}$ is called *the Weyl system associated with the pseudo-trivialization λ and the endomorphism τ.*

These operators satisfy for all $\xi = (x,\chi), \eta = (y,\kappa) \in \Xi$ the relations

$$W_\tau^\lambda(x,\chi)\,W_\tau^\lambda(y,\kappa) = r\{\chi[\tau y]\,\kappa[(\tau-1)x]\,\omega(x,y)\}\,W_\tau^\lambda(x+y,\chi\cdot\kappa). \tag{3.5}$$

In fact this is part of a more comprehensive assertion:

Proposition 3.3. (a) *If $(r,\mathcal{H},T^\lambda)$ is a covariant representation of $(\mathcal{A},\theta,\omega,X)$, then $(r,\mathcal{H},W_\tau^\lambda)$ is a covariant representation of the twisted dynamical system $(\mathcal{A},\Theta,\Omega_\tau,\Xi)$, where $\Xi = X\times X^\sharp$, $\left[\Theta_{(x,\chi)}(a)\right](y) = \left[\theta_x(a)\right](y) = a(y+x)$ and $\Omega_\tau[(x,\chi),(y,\kappa)] := \chi[\tau y]\,\kappa[(\tau-1)x]\,\omega(x,y)$.*

(b) *If μ is another element of $C^1\big(X;\mathcal{C}(X,\mathbb{T})\big)$ such that $\delta^1(\mu) = \omega$, then there exists $c \in \mathcal{C}(X;\mathbb{T})$ such that $W_\tau^\mu(\xi) = r(c^{-1})\,W_\tau^\lambda(\xi)\,r(c)$ for all $\xi \in \Xi$.*

(c) *For $\tau,\tau' \in \mathfrak{End}(X)$, the 2-cocycles Ω_τ and $\Omega_{\tau'}$ on Ξ are cohomologous and the corresponding Weyl systems are connected by $W_{\tau'}^\lambda(x,\chi) = \chi[(\tau-\tau')x]\,W_\tau^\lambda(x,\chi)$ for all x and χ.*

Proof. (a) Simple verifications.

(b) This follows from Proposition 2.16 (b) or by direct calculation.

(c) One finds immediately that $\Omega_{\tau'}[(x,\chi),(y,\kappa)] = \chi[(\tau'-\tau)y]\kappa[(\tau'-\tau)x]\Omega_\tau[(x,\chi),(y,\kappa)]$, which can be written $\Omega_{\tau'} = \delta^1(\Lambda_{\tau,\tau'})\Omega_\tau$ for $\Lambda_{\tau,\tau'}(x,\chi) = \chi[(\tau-\tau')x]$. The relation between W_τ^λ and $W_{\tau'}^\lambda$ follows then from (3.5) or is deduced directly from the explicit formula (3.4). ☐

The inflated twisted dynamical system $(\mathcal{A},\Theta,\Omega_\tau,\Xi)$ may be used to construct twisted crossed product C^*-algebras. We do not pursue this here. The point (c) shows that the correlation between the structures defined by different τ's is once again a matter of cohomology.

Let us denote by $\mathcal{F}_\Xi$ the "symplectic" Fourier transformation defined on $L^1(\Xi)$ by

$$(\mathcal{F}_\Xi g)(x,\chi) := \int_X\int_{X^\sharp} dy\,d\kappa\;\chi(y)\overline{\kappa(x)}g(y,\kappa),$$

which can be expressed as $\mathcal{F}_\Xi = \mathcal{I}\circ\big(\overline{\mathcal{F}}_X\otimes\mathcal{F}_{X^\sharp}\big)$, with $(\mathcal{I}h)(x,\chi) := h(\chi,x)$. One easily checks that $\mathcal{F}_\Xi^{-1} = \mathcal{F}_\Xi$. It is natural to define for $f \in \mathcal{F}_\Xi L^1(\Xi)$

$$\widetilde{\mathfrak{Op}}_\tau^\lambda(f) = \int_\Xi d\xi\;(\mathcal{F}_\Xi^{-1}f)(\xi)W_\tau^\lambda(\xi). \tag{3.6}$$

This is intended to be a sort of functional calculus generalizing (1.7), the idea being to mimick once again a formula that works well in the simple, commutative case. A certain convention used in constructing our W_τ^λ asks for the "symplectic" Fourier transformation.

The next result will show that we have already constructed this functional calculus. We need to take into account the algebraic tensor product (finite linear combinations of elementary tensors) $\mathfrak{L} := \mathcal{F}_{X^\sharp}L^1(X^\sharp)\odot\overline{\mathcal{F}}_X L^1(X)$. It can be naturally viewed as a subspace of $\mathcal{C}_0(X)\odot\mathcal{C}_0(X^\sharp)$.

It is simple to check that $\mathfrak{L}$ is a subspace of $\mathcal{F}_\Xi L^1(\Xi)$ (on which $\widetilde{\mathfrak{Op}}_\tau^\lambda$ is defined). If $\mathcal{C}_0(X)\subset\mathcal{A}$, then $\mathfrak{L}$ is also a subspace of $\mathcal{A}\odot\overline{\mathcal{F}}_X L^1(X)\subset\mathfrak{F}L^1(X;\mathcal{A})$ (on which $\mathfrak{Op}_\tau^\lambda$ is defined). We show that $\widetilde{\mathfrak{Op}}_\tau^\lambda(f)=\mathfrak{Op}_\tau^\lambda(f)$ for the elementary vector $f=(\mathcal{F}_{X^\sharp}a)\otimes\big(\overline{\mathcal{F}}_X b\big)$, with $a\in L^1(X^\sharp)$ and $b\in L^1(X)$. Note first that

$$\mathcal{F}_\Xi^{-1}f=\big(\overline{\mathcal{F}}_X\otimes\mathcal{F}_{X^\sharp}\big)^{-1}\left\{\mathcal{I}^{-1}\left[(\mathcal{F}_{X^\sharp}a)\otimes\big(\overline{\mathcal{F}}_X b\big)\right]\right\}=\big(\mathcal{F}_{X^\sharp}\otimes\overline{\mathcal{F}}_X\big)\left\{\big(\overline{\mathcal{F}}_X b\big)\otimes(\mathcal{F}_{X^\sharp}a)\right\}=b\otimes a.$$

Then

$$\widetilde{\mathfrak{Op}}_\tau^\lambda(f)=\int_\Xi d\xi\,(b\otimes a)(\xi)W_\tau^\lambda(\xi)=\int_X dy\; r\left[b(y)\int_{X^\sharp}d\chi\, a(\chi)\chi(-\tau y)\overline{\chi}\right]T^\lambda(y)=$$

$$=\int_X dy\; r\left[\theta_{\tau y}\big([\mathcal{F}_{X^\sharp}a\otimes b](y)\big)\right]T^\lambda(y)=\mathfrak{Op}_\tau^\lambda(f).$$

Thus we have proved

Proposition 3.4. *Assume that the abelian X-algebra $\mathcal{A}$ contains $\mathcal{C}_0(X)$. Then both $\mathfrak{Op}_\tau^\lambda$ and $\widetilde{\mathfrak{Op}}_\tau^\lambda$ are well-defined on $\mathfrak{L}$ and they coincide on this set.*

We regard (3.1) and (3.6) as special instances of the same object that makes sense for more general classes of symbols f, maybe in a weaker sense.

3.3 Extensions

One would like to extend the composition laws $\diamond_\tau^\omega$ and $\circ_\tau^\omega$ and the representations $\mathfrak{Rep}_\tau^\lambda$ and $\mathfrak{Op}_\tau^\lambda$ to more general symbols. We shall indicate only the extension results making use of multiplier algebras. As a rule, the extensions will be denoted by the same letters as before.

The general theory says that any C^*-algebra $\mathfrak{C}$ is embedded as an essential ideal in the (maximal) multiplier C^*-algebra $\mathfrak{M}(\mathfrak{C})$ and that any non-degenerate representation of $\mathfrak{C}$ extends to a representation of $\mathfrak{M}(\mathfrak{C})$. This should be applied respectively to the C^*-algebra $\mathcal{A}\rtimes_{\theta,\tau}^\omega X$ with the representation $\mathfrak{Rep}_\tau^\lambda$ and to the C^*-algebra $\mathfrak{C}_{\mathcal{A},\tau}^\omega$ with the representation $\mathfrak{Op}_\tau^\lambda$. We shall spell out only the case $\mathcal{A}=\mathcal{C}_0(X)$, that has some specific features.

Let us set $\mathcal{N}_{\diamond_\tau^\omega}:=\mathfrak{M}\left[\mathcal{C}_0(X)\rtimes_{\theta,\tau}^\omega X\right]$ and $\mathcal{N}_{\circ_\tau^\omega}:=\mathfrak{M}\left[\mathfrak{C}_{\mathcal{C}_0(X),\tau}^\omega\right]$. The partial Fourier transform $\mathfrak{F}$ extends to an isomorphism between these two C^*-algebras. By (b) and (d) of Proposition 2.17, $\mathfrak{Rep}_\tau^\lambda$ defines an isomorphism between $\mathcal{C}_0(X)\rtimes_{\mathcal{A},\tau}^\omega X$ and the ideal $\mathcal{K}(\mathcal{H})$ of all compact operators in $\mathcal{H}=L^2(X)$. So it extends to an isomorphism between the corresponding multiplier algebras. But the multiplier algebra of $\mathcal{K}(\mathcal{H})$ is the entire $\mathcal{B}(\mathcal{H})$. Concluding, the C^*-algebras $\mathcal{N}_{\diamond_\tau^\omega}$ and $\mathcal{N}_{\circ_\tau^\omega}$ are both represented faithfully and surjectively on $\mathcal{B}(\mathcal{H})$ respectively by $\mathfrak{Rep}_\tau^\lambda$ and $\mathfrak{Op}_\tau^\lambda$, the representations being connected to each other by the extension of the isomorphism $\mathfrak{F}$.

Now, of course, $\mathcal{N}_{\diamond_\tau^\omega}$ and $\mathcal{N}_{\circ_\tau^\omega}$ are defined only in a very implicit way. Even in simple case ($X=\mathbb{R}^N$, $\omega=1$) one only knows that certain classes of symbols belong to them. The Calderón-Vaillancourt Theorem (cf. [Fo]) is such a statement and a generalization to the case of a non-trivial ω would be interesting. For our general situation we shall give only rather simple results. The strategy is to extend explicitly $\mathfrak{Op}_\tau^\lambda$ or $\mathfrak{Rep}_\tau^\lambda$ to the desired space of functions or measures, by imposing that the corresponding operators are bounded.

Proposition 3.5. *The space $\mathcal{F}_\Xi\mathbb{M}(\Xi)$ of all (symplectic) Fourier transforms of bounded, complex measures is contained in $\mathcal{N}_{\circ_\tau^\omega}$.*

Proof. For $F\in\mathcal{F}_\Xi\mathbb{M}(\Xi)$, one defines $\mathfrak{Op}_\tau^\lambda(F)=\int_\Xi\big(\mathcal{F}_\Xi^{-1}F\big)(d\xi)W_\tau^\lambda(\xi)$ in a weak, dual sense: for $u,v\in\mathcal{H}$, $\left\langle v,\mathfrak{Op}_\tau^\lambda(F)u\right\rangle$ is obtained by applying the bounded complex measure $\mathcal{F}_\Xi^{-1}F$ to the bounded continuous function $\langle v,W_\tau^\lambda(\cdot)u\rangle$. This defines bounded operators. $\square$

Let us consider "the exponential functions" $\{F_{\xi_0}\}_{\xi_0\in\Xi}$, where for $\xi_0=(x_0,\chi_0)\in\Xi$ one sets $F_{\xi_0}(x,\chi)=\chi(x_0)\overline{\chi_0(x)}$. They are symplectic Fourier transforms of Dirac measures, $F_{\xi_0}=\mathcal{F}_\Xi\delta_{\xi_0}$, thus elements of $\mathcal{N}_{\circ_\tau^\omega}$. The Weyl system is obtained by applying $\mathfrak{Op}_\tau^\lambda$ to them: $\mathfrak{Op}_\tau^\lambda(F_{\xi_0})=W_\tau^\lambda(\xi_0)$, $\forall\xi_0\in\Xi$. A simple calculation shows that linear combinations of functions F_{ξ_0} do not form an algebra.

3.4 Example: $X = \mathbb{Z}^2$

The case $X = \mathbb{Z}^2$ leads, under some simplifying assumptions, to intensively studied objects as the rotation algebra and almost Mathieu operators. Our aim is to show that these objects emerge naturally and to allow comparisons with Section 4, in which $X = \mathbb{R}^N$.

We may work with any translational invariant C^*-subalgebra $\mathcal{A}$ of the C^*-algebra of bounded complex functions on $\mathbb{Z}^2$. In most of our arguments we shall take $\mathcal{A} = \mathbb{C}$ and this allows only the trivial action $\theta_x = id$, $\forall x \in \mathbb{Z}^2$. Then one deals with 2-cocycles $\omega : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T} \subset \mathbb{C}$. It follows from Prop. 6.2 of [Gui] that any such 2-cocycle is cohomologous with one of the form $\omega^B(x,y) = \exp[-iB(x,y)]$, where $B : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ is an antisymmetric biadditive map (with a bit of imagination one could call it *a constant magnetic field on the lattice*). To make the connection with standard notations, one sets for some $\alpha \in \mathbb{R}$

$$B(x,y) = \pi\alpha(x_1 y_2 - x_2 y_1) \equiv \pi\alpha x \wedge y. \tag{3.7}$$

and write ω_α instead of ω^B. Note that $\omega_\alpha(x,-x) = 0$, $\forall x \in \mathbb{Z}^2$.

Our twisted dynamical system is $(\mathbb{C}, id, \omega_\alpha, \mathbb{Z}^2)$. The space $L^1(\mathbb{Z}^2;\mathbb{C}) \equiv L^1(\mathbb{Z}^2)$, endowed with the structure given by

$$(\varphi \diamond \psi)(x) = \sum_{y\in\mathbb{Z}^2} \omega_\alpha(y,x)\varphi(y)\psi(x-y) = \sum_{y\in\mathbb{Z}^2} \exp\{-i\pi\alpha(y\wedge x)\}\varphi(y)\psi(x-y),$$

$$\varphi^\diamond(x) = \overline{\varphi(-x)}, \qquad \|\!|\varphi|\!\| = \sum_{y\in\mathbb{Z}^2} |\varphi(y)|$$

is a Banach *-algebra (the action θ being trivial, τ plays no role at all here). We denote by $C^*_{\omega_\alpha}(\mathbb{Z}^2) \equiv C^*_\alpha(\mathbb{Z}^2) := \mathbb{C} \rtimes^{\omega_\alpha}_{id} \mathbb{Z}^2$ the associated twisted crossed product. It is the twisted group C^*-algebra associated with the group $\mathbb{Z}^2$ and the multiplier ω_α. Traditionally it is called *the rotation algebra assigned to the real number* α and the usual notation is something like $\mathfrak{A}_\alpha$. It can also be defined as the untwisted crossed product of $\mathcal{C}(\mathbb{T})$ by a suitable action of $\mathbb{Z}$. For $\alpha = 0$ we get the group C^*-algebra of $\mathbb{Z}^2$; its spectrum is the 2-torus $\mathbb{T}^2$. For $\alpha \neq 0$ one obtains the so-called *noncommutative tori of dimension 2*. Similar objects may be defined for higher dimensions N (i.e. for $X = \mathbb{Z}^N$), cf. [Ri2]. In these cases the relevant input is an $N \times N$ real, antisymmetric matrix $(\alpha_{jk})_{j,k=1,\dots,N}$.

The group $\mathbb{Z}^2$ being discrete, the C^*-algebra $C^*_\alpha(\mathbb{Z}^2)$ is generated by the elements $\{\delta_x\}_{x\in\mathbb{Z}^2}$ with $\delta_x(y) := 0$ for $y \neq x$ and $\delta_x(x) := 1$, and is unital, with $\mathbf{1} = \delta_0$. These generators satisfy the relations: $\delta_x \diamond \delta_y = \omega_\alpha(x,y)\delta_{x+y}$, $\delta_x^\diamond = \delta_{-x}$. In fact, since $\mathbb{Z}^2$ is generated as a group by the elements $(1,0), (0,1)$, we recover easily the usual definition of the rotation algebra as the universal C^*-algebra generated by two unitary elements $u := \delta_{(1,0)}$ and $v := \delta_{(0,1)}$ satisfying $u \diamond v = e^{2\pi i\alpha} v \diamond u$ (cf. [Ri1], [Be1], [Bo] for instance).

One finds inside the rotation algebra elements with interesting spectral properties (see for instance [Be2], [Bo], [Sh2] and references therein). Let ν, μ be real numbers and u, v the elements introduced above. Then

$$h(\alpha,\nu,\mu) := u + u^* + \nu\left(e^{2\pi i\mu} v + e^{-2\pi i\mu} v^*\right) \in L^1(\mathbb{Z}^2) \subset C^*_\alpha(\mathbb{Z}^2)$$

is called *the almost Mathieu Hamiltonian*, and the simplified version $h(\alpha) := h(\alpha,1,0)$ is called *the Harper Hamiltonian.*

The example of **2.4** shows that for $\alpha \neq 0$ the 2-cocycles we consider are not trivial. But since our twisted dynamical system is standard, by Proposition 2.14, all ω_α's are pseudo-trivial, i.e. they are coboundaries with respect to the larger Polish module $\mathcal{C}(\mathbb{Z}^2;\mathbb{T})$. Thus we can find a function $\lambda \in C^1\big(\mathbb{Z}^2;\mathcal{C}(\mathbb{Z}^2;\mathbb{T})\big)$ such that $\delta^1(\lambda) = \omega_\alpha$. In fact one may take $[\lambda(y)](q) := \omega_\alpha(q,y)$ $\big($this is exactly the choice (2.7)$\big)$. We set $\mathcal{H} = L^2(\mathbb{Z}^2)$ (with the counting measure) and consider the covariant representation $(\mathcal{H}, r, T^\lambda)$ given for $u \in \mathcal{H}$ by:

$$r : \mathbb{C} \to \mathcal{B}(\mathcal{H}), \quad r(\nu)u := \nu u,$$

$$T^\lambda : \mathbb{Z}^2 \to \mathcal{U}(\mathcal{H}), \qquad [T^\lambda(y)u](x) := \omega_\alpha(x,y)u(y+x).$$

Then we get the associated representation $\mathfrak{Rep}^\lambda := r \rtimes T^\lambda : L^1(\mathbb{Z}^2) \to \mathcal{B}(\mathcal{H})$

$$\left[\mathfrak{Rep}^\lambda(\varphi)u\right](x) := \sum_{y\in\mathbb{Z}^2} \omega_\alpha(x,y)\varphi(y)u(x+y),$$

which extends to a representation of $C^*_\alpha(\mathbb{Z}^2)$.

The map $\mathbb{R}^2 \ni p \mapsto \epsilon_p \in \left(\mathbb{Z}^2\right)^\sharp$ given by $\epsilon_p(x) := e^{2\pi i x\cdot p} \equiv \exp\{2\pi i \sum_{j=1,2} x_j p_j\}$ is a surjective group morphism with kernel $\mathbb{Z}^2$. It follows that the dual group $\left(\mathbb{Z}^2\right)^\sharp$ can be identified with the quotient $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$.

We denote by $\mathfrak{C}^{\omega_\alpha}(\mathbb{Z}^2)$ the envelopping C^*-algebra of $\overline{\mathcal{F}}_{\mathbb{Z}^2}\big(L^1(\mathbb{Z}^2)\big)$ and define $\mathfrak{Op}^\lambda = \mathfrak{Rep}^\lambda \circ \overline{\mathcal{F}}_{\mathbb{Z}^2}^{-1}$. Then for $f \in C(\mathbb{T}^2)$, denoting by $\hat{f}_y$ its Fourier coefficient in y, we have the following representation on $\mathcal{H}$:

$$\left[\mathfrak{Op}^\lambda(f)u\right](x) = \sum_{y\in\mathbb{Z}^2} \omega_\alpha(x,y)\hat{f}_y u(x+y)$$

For the Moyal product we obtain:

$$(f \circ^{\omega_\alpha} g)(\theta) := \sum_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}\int_{\mathbb{T}^2} d\tau \int_{\mathbb{T}^2} d\gamma\, e^{2\pi i\theta\cdot x}\, e^{-2\pi i\tau\cdot y}\, e^{-2\pi i\gamma\cdot(x-y)}\,\omega_\alpha(y,x)\, f(\tau)\, g(\gamma) =$$

$$= \sum_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2} e^{2\pi i\theta\cdot x}\,\omega_\alpha(y,x)\hat{f}_y\,\hat{g}_{x-y}.$$

The dependence on α of the almost Mathieu Hamiltonian was hidden; but of course its spectral properties depend heavily on the C^*-algebra $C^*_\alpha(\mathbb{Z}^2)$ in which we consider it embedded. By representing it via $\mathfrak{Rep}^\lambda$ (other representations are also interesting), the α-dependence becomes explicit. The study of the resulting operators is one of the main trends in modern spectral theory.

We do not have much to say about cases more complicated than $\mathcal{A} = \mathbb{C}$, because we do not understand 2-cocycles in these situations (in contrast whith $X = \mathbb{R}^N$, for which $\mathcal{A}$-smooth magnetic fields define 2-cocycles; see Section 4). We just note that "weighted convolution operators" with very general weights are within reach. Let $\lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{T}$ be such that $\omega(q;x,y) = \frac{\lambda(q;x)\lambda(q+x;y)}{\lambda(q;x+y)}$ behaves like a function in $\mathcal{A}$ in the variable q, for x and y fixed. Let b be a complex function on $\mathbb{T}^N$. Then

$$[\mathfrak{op}^\lambda(b)u](x) = \sum_{y\in\mathbb{Z}^N} \lambda(x;y-x)\hat{b}_{y-x}u(y)$$

can be given a meaning under various conditions on b; see [Ne] for example. This is only a particular case of the pseudodifferential operators with symbols defined on $\mathbb{Z}^N \times \mathbb{T}^N$ that are covered by our formalism.

4 The magnetic case

We shall outline now how the setting and results of Sections 2 and 3 serve in the quantization of a non-relativistic particle without spin in a variable magnetic field.

We consider a quantum particle without internal structure moving in $X = \mathbb{R}^N$, in the presence of a variable magnetic field. The *magnetic field* is described by a closed continuous field of 2-forms B defined on $\mathbb{R}^N$. It is well-known that any such field B may be written as the differential dA of a field of 1-forms A, *a vector potential*, that is highly non-unique (the gauge ambiguity). By using coordinates, one has $B_{jk} = \partial_j A_k - \partial_k A_j$ for each $j,k = 1,\dots,N$.

In the presence of the field $B = dA$, the prescription (1.2) has to be modified. This topic was very rarely touched in the literature and the following wrong solution appears: The minimal coupling principle says roughly that the momentum p should be replaced with the magnetic momentum $\pi^A := p - A$. This originated in Lagrangian Classical Mechanics and works well also at the quantum level as long as the expressions are polynomials of order less or equal to 2. But if one just replaces in (1.2) $f\left(\frac{x+y}{2}, p\right)$ by $f\left[\frac{x+y}{2}, p - A\left(\frac{x+y}{2}\right)\right]$ one gets a formula which misses the right gauge covariance. Let us denote the result of this procedure for some function f in phase space by $\mathfrak{Op}_A(f)$. If another vector potential A' is chosen such that $A' = A + \nabla\rho$ with ρ a scalar function, then $dA' = dA$. But the expected formula $\mathfrak{Op}_{A'}(f) = e^{i\rho}\mathfrak{Op}_A(f)e^{-i\rho}$ is verified for some simple cases (A, A' linear and f arbitrary, or f polynomial of order strictly less than 3 in p and A, A' arbitrary), but in general it fails.

In fact a partial solution was offered long ago in [Lu], in connection with the Peierls substitution, for the case of functions depending only of p and periodic in this argument. It seems that it was forgotten, maybe because the most studied case is that of a constant field, for which the choice of a linear vector potential cannot lead to any trouble. In [KO1], [KO2], [MP2] these matters were tackled systematically, leading to a new, gauge invariant, pseudodifferential calculus, which was called *the magnetic Weyl calculus*. We review it here as a special, distinguished case of the preceding developments, underlining its connection with twisted dynamical systems, cf. also [MP1].

4.1 Magnetic twisted dynamical systems

From now on X will be the vector space $\mathbb{R}^N$; it is a particular instance of an abelian, second countable locally compact group. It is a standard fact that the group dual $X^\sharp$ of X can be identified with the vector space dual $X^\star$. Let us denote by $(x,p) \to x \cdot p$ the duality between X and $X^\star$. Then $\epsilon : X^\star \to X^\sharp$, $\epsilon_p(x) := e^{ix\cdot p}$ is the isomorphism (there are many others but we choose this one).

Particular 2-cocycles on X will be given by magnetic fields. *A magnetic field on X* is a closed continuous 2-form B. Since on $X = \mathbb{R}^N$ we have canonical global coordinates, we shall speak freely of the components B_{jk} of B; they are continuous real functions on X satisfying $B_{kj} = -B_{jk}$ and $\partial_j B_{kl} + \partial_l B_{jk} + \partial_k B_{lj} = 0$, $\forall j,k,l = 1,\dots,N$. It is well-known that $B = dA$ for some 1-form A on X, called *a vector potential*. The differential is viewed in a distributional sense. The vector potential is highly non-unique. We restrict only to continuous A; this is always possible, at least by *the transversal gauge*

$$A_j(x) := -\sum_{k=1}^{N} \int_0^1 ds\, B_{jk}(sx)sx_k. \tag{4.1}$$

Given a k-form C on X and a compact k-surface $\gamma \subset X$, we define

$$\Gamma^C(\gamma) := \int_\gamma C,$$

this integral having a well-defined parametrization independent meaning. We shall mainly encounter circulations of 1-forms along linear segments $\gamma = [x,y]$ and fluxes of 2-forms through triangles $\gamma = < x,y,z >$.

For a continuous magnetic field B one defines

$$\omega^B(q;x,y) := e^{-i\Gamma^B(<q,q+x,q+x+y>)} \qquad \text{for all } x,y,q \in X. \tag{4.2}$$

Let us now fix a separable X-algebra $\mathcal{A}$ with spectrum $S_\mathcal{A}$. Functions of type $\mathcal{A}$ (see Definition 2.12) are continuous on X, but one observes that they can be unbounded if $\mathcal{A}$ is not unital. The simplest example is obtained by considering $\mathcal{A} = \mathcal{C}_0(X)$ with $S_\mathcal{A}$ equal to X and $\delta^\mathcal{A}$ the identity on X. In the sequel we shall supress the notational difference between b and $\tilde{b}$ from Definition 2.12.

Definition 4.1. *A magnetic field B is of type $\mathcal{A}$ if all its components $\{B_{jk}\}_{j,k=1,\dots,N}$ are of type $\mathcal{A}$.*

Lemma 4.2. *If B is a magnetic field of type $\mathcal{A}$, then $(\mathcal{A},\theta,\omega^B,X)$ is a standard twisted dynamical system.*

Proof. The proof that ω^B is a normalized 2-cocyle, i.e. it satisfies relations (2.1) and (2.2), follows easily by direct calculations (for the first one use the Stokes Theorem for the closed 2-form B and the tetrahedron of vertices $q, q+x, q+x+y, q+x+y+z$).

We show that ω^B has the right continuity properties. It should define a mapping

$$X \times X \ni (x,y) \to \left[\omega^B(x,y)\right](\cdot) \equiv \omega^B(\cdot;x,y) \in \mathcal{C}(S_\mathcal{A};\mathbb{T}), \tag{4.3}$$

continuous with respect to the topology of uniform convergence on compact subsets of $S_\mathcal{A}$. But this is equivalent to the fact that ω^B defines un element of $\mathcal{C}(S_\mathcal{A} \times X \times X;\mathbb{T})$; this type of statement has already appeared in the proof of Lemma 2.9. Taking into account obvious properties of the exponential, this amounts to the fact that the function

$$\varphi^B : X \times X \times X \to \mathbb{R}, \quad \varphi^B(q;x,y) := \Gamma^B(< q, q+x, q+x+y >)$$

can be viewed (using the map $\delta^\mathcal{A}$ at the level of the first variable) as a continuous function on $S_\mathcal{A} \times X \times X$.

We use the parametrization

$$\varphi^B(q;x,y) = \sum_{j,k=1}^{N} x_j\, y_k \int_0^1 dt \int_0^1 ds\, s\, B_{jk}(q + sx + sty).$$

The continuous action θ defines a continuous mapping $\tilde{\theta} : X \times S_\mathcal{A} \to S_\mathcal{A}$, so one has the continuous correspondence $S_\mathcal{A} \times X \times X \ni (q;x,y) \to q + sx + sty = \tilde{\theta}_{sx+sty}(q) \in S_\mathcal{A}$. Since B_{jk} is seen as a continuous function: $S_\mathcal{A} \to \mathbb{R}$, the assertion follows easily. $\square$

We call $(\mathcal{A}, \theta, \omega^B, X)$ *the twisted dynamical system associated with the abelian algebra* $\mathcal{A}$ *and the magnetic field* B. In most of the cases the 2-cocycle $\omega^B \in Z^2\big(X; \mathcal{U}(\mathcal{A})\big)$ is not trivial. But as Proposition 2.14 shows, it is pseudo-trivial. In fact, its pseudo-trivialization can be achieved by a vector potential. Any continuous 1-form A defines a 1-cochain $\lambda^A \in C^1\big(X; \mathcal{C}(X; \mathbb{T})\big)$ via its circulation:

$$\left[\lambda^A(x)\right](q) \equiv \lambda^A(q; x) = e^{-i\Gamma^A([q,q+x])} = e^{-ix\cdot\int_0^1 ds\, A(q+sx)}. \tag{4.4}$$

As soon as $dA = B$, we have $\delta^1(\lambda^A) = \omega^B$ $\big($a priori with respect to $\mathcal{C}(X;\mathbb{T})\big)$, by a suitable version of Stokes Lemma. As said above, the transversal gauge offers a continuous vector potential corresponding to a given B. Actually, this is consistent with the choice (2.7) of a pseudo-trivialization of ω^B: for $q, x \in X$, $\lambda(q; x) := \omega^B(0; q, x) = e^{-i\Gamma^B <0,q,q+x>}$ and it follows immediatly that $\Gamma^B(<0, q, q + x>) = \Gamma^A([q, q + x])$, with A given by (4.1).

Depending on the X-algebra $\mathcal{A}$, different magnetic fields can give cohomologous 2-cocycles:

Definition 4.3. Let B_1, B_2 be two magnetic fields of type $\mathcal{A}$. We say that they are $\mathcal{A}$-*equivalent* if there exists a vector potential A with components of type $\mathcal{A}$ such that $B_2 - B_1 = dA$.

Proposition 4.4. *If* B_1, B_2 *are* $\mathcal{A}$*-equivalent, then* ω^{B_1} *and* ω^{B_2} *are cohomologous.*

Proof. If the components of A are of type $\mathcal{A}$, then considerations as in the proof of Lemma 4.2 show that $\lambda^A \in C^1\big(X; \mathcal{U}(\mathcal{A})\big) = C^1\big(X; \mathcal{C}(S_{\mathcal{A}}; \mathbb{T})\big)$. In addition $\omega^{B_2} = \omega^{B_1}\delta^1(\lambda^A)$. □

We explained above that any two magnetic fields that are continuous on X $\big($this means exactly that they are of type $\mathcal{C}_0(X)\big)$ are also $\mathcal{C}_0(X)$-equivalent. On the other hand two constant magnetic fields (of type $\mathbb{C}$) are $\mathbb{C}$-equivalent only if they coincide.

Let us point out that the natural approach, in the framework of twisted dynamical systems associated with magnetic fields, would be to start with a given magnetic field and consider the translational invariant C^*-algebra $\mathcal{A}$ that it generates. The magnetic field will be of type $\mathcal{A}$ by construction. Then, one may enlarge this C^*-algebra in order to fit some other asymptotic behaviours associated with the operators one intends to consider (see also [GI1], [MP1]).

Another interesting subject would be to understand to what extent the 2-cocycles and the group cohomology discussed in Subsection **2.3** may be related to magnetic fields (as above) and with de Rham cohomology. It is evident that requiring the cocycles to be multipliers on some given C^*-algebra induces important obstructions. Moreover, although de Rham cohomology of $\mathbb{R}^N$ is trivial, the vector potentials associated with some bounded magnetic field may no longer define good multipliers.

Now, specific standard twisted dynamical systems being constructed, the whole formalism of the preceding sections is released. *The twisted crossed product* $\mathcal{A} \rtimes_{\theta,\tau}^{\omega^B} X$, denoted slightly simpler by $\mathcal{A} \rtimes_{\theta,\tau}^{B} X$, *is said to be assigned to the* X*-algebra* $\mathcal{A}$*, the magnetic field* B *and the endomorphism* τ*.* As always, the dependence on τ is within isomorphism. If one replaces B by some $\mathcal{A}$-equivalent B', the C^*-algebras $\mathcal{A} \rtimes_{\theta,\tau}^{B} X$ and $\mathcal{A} \rtimes_{\theta,\tau}^{B'} X$ will be canonically isomorphic. For any continuous B the C^*-algebra $\mathcal{C}_0(X) \rtimes_{\theta,\tau}^{B} X$ is isomorphic to $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators in $\mathcal{H} = L^2(X)$.

The fact that the magnetic 2-cocycle ω^B satisfies

$$\omega^B(q; sx, tx) = 1, \quad \forall q, x \in X \quad \text{and} \quad \forall s, t \in \mathbb{R} \tag{4.5}$$

leads directly to the magnetic momenta. Let us fix some continuous A such that $dA = B$, thus $\delta^1(\lambda^A) = \omega^B$ $\big($pseudo-trivialization with respect to $\mathcal{C}(X;\mathbb{T})\big)$. Then λ^A will satisfy for all $q, x \in X$ and all $s, t \in \mathbb{R}$: $\lambda^A(q; sx + tx) = \lambda^A(q; sx)\lambda^A(q + sx; tx)$ (in general, if λ is not the exponential of *a circulation* this will not be true). We consider *the Schrödinger covariant representation* $(\mathcal{H}, r, T^{\lambda^A} \equiv T^A)$ *defined by* A, with $\mathcal{H} = L^2(X)$, $r(a) = a(Q)$ and

$$[T^A(y)u](x) = \lambda^A(x; y)u(x + y), \quad x, y \in X, \quad u \in \mathcal{H}.$$

The unitary operators $\{T^A(y)\}_{y\in X}$ are called *magnetic translations*. They appear often in the physical literature; we refer to [Lu] and [Za] for example. One has, by a short calculation,

$$T^A(sx + tx) = T^A(sx)T^A(tx), \quad \forall x \in X, \quad \forall s, t \in \mathbb{R} \tag{4.6}$$

and this also implies $T^A(-x) = T^A(x)^{-1}$ $\big(= T^A(x)^*\big)$, $\forall x \in X$. In fact, the formula

$$T^A(y)T^A(z) = r[\omega^B(y, z)]T^A(y + z), \quad y, z \in X$$

shows that (4.6) is equivalent with (4.5). For $t \in \mathbb{R}$ and $x \in X$, let us set $T_t^A(x) := T^A(tx)$. By (4.6), $\{T_t^A(x)\}_{t\in\mathbb{R}}$ is an evolution group in $\mathcal{H}$ for any x. Thus, by Stone Theorem, it has a self-adjoint generator that moreover depends linearly (as a linear operator on $\mathcal{H}$) on the vector $x \in X$; thus we denote it by $x \cdot \Pi^A$ and call it *the projection on x of the magnetic momentum associated with the vector potential A*. For any index $j \in \{1, ..., N\}$ we denote $\Pi_j^A := e_j \cdot \Pi^A$ the projection of the magnetic momentum on the j'th vector of the canonical base in X. A direct calculation shows that on $C_c^\infty(X)$ one has $\Pi_j^A = -i\partial_j - A_j(Q)$.

4.2 Magnetic pseudodifferential operators

One remote origin of the present work is the observation that a certain tentative to quantize systems in a variable magnetic field fails, lacking gauge-covariance.

We recall from **1.1** the formula (1.2) giving the Weyl prescription to quantize a symbol f defined on the phase-space $\Xi = X \times X^\star$. The background is a physical system composed of a non-relativistic, spinless particle moving in the configurational space X in the absence of any magnetic field. When a magnetic field B is turned on, a tentative to incorporate it was via the minimal coupling principle. This states, rather vaguely, that B can be taken into account by choosing a vector potential A corresponding to B and replacing the canonical variables (x, p) by $\big(x, p - A(x)\big)$. This would lead to the formula

$$[\mathfrak{Op}_A(f)u](x) := \int_{\mathbb{R}^{2N}} dy\, dp\, e^{i(x-y)\cdot p} f\left[\frac{x+y}{2}, p - A\left(\frac{x+y}{2}\right)\right] u(y) \tag{4.7}$$

which, in its turn, imposes a certain symbolic calculus, modifying the formula (1.3). In fact this cannot be correct, since under a change of the vector potential $A' = A + \nabla\rho$ one fails to obtain the natural covariance relation $\mathfrak{Op}_{A'}(f) = e^{i\rho(Q)}\mathfrak{Op}_A(f)e^{-i\rho(Q)}$.

The solution was offered (independently) in [KO1] and [MP2]. We refer also to [MP1] and [KO2] to some related works and note that the right attitude (undeveloped and stated only within a restricted setting) has already appeared in [Lu]. Nevertheless the formula (4.7) still works in two important particular cases, which were studied mostly. One is that of a linear vector potential A (giving a constant magnetic field) and an arbitrary (reasonable) symbol f. The other is obtained by taking an arbitrary (continuous) A but restricting f to be a polynomial of order ≤ 2 in p with constant coefficients. We refer to [MP2], Subsection 3.4 for a discussion on this point.

In the present context, quantizing the above physical system in the presence of the magnetic field follows as a particular case of the developments in Sections **2** and **3**. At the level of twisted dynamical systems and twisted crossed products the particularization was initiated in **4.1**. The situation in pseudodifferential terms will be discussed now. This could serve as a useful résumé for the reader. Everything has already been proved at a more general level.

So let B be a continuous magnetic field and A a corresponding continuous vector potential. The 2-cocycle ω^B and the 1-cochain λ^A are defined, repectively, in (4.3) and (4.4). In our case $X = \mathbb{R}^N$ and $\tau = 1/2$ is an endomorphism of X. It leads to the most symmetric formulae, so we shall restrict to $\tau = 1/2$ for simplicity. It was shown before that other choices lead to isomorphic formalisms.

The magnetic Weyl system. For $X = \mathbb{R}^N$, $\Xi := X \times X^\star$ is the usual phase space with the symplectic structure defined by the canonical symplectic form

$$\sigma[(x,p),(y,k)] := y \cdot p - x \cdot k, \qquad (x,p),(y,k) \in \Xi.$$

Associated with the Schrödinger covariant representation $(\mathcal{H}, r, T^A)$ defined above, we can define now the *magnetic Weyl system* W^A

$$\Xi \ni (x,p) \mapsto W^A\big((x,p)\big) := e^{-\frac{i}{2}x\cdot p} e^{-iQ\cdot p} T^A(x) \in \mathcal{U}(\mathcal{H}).$$

These unitary operators define then a projective representation satisfying

$$W^A(\xi)\, W^A(\eta) = e^{\frac{i}{2}\sigma(\xi,\eta)}\, \omega^B(x,y)\, W^A(\xi+\eta)$$

where $\xi = (x,p)$, $\eta = (y,k)$.

The symplectic structure comes into play at a rather late stage because at the beginning we were placed in a very general setting. Anyway, our starting point was "a twisted action of the group X on an abelian algebra of position observables" and this is nearly an extension of the framework of "imprimitivity systems", cf. [Va]. But it is perfectly natural (and well-suited to the classical picture) to start from a symplectic formalism. This is done in [KO1] and [KO2]. The canonical symplectic form and the magnetic field are melt together into a B-depending symplectic form and this is at the basis of the quantization procedure. A beautiful geometric picture emerges, for which we refer to the quoted papers.

The magnetic Weyl calculus. For any $f \in \mathcal{S}(\Xi)$ we define the operator $\mathfrak{Op}^A(f) := \mathfrak{Op}^{\lambda^A}_{1/2}(f)$ in $\mathcal{H} = L^2(X)$ and obtain

$$\left[\mathfrak{Op}^A(f)u\right](x) = \int_X \int_{X^\star} dy\, dp\, e^{i(x-y)\cdot p} e^{-i\Gamma^A([x,y])} f\left(\frac{x+y}{2}, p\right) u(y).$$

It is easy to observe that this is an integral operator with kernel

$$K^A := \tilde{\lambda}^A S^{-1} (\mathbf{1} \otimes \overline{\mathcal{F}}_{X^\star})$$

where $\tilde{\lambda}^A(x,y) := \lambda^A(x; y-x)$ and $\left(S^{-1}h\right)(x,y) = h\left(\frac{x+y}{2}, x-y\right)$. It is now easy to extend the map K^A and thus define $\mathfrak{Op}^A(F)$ for any $F \in \mathcal{S}'(\Xi)$ as the integral operator with kernel $K^A(F)$, defined on $\mathcal{S}(X)$ with values in $\mathcal{S}'(X)$. It seems legitimate to view the correspondence $f \to \mathfrak{Op}^A(f)$ as a functional calculus for the family of self-adjoint operators $Q_1, \dots, Q_N, \Pi_1^A, \dots, \Pi_N^A$. The high degree of non-commutativity of these $2N$ operators stays at the origin of the sophistication of the symbolic calculus. The commutation relations

$$i[Q_j, Q_k] = 0, \quad i[\Pi_j^A, Q_k] = \delta_{jk}, \quad i[\Pi_j^A, \Pi_k^A] = -B_{jk}(Q), \quad j,k = 1, \dots, N \tag{4.8}$$

collapse for $B = 0$ to the canonical commutation relations satisfied by Q and P. But they are much more complicated, especially when B is not a polynomial. In particular, no simple analogue of the Heisenberg group is available. The main mathematical miracle that allows, however, a nice treatment is the fact that (4.8) can be recast in the form of a covariant representation of a twisted dynamical system. And this is connected to its symplectic flavour.

Let us emphasis here that the functional calculus that we have defined is gauge covariant, in the sense that it satisfies the property: If $A' = A + \nabla\varphi$ with $\varphi : X \to \mathbb{R}$ continuous, then $\mathfrak{Op}^{A'}(f) = e^{i\varphi(Q)} \mathfrak{Op}^A(f) e^{-i\varphi(Q)}$. This gauge covariance property may be seen as a special instance of Proposition 3.1 (b).

The magnetic symbolic calculus. The usual product and adjoint operation on $\mathcal{B}(\mathcal{H})$ allows now to define on $\mathcal{S}(\Xi)$ a composition and an involution:

$$(f \circ^B g)(\xi) = 4^N \int_\Xi \int_\Xi d\eta\, d\zeta\, e^{-2i\sigma(\xi-\eta, \xi-\zeta)} e^{-i\Gamma^B(<q-y+x, x-q+y, y-x+q>)} f(\eta) g(\zeta), \tag{4.9}$$

$\big($for $\xi = (q,l)$, $\eta = (x,p)$ and $\zeta = (y,k)\big)$

$$f^{\circ^B}(\xi) = \overline{f(\xi)}, \quad \forall \xi \in \Xi$$

such that

$$\mathfrak{Op}^A(f \circ^B g) = \mathfrak{Op}^A(f)\mathfrak{Op}^A(g), \quad \mathfrak{Op}^A(f^{\circ^B}) = \mathfrak{Op}^A(f)^*, \quad (f \circ^B g)^{\circ^B} = g^{\circ^B} \circ^B f^{\circ^B}.$$

Let us remark that the involution ${}^{\circ^B}$ and the product $\circ^B$ are defined intrinsically, without any choice of a vector potential. The choice is only needed when we represent the resulting structures in a Hilbert space. We call (4.9) *the magnetic Moyal product*. The involution ${}^{\circ^B}$ does not depend on B at all. This is no longer true if $\tau \neq 1/2$. The property $\omega^B(x, -x) = 1$, $\forall x \in X$ is also used to get the simple form of ${}^{\circ^B}$.

Let us assume now that B is of type $\mathcal{A}$ for some X-algebra $\mathcal{A}$. The C^*-algebra $\mathfrak{C}^{\omega^B}_{\mathcal{A},1/2}$, introduced in **3.1**, will be denoted by $\mathfrak{C}^B_{\mathcal{A}}$. We call it *the C^*-algebra of pseudodifferential symbols of class $\mathcal{A}$ associated with B*. We recall that it is essentially a partial Fourier transform of the twisted crossed product $\mathcal{A} \rtimes^B_{\theta,1/2} X$. The formulae defining the magnetic Weyl calculus make sense at least on the dense subset $\mathfrak{F}L^1(X;\mathcal{A})$, with iterated integrals. The extension of $\mathfrak{Op}^A$ is a faithful representation of the C^*-algebra $\mathfrak{C}^B_{\mathcal{A}}$ for any continuous A with $dA = B$. If $\mathcal{C}_0(X) \subset \mathcal{A}$, then $\mathfrak{Op}^A$ is irreducible.

Remark 4.5. (1) If B and B' are of type $\mathcal{A}$ and $\mathcal{A}$-equivalent, then the C^*-algebras $\mathfrak{C}^B_{\mathcal{A}}$ and $\mathfrak{C}^{B'}_{\mathcal{A}}$ are isomorphic. (2) In particular, $\mathfrak{C}^B_{\mathcal{C}_0(X)}$ and $\mathfrak{C}^{B'}_{\mathcal{C}_0(X)}$ are always isomorphic if B and B' are of type $\mathcal{C}_0(X)$ (and this means just that the components B_{jk}, B'_{jk} are continuous functions on X, maybe unbounded). In this case, for any continuous A defining B, $\mathfrak{Op}^A$ sends $\mathfrak{C}^B_{\mathcal{C}_0(X)}$ isomorphically on $\mathcal{K}[L^2(X)]$.

It is very useful that the usual pseudodifferential calculus can be extended to unbounded symbols. This works well for the magnetic version. The modest extension strategy of **3.3** can now be greatly improved; in $X = \mathbb{R}^N$ there are plenty of extra structures that can be exploited. First of all, one notes that $\mathfrak{Op}^A$ and $\mathfrak{Rep}^A$ are integral operators with L^2 kernels. But if the magnetic field is of class $\mathcal{C}^\infty_{\rm pol}$ (i.e. it is $\mathcal{C}^\infty$ and all its derivatives verify polynomial bounds) then one can choose A in the same class and the kernel will have an improved behaviour. It follows (see [MP2], Subsection 3.2) that one can extend, in a suitable weak sense, $\mathfrak{Op}^A$ to temperate distributions; the same will be true for $\mathfrak{Rep}^A$.

It is more difficult to extend significantly the product. Usually, in a pseudodifferential setting, one tries oscillatory integral techniques. This gives refined results but the details are sometimes cumbersome. In [GBV1] and [GBV2] it is shown how to extend the standard Moyal product (1.3) by duality methods, such to obtain large *-algebras of distributions. Under the assumption that B is in $\mathcal{C}^\infty_{\rm pol}$, this can also be done for the more complicated magnetic Moyal product (4.9). This is the subject of Section 4 of [MP2]. By a partial Fourier transform or by an analogous direct approach, this works also for the composition law $\diamond^B \equiv \diamond^{\omega^B}_{1/2}$. We do not give details here.

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