The Method of the Weakly Conjugate Operator: Extensions and Applications to Operators on Graphs and Groups

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Abstract

In this review we present some recent extensions of the method of the weakly conjugate operator. We illustrate these developments through examples of operators on graphs and groups.

Introduction

In spectral analysis, one of the most powerful tools is the method of the conjugate operator, also called Mourre's commutator method after the seminal work of Mourre in the early eighties. This approach has reached a very high degree of precision and abstraction in [1]; see also [14] for further developments. In order to study the nature of the spectrum of a selfadjoint operator H, the main idea of the standard method is to find an auxiliary selfadjoint operator A such that the commutator $i[H, A]$ is strictly positive when localized in some interval of the spectrum of H. More precisely, one looks for intervals J of $\mathbb{R}$ such that

$$E(J)i[H, A]E(J) \geq aE(J) \tag{1}$$

for some strictly positive constant a that depends on J, where $E(J)$ denotes the spectral projection of H on the interval J. An additional compact contribution to (1) is allowed, greatly enlarging the range of applications.

When strict positivity is not available, one can instead look for an A such that the commutator is positive and injective, *i.e.*

$$i[H, A] > 0. \tag{2}$$

This requirement is close to the one of the Kato-Putnam theorem, *cf.* [29, Thm. XIII.28]. A new commutator method based on such an inequality was proposed in [7, 8]. By analogy to the

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method of the conjugate operator, it has been called the method of the weakly conjugate operator (MWCO). Under some technical assumptions, both approaches lead to a limiting absorption principle, that is, a control of the resolvent of H near the real axis. In the case of the usual method of the conjugate operator, this result is obtained locally in J, and away from thresholds. The MWCO establishes the existence of the boundary value of resolvent also at thresholds, but originally applies only to situations where the operator H has a purely absolutely continuous spectrum. This drawback limits drastically the range of applications. However, in some recent works [26, 27, 30] the MWCO has also been applied successfully to examples with point spectrum. This review intend to present and illustrate some of these extensions through applications to the spectral theory of operators acting on groups and graphs.

Compared to the huge number of applications based on an inequality of the form (1), the number of papers that contain applications of the MWCO is very small. Let us cite for example the works [13, 24, 25] that deal with the original form of the theory, and the papers [12, 20] that contain very close arguments. The derivation of the limiting absorption principle of [12] has been abstracted in [30]. The framework of [30] is still the one of the MWCO, but since its result applies a certain class of two-body Schrödinger operators which have bound states below zero, it can also be considered as the first extension of the MWCO dealing with operators that are not purely absolutely continuous. The main idea of [30] is that H itself can add some positivity to (2). The new requirement is the existence of a constant $c \geq 0$ such that

$$-cH + i[H, A] > 0.$$

This inequality, together with some technical assumptions, lead to a limiting absorption principle which is either uniform on $\mathbb{R}$ if $c = 0$ or uniform on $[0, \infty)$ if $c > 0$.

The extensions of the MWCO developed in [26, 27] are of a different nature. In these papers, the operators H under consideration admit a natural conjugate operator A that fulfills the inequality $i[H, A] \geq 0$, namely, the commutator is positive but injectivity may fail. In that situation, the authors considered a decomposition $\mathcal{H} := \mathcal{K} \oplus \mathcal{G}$, and the restrictions of H and $i[H, A]$ to these subspaces. In favorable circumstances the injectivity can be restored in one of the subspace, and a comprehensible description of the vectors of the second subspace can be given. This decomposition leads again to statements that are close to the ones of the MWCO, but which apply to operators with arbitrary spectrum. This extension is described in Section The extension.

We would also like to mention the references [2, 3]. They pereform what can be considered as a unitary version of the MWCO and extend the Kato-Putnam analysis of unitary operators to the case of unbounded conjugate operators. The main applications concern time-depending propagators.

The content of this review paper is the following. In Section The Method of the Weakly Conjugate Operator we recall the original method of the weakly conjugate operator, and then present an abstract version of the approach used in [26, 27]. In Section Spectral Analysis for Adjacency Operators on Graphs we present applications of this approach to the study of adjacency operators on graphs. A similar analysis for operators of convolution on locally compact groups is performed in Section Convolution Operators on Locally Compact Groups.

Let us finally fix some notations. Given a selfadjoint operator H in a Hilbert space $\mathcal{H}$, we write $\mathcal{H}_{\mathrm{c}}(H)$, $\mathcal{H}_{\mathrm{ac}}(H)$, $\mathcal{H}_{\mathrm{sc}}(H)$, $\mathcal{H}_{\mathrm{s}}(H)$ and $\mathcal{H}_{\mathrm{p}}(H)$ respectively for the continuous, absolutely continuous, singularly continuous, singular and pure point subspaces of $\mathcal{H}$ with respect to H. The corresponding parts of the spectrum of H are denoted by $\sigma_{\mathrm{c}}(H)$, $\sigma_{\mathrm{ac}}(H)$, $\sigma_{\mathrm{sc}}(H)$, $\sigma_{\mathrm{s}}(H)$ and $\sigma_{\mathrm{p}}(H)$.

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The Method of the Weakly Conjugate Operator

In this section we recall the basic characteristics of the method of the weakly conjugate operator, as originally introduced and applied to partial differential operators in [7, 8]. We then present the abstract form of the extension developed in [26, 27]. The method works for unbounded operators, but for our purposes it is enough to assume H bounded.

The standard theory

We start by introducing some notations. The symbol $\mathcal{H}$ stands for a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)$ the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and put $\mathscr{B}(\mathcal{H}):=\mathscr{B}(\mathcal{H},\mathcal{H})$. We assume that $\mathcal{H}$ is endowed with a strongly continuous unitary group $\{W_t\}_{t\in\mathbb{R}}$. Its selfadjoint generator is denoted by A and has domain $D(A)$. In most of the applications A is unbounded.

Definition 1. *A bounded selfadjoint operator H in $\mathcal{H}$ belongs to $C^1(A;\mathcal{H})$ if one of the following equivalent condition is satisfied:*

(i) the map $\mathbb{R}\ni t\mapsto W_{-t}HW_t\in\mathscr{B}(\mathcal{H})$ is strongly differentiable,

(ii) the sesquilinear form

$$D(A)\times D(A)\ni(f,g)\mapsto i\langle Hf,Ag\rangle-i\langle Af,Hg\rangle\in\mathbb{C}$$

is continuous when $D(A)$ is endowed with the topology of $\mathcal{H}$.

We denote by B the strong derivative in (i), or equivalently the bounded selfadjoint operator associated with the extension of the form in (ii). The operator B provides a rigorous meaning to the commutator $i[H,A]$. We write $B>0$ if B is positive and injective, namely if $\langle f,Bf\rangle>0$ for all $f\in\mathcal{H}\setminus\{0\}$.

Definition 2. *The operator A is* weakly conjugate to *the bounded selfadjoint operator H if $H\in C^1(A;\mathcal{H})$ and $B\equiv i[H,A]>0$.*

For $B>0$ let us consider the completion $\mathcal{B}$ of $\mathcal{H}$ with respect to the norm $\|f\|_{\mathcal{B}}:=\langle f,Bf\rangle^{1/2}$. The adjoint space $\mathcal{B}^*$ of $\mathcal{B}$ can be identified with the completion of $B\mathcal{H}$ with respect to the norm $\|g\|_{\mathcal{B}^*}:=\langle g,B^{-1}g\rangle^{1/2}$. One has then the continuous dense embeddings $\mathcal{B}^*\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{B}$, and B extends to an isometric operator from $\mathcal{B}$ to $\mathcal{B}^*$. Due to these embeddings it makes sense to assume that $\{W_t\}_{t\in\mathbb{R}}$ restricts to a C_0-group in $\mathcal{B}^*$, or equivalently that it extends to a C_0-group in $\mathcal{B}$. Under this assumption (tacitly assumed in the sequel) we keep the same notation for these C_0-groups. The domain of the generator of the C_0-group in $\mathcal{B}$ (resp. $\mathcal{B}^*$) endowed with the graph norm is denoted by $D(A,\mathcal{B})$ (resp. $D(A,\mathcal{B}^*)$). In analogy with Definition 1 the requirement $B\in C^1(A;\mathcal{B},\mathcal{B}^*)$ means that the map $\mathbb{R}\ni t\mapsto W_{-t}BW_t\in\mathscr{B}(\mathcal{B},\mathcal{B}^*)$ is strongly differentiable, or equivalently that the sesquilinear form

$$D(A,\mathcal{B})\times D(A,\mathcal{B})\ni(f,g)\mapsto i\langle f,BAg\rangle-i\langle Af,Bg\rangle\in\mathbb{C}$$

is continuous when $D(A,\mathcal{B})$ is endowed with the topology of $\mathcal{B}$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{B}$ and $\mathcal{B}^*$. Finally let $\mathcal{E}$ be the Banach space $\big(D(A,\mathcal{B}^*),\mathcal{B}^*\big)_{1/2,1}$ defined by real interpolation, see for example [1, Proposition 2.7.3]. One has then the natural continuous embeddings $\mathscr{B}(\mathcal{H})\subset\mathscr{B}(\mathcal{B}^*,\mathcal{B})\subset\mathscr{B}(\mathcal{E},\mathcal{E}^*)$ and the following results [8, Theorem 2.1]:

Theorem 3. *Assume that A is weakly conjugate to H and that $B\equiv i[H,A]$ belongs to $C^1(A;\mathcal{B},\mathcal{B}^*)$. Then there exists a constant $\mathrm{c}>0$ such that*

$$\big|\big\langle f,(H-\lambda\mp i\mu)^{-1}f\big\rangle\big|\leq \mathrm{c}\|f\|_{\mathcal{E}}^2$$

for all $\lambda\in\mathbb{R}$, $\mu>0$ and $f\in\mathcal{E}$. In particular the spectrum of H is purely absolutely continuous.

The global limiting absorption principle plays an important role for partial differential operators, *cf.* [7, 13, 24, 30]. However, for the examples included sections 3 and 4, the space $\mathcal{E}$ involved has a rather obscure meaning and cannot be greatly simplified, so we shall only state the spectral results.

The extension

The extension proposed in [26, 27] relies on the following observation. Assume that H is a bounded selfadjoint operator in a Hilbert space $\mathcal{H}$. Assume also that there exists a selfadjoint operator A such that $H\in C^1(A;\mathcal{H})$, and that

$$B\equiv i[H,A]=K^2 \tag{3}$$

for some selfadjoint operator K, which is bounded by hypothesis. It follows that $i[H,A]\geq 0$ but injectivity may not be satisfied. So let us introduce the decomposition of the Hilbert space $\mathcal{H}:=\mathcal{K}\oplus\mathcal{G}$ with $\mathcal{K}:=\ker(K)$. The operator B is reduced by this decomposition and its restriction B_0 to $\mathcal{G}$ satisfies $B_0>0$. Formally, the positivity and injectivity of B_0 are rather promising, but are obviously not sufficient for a direct application of the MWCO to B_0.
However, let us already observe that some informations on $\mathcal{H}_{\mathrm{p}}(H)$ can be inferred from (3). Indeed, it follows from the Virial Theorem [1, Prop. 7.2.10] that any eigenvector f of H satisfies $\langle f,i[H,A]f\rangle=0$. So $0=\big\langle f,K^2f\big\rangle=\|Kf\|^2$, *i.e.* $f\in\ker(K)$, and one has proved:

Lemma 4. *If $H\in C^1(A;\mathcal{H})$ and $i[H,A]=K^2$, then $\mathcal{H}_{\mathrm{p}}(H)\subset\mathcal{K}=\ker(K)$.*

Let us now come back to the analysis of B_0. The space $\mathcal{B}$ can still be defined in analogy with what has been presented in the previous section: $\mathcal{B}$ is the completion of $\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{B}}:=\langle f,Bf\rangle^{1/2}$. Then, the adjoint space $\mathcal{B}^*$ of $\mathcal{B}$ can be identified with the completion of $B\mathcal{G}$ with respect to the norm $\|g\|_{\mathcal{B}^*}:=\big\langle g,B^{-1}g\big\rangle^{1/2}$. But in order to go further on, some compatibility has be imposed between the decomposition of the Hilbert space and the operators H and A. Let us assume that both operators H and A are reduced by the decomposition $\mathcal{K}\oplus\mathcal{G}$ of $\mathcal{H}$, and let H_0 and A_0 denote their respective restriction to $\mathcal{G}$. It clearly follows from the above lemma that H_0 has no point spectrum.
We are now in a suitable position to rephrase Theorem 3 in our present framework. We freely use the notations introduced above.

Theorem 5. *Assume that $H\in C^1(A;\mathcal{H})$ and that $B\equiv i[H,A]=K^2$. Assume furthermore that both operators H and A are reduced by the decomposition $\ker(K)\oplus\ker(K)^\perp$ of $\mathcal{H}$ and that B_0 belongs to $C^1(A_0;\mathcal{B},\mathcal{B}^*)$. Then a limiting absorption principle holds for H_0 uniformly on $\mathbb{R}$, and in particular the spectrum of H_0 is purely absolutely continuous.*

A straightforward consequence of this statement is that

$$\mathcal{H}_{\mathrm{sc}}(H) \subset \mathcal{H}_{\mathrm{s}}(H) \subset \mathcal{K} = \ker(K).$$

We will see in the applications below that Theorem 5 applies to various situations, and that it really is a useful extension of the original method of the weakly conjugate operator.

Spectral Analysis for Adjacency Operators on Graphs

We present in this section the results of [26] on the spectral analysis for adjacency operators on graphs. We follow the notations and conventions of this paper regarding graph theory.
A *graph* is a couple $(X, \sim)$ formed of a non-void countable set X and a symmetric relation $\sim$ on X such that $x \sim y$ implies $x \neq y$. The points $x \in X$ are called vertices and couples $(x, y) \in X \times X$ such that $x \sim y$ are called edges. So, for simplicity, multiple edges and loops are forbidden in our definition of a graph. Occasionally $(X, \sim)$ is said to be a simple graph. For any $x \in X$ we denote by $N(x) := \{y \in X : y \sim x\}$ the set of neighbours of x. We write $\deg(x) := \# N(x)$ for the degree or valence of the vertex x and $\deg(X) := \sup_{x \in X} \deg(x)$ for the degree of the graph. We also suppose that $(X, \sim)$ is uniformly locally finite, *i.e.* that $\deg(X) < \infty$. A path from x to y is a sequence $p = (x_0, x_1, \ldots, x_n)$ of elements of X, usually denoted by $x_0 x_1 \ldots x_n$, such that $x_0 = x$, $x_n = y$ and $x_{j-1} \sim x_j$ for each $j \in \{1, \ldots, n\}$.
Throughout this section we restrict ourselves to graphs $(X, \sim)$ which are simple, infinite countable and uniformly locally finite. Given such a graph we consider the adjacency operator H acting in the Hilbert space $\mathcal{H} := \ell^2(X)$ as

$$(Hf)(x) := \sum_{y \sim x} f(y), \quad f \in \mathcal{H},\ x \in X.$$

Due to [28, Theorem 3.1], H is a bounded selfadjoint operator with $\|H\| \leq \deg(X)$ and spectrum $\sigma(H) \subset [-\deg(X), \deg(X)]$.
Results on the nature of the spectrum of adjacency operators on graphs are quite sparse. Some absolutely continuous examples are given in [28], including the lattice $\mathbb{Z}^n$ and homogeneous trees. For cases in which singular components are present we refer to [11], [17], [31] and [32].
We now introduce the key concept of [26]. Sums over the empty set are zero by convention.

Definition 6. *A function* $\Phi : X \to \mathbb{R}$ *is* semi-adapted to the graph $(X, \sim)$ *if*

(i) there exists $\mathrm{c} \geq 0$ *such that* $|\Phi(x) - \Phi(y)| \leq \mathrm{c}$ *for all* $x, y \in X$ *with* $x \sim y$,

(ii) for any $x, y \in X$ *one has*

$$\sum_{z \in N(x) \cap N(y)} [2\Phi(z) - \Phi(x) - \Phi(y)] = 0. \tag{4}$$

If in addition for any $x, y \in X$ *one has*

$$\sum_{z \in N(x) \cap N(y)} [\Phi(z) - \Phi(x)]\,[\Phi(z) - \Phi(y)]\,[2\Phi(z) - \Phi(x) - \Phi(y)] = 0, \tag{5}$$

then Φ *is* adapted to the graph $(X, \sim)$.

For a function Φ semi-adapted to the graph $(X,\sim)$ we consider in $\mathcal{H}$ the operator K given by

$$(Kf)(x) := i\sum_{y\sim x}\left[\Phi(y)-\Phi(x)\right]f(y), \quad f\in\mathcal{H},\ x\in X.$$

The operator K is selfadjoint and bounded due to the condition (i) of Definition 6. It commutes with H, as a consequence of Condition (4). We also decompose the Hilbert space $\mathcal{H}$ into the direct sum $\mathcal{H}=\mathcal{K}\oplus\mathcal{G}$, where $\mathcal{G}$ is the closure of the range $K\mathcal{H}$ of K, thus the orthogonal complement of the closed subspace

$$\mathcal{K} := \ker(K) = \left\{f\in\mathcal{H} : \textstyle\sum_{y\in N(x)}\Phi(y)f(y) = \Phi(x)\sum_{y\in N(x)}f(y)\ \text{ for each } x\in X\right\}.$$

It is shown in [26, Sec. 4] that H and K are reduced by this decomposition, and that their restrictions H_0 and K_0 to the Hilbert space $\mathcal{G}$ are bounded selfadjoint operators.
A rather straightforward application of the general theory presented in section The extension gives

Theorem 7 (Theorem 3.2 of [26])**.** *Assume that Φ is a function semi-adapted to the graph $(X,\sim)$. Then H_0 has no point spectrum.*

Theorem 8 (Theorem 3.3 of [26])**.** *Let Φ be a function adapted to the graph $(X,\sim)$. Then the operator H_0 has a purely absolutely continuous spectrum.*

The role of the weakly conjugate operator is played by $A:=\frac{1}{2}(\Phi K+K\Psi)$ and the assumptions imposed on Φ make the general theory work.
For a certain class of admissible graphs, the result of Theorem 8 on the restriction H_0 can be turned into a statement on the original adjacency operator H. The notion of admissibility requires (among other things) the graph to be directed. Thus the family of neighbors $N(x):=\{y\in X : y\sim x\}$ is divided into two disjoint sets $N^-(x)$ (fathers) and $N^+(x)$ (sons), $N(x)=N^-(x)\sqcup N^+(x)$. We write $y<x$ if $y\in N^-(x)$ and $x<y$ if $y\in N^+(x)$. On drawings, we set an arrow from y to x ($x\leftarrow y$) if $x<y$, and say that the edge from y to x is positively oriented.
We assume that the directed graph subjacent to X, from now on denoted by $(X,<)$, is *admissible* with respect to these decompositions, *i.e.* (i) it admits a position function and (ii) it is uniform. *A position function* is a function $\Phi : X\to\mathbb{Z}$ such that $\Phi(y)+1=\Phi(x)$ whenever $y<x$. It is easy to see that it exists if and only if all paths between two points have the same index (which is the difference between the number of positively and negatively oriented edges). The directed graph $(X,<)$ is called *uniform* if for any $x,y\in X$ the number $\#\left[N^-(x)\cap N^-(y)\right]$ of common fathers of x and y equals the number $\#\left[N^+(x)\cap N^+(y)\right]$ of common sons of x and y. Thus the admissibility of a directed graph is an explicit property that can be checked directly, without making any choice. The graph $(X,\sim)$ is admissible if there exists an admissible directed graph subjacent to it.

Theorem 9 (Theorem 1.1 of [26])**.** *The adjacency operator of an admissible graph $(X,\sim)$ is purely absolutely continuous, except at the origin, where it may have an eigenvalue with eigenspace*

$$\ker(H) = \left\{f\in\mathcal{H} : \textstyle\sum_{y<x}f(y) = 0 = \sum_{y>x}f(y)\ \textit{for each } x\in X\right\}. \tag{6}$$

Many examples of periodic graphs, both admissible and non-admissible, are presented in [26, Sec. 6]. In particular, it is explained that periodicity does not lead automatically to absolute continuity, especially (but not only) if the number of orbits is infinite. D-products of graphs, as well as the graph associated with the one-dimensional XY Hamiltonian, are also treated in [26]. We recall in Figures 1, 2, and 5 some two-dimensional $\mathbb{Z}$-periodic examples taken from [26, Sec. 6]. More involved, $\mathbb{Z}^n$-periodic situations are also available.

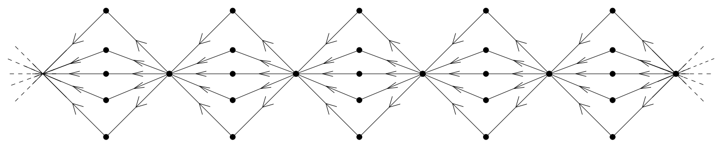

Fig. 1: Example of an admissible directed graph with $\ker(H) \neq \{0\}$

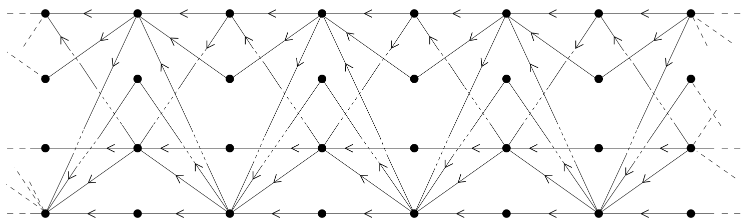

Fig. 2: Example of an admissible directed graph

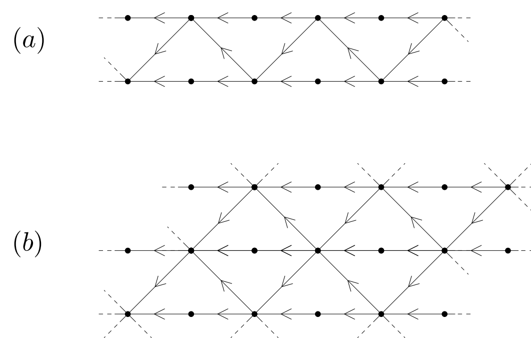

Fig. 3: Examples of admissible, directed graphs with $\ker(H) = \{0\}$

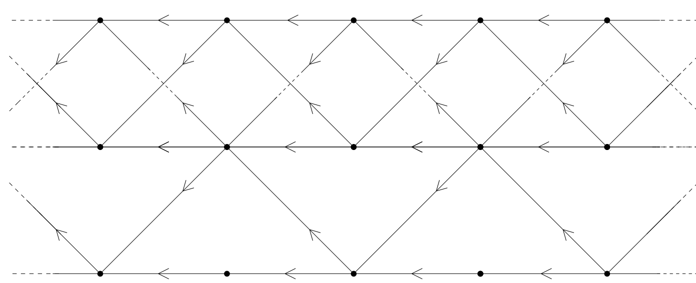

Fig. 4: Example of an admissible, directed graph with $\ker(H) \neq \{0\}$

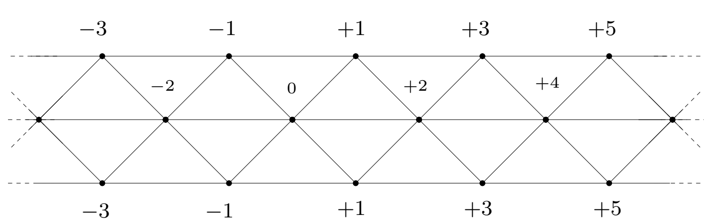

Fig. 5: Example of a non-admissible, adapted graph (with function Φ as indicated)

Convolution Operators on Locally Compact Groups

In this section we consider locally compact groups X, abelian or not, and convolution operators H_μ, acting on $\mathsf{L}^2(X)$, defined by suitable measures μ belonging to $\mathsf{M}(X)$, the Banach *-algebra of complex bounded Radon measures on X. Using the method of the weakly conjugate operator, we determine subspaces $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^2$ of $\mathsf{L}^2(X)$, explicitly defined in terms of μ and the family $\mathrm{Hom}(X,\mathbb{R})$ of continuous group morphisms $\Phi : X \to \mathbb{R}$, such that $\mathcal{H}_\mathrm{p}(H_\mu) \subset \mathcal{K}_\mu^1$ and $\mathcal{H}_\mathrm{s}(H_\mu) \subset \mathcal{K}_\mu^2$. This result, obtained in [27], supplements other works on the spectral theory of operators on groups and graphs [4, 5, 6, 9, 10, 11, 15, 17, 18, 19, 21, 22, 23, 31].

Let X be a locally compact group (LCG) with identity e, center $Z(X)$ and modular function Δ. Let us fix a left Haar measure λ on X, using the notation $\mathrm{d}x := \mathrm{d}\lambda(x)$. On discrete groups the counting measure (assigning mass 1 to every point) is considered. The notation *a.e.* stands for "almost everywhere" and refers to the Haar measure λ.

We consider in the sequel the convolution operator H_μ, $\mu \in \mathsf{M}(X)$, acting in the Hilbert space $\mathcal{H} := \mathsf{L}^2(X, \mathrm{d}\lambda)$, *i.e.*

$$H_\mu f := \mu * f,$$

where $f \in \mathcal{H}$ and

$$(\mu * f)(x) := \int_X \mathrm{d}\mu(y)\, f(y^{-1}x) \quad \text{for } a.e.\ x \in X.$$

The operator H_μ is bounded with norm $\|H_\mu\| \le \|\mu\|$, and it admits an adjoint operator H_μ^* equal to H_{μ^*}, the convolution operator by $\mu^* \in \mathsf{M}(X)$ defined by $\mu^*(E) = \overline{\mu(E^{-1})}$. If the measure μ is absolutely continuous w.r.t. the left Haar measure λ, so that $\mathrm{d}\mu = a\,\mathrm{d}\lambda$ with $a \in \mathsf{L}^1(X)$, then μ^* is also absolutely continuous w.r.t. λ and $\mathrm{d}\mu^* = a^*\mathrm{d}\lambda$, where $a^*(x) := \overline{a(x^{-1})}\Delta(x^{-1})$ for *a.e.* $x \in X$. In such a case we simply write H_a for $H_{a\,\mathrm{d}\lambda}$. We shall always assume that H_μ is selfadjoint, that is $\mu = \mu^*$.

Given $\mu \in \mathsf{M}(X)$, let $\varphi : X \to \mathbb{R}$ be such that the linear functional

$$F : C_0(X) \to \mathbb{C}, \quad g \mapsto \int_X \mathrm{d}\mu(x)\, \varphi(x) g(x)$$

is bounded. Then there exists a unique measure in $M(X)$ associated to F, due to the Riesz-Markov representation theorem. We write $\varphi\mu$ for this measure, and we simply say that φ is such that $\varphi\mu \in \mathsf{M}(X)$. We call *real character* any continuous group morphism $\Phi : X \to \mathbb{R}$.

Definition 10. *Let $\mu = \mu^* \in \mathsf{M}(X)$.*

(a) A real character Φ is semi-adapted to μ *if $\Phi\mu, \Phi^2\mu \in \mathsf{M}(X)$, and $(\Phi\mu) * \mu = \mu * (\Phi\mu)$. The set of real characters that are semi-adapted to μ is denoted by $\mathrm{Hom}_\mu^1(X,\mathbb{R})$.*

(b) A real character Φ is adapted to μ *if Φ is semi-adapted to μ, $\Phi^3\mu \in \mathsf{M}(X)$, and $(\Phi\mu) * (\Phi^2\mu) = (\Phi^2\mu) * (\Phi\mu)$. The set of real characters that are adapted to μ is denoted by $\mathrm{Hom}_\mu^2(X,\mathbb{R})$.*

Let $\mathcal{K}_\mu^j := \bigcap_{\Phi \in \mathrm{Hom}_\mu^j(X,\mathbb{R})} \ker(H_{\Phi\mu})$, for $j = 1, 2$; then the main result is the following.

Theorem 11 (Theorem 2.2 of [27])**.** *Let X be a LCG and let $\mu = \mu^* \in \mathsf{M}(X)$. Then*

$$\mathcal{H}_\mathrm{p}(H_\mu) \subset \mathcal{K}_\mu^1 \qquad \text{and} \qquad \mathcal{H}_\mathrm{s}(H_\mu) \subset \mathcal{K}_\mu^2.$$

The cases $\mathcal{K}_\mu^1 = \{0\}$ or $\mathcal{K}_\mu^2 = \{0\}$ are interesting; in the first case H_μ has no eigenvalues, and in the second case H_μ is purely absolutely continuous. A more precise result is obtained in a particular situation.

Corollary 12 (Corollary 2.3 of [27]). *Let X be a LCG and let $\mu = \mu^* \in \mathsf{M}(X)$. Assume that there exists a real character Φ adapted to μ such that Φ^2 is equal to a nonzero constant on $\mathrm{supp}(\mu)$. Then H_μ has a purely absolutely continuous spectrum, with the possible exception of an eigenvalue located at the origin, with eigenspace $\ker(H_\mu) = \ker(H_{\Phi\mu})$.*

Corollary 12 specially applies to adjacency operators on certain classes of Cayley graphs, which are Hecke-type operators in the regular representation, thus convolution operators on discrete groups.
To see how the method of the weakly conjugate operator comes into play, let us sketch the proof of the second inclusion in Theorem 11. In quantum mechanics in $\mathbb{R}^d$, the position operators Q_j and the momentum operators P_j satisfy the relation $P_j = i[H, Q_j]$ with $H := -\frac{1}{2}\Delta$, and the usual conjugate operator is the generator of dilations $D := \frac{1}{2}\sum_j (Q_j P_j + P_j Q_j)$. So if we regard $\Phi \in \mathrm{Hom}^2_\mu(X, \mathbb{R})$ as a position operator on X, it is reasonable to think of $K := i[H_\mu, \Phi]$ as the corresponding momentum operator and to use $A := \frac{1}{2}(\Phi K + K\Phi)$ as a tentative conjugate operator. In fact, simple calculations using the hypotheses of Definition 10.(b) show that

$$K = -iH_{\Phi\mu} \in \mathscr{B}(\mathcal{H}) \qquad \text{and} \qquad i[H_\mu, A] = K^2.$$

Therefore $H_\mu \in C^1(A; \mathcal{H})$ and the commutator $i[H_\mu, A]$ is a positive operator. However, in order to apply the theory of Section The Method of the Weakly Conjugate Operator, we need strict positivity. So, we consider the subspace $\mathcal{G} := [\ker(K)]^\perp$ of $\mathcal{H}$, where $i[H_\mu, A] \equiv K^2$ is strictly positive. The orthogonal decomposition $\mathcal{H} := \mathcal{K} \oplus \mathcal{G}$ with $\mathcal{K} := \ker(K)$, reduces H_μ, K, and A, and their restrictions H_0, K_0, and A_0 to $\mathcal{G}$ are selfadjoint. It turns out that all the other conditions necessary to apply Theorem 5 can also be verified in the Hilbert space $\mathcal{G}$. So we get the inclusion

$$\mathcal{H}_{\mathrm{s}} \subset \mathcal{K} = \ker(H_{\Phi\mu}).$$

Since $\Phi \in \mathrm{Hom}^2_\mu(X, \mathbb{R})$ is arbitrary, this implies the second inclusion of Theorem 11.

Examples

The construction of weakly conjugate operators for H_μ relies on real characters. So a small vector space $\mathrm{Hom}(X, \mathbb{R})$ is an obstacle to applying the method. A real character Φ maps compact subgroups of X to the unique compact subgroup $\{0\}$ of $\mathbb{R}$. Consequently, abundancy of compact elements (elements $x \in X$ generating compact subgroups) prevents us from constructing weakly conjugate operators. The extreme case is when X is itself compact, so that $\mathrm{Hom}(X, \mathbb{R}) = \{0\}$. Actually in such a case all convolution operators H_μ have pure point spectrum. We review now briefly some of the groups for which we succeeded in [27] to apply the MWCO.
If the group X is unimodular, one can exploit commutativity in a non-commutative setting by using central measures (*i.e.* elements of the center $Z[\mathsf{M}(X)]$ of the convolution Banach *-algebra $\mathsf{M}(X)$). For instance, in the case of central groups [16], we have the following result ($\mathscr{B}(X)$ stands for the closed subgroup generated by the set of compact elements of X):

Proposition 13 (Proposition 4.2 of [27]). *Let X be a central group and $\mu_0 = \mu_0^* \in \mathsf{M}(X)$ a central measure such that $\mathrm{supp}(\mu_0)$ is compact and not included in $\mathscr{B}(X)$. Let $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset \mathscr{B}(X)$ and set $\mu := \mu_0 + \mu_1$. Then $\mathcal{H}_{\mathrm{ac}}(H_\mu) \neq \{0\}$.*

Let us recall three examples deduced from Proposition 13 taken from [27].

Example 14. *Let $X := S_3 \times \mathbb{Z}$, where S_3 is the symmetric group of degree 3. The group S_3 has a presentation $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$, and its conjugacy classes are $E_1 = E_1^{-1} = \{e\}$, $E_2 = E_2^{-1} = \{a, b, aba\}$ and $E_3 = E_3^{-1} = \{ab, ba\}$. Set $\mathcal{E} := \{E_2, E_3\}$ and choose I_{E_1}, I_{E_2} two finite symmetric subsets of $\mathbb{Z}$, each of them containing at least two elements. Then $\mathcal{H}_{\rm ac}(H_{\chi_S}) \neq \{0\}$ if $S := \bigcup_{E \in \mathcal{E}} E \times I_E$.*

Example 15. *Let $X := \mathrm{SU}(2) \times \mathbb{R}$, where $\mathrm{SU}(2)$ is the group (with Haar measure λ_2) of 2×2 unitary matrices of determinant $+1$. For each $\vartheta \in [0, \pi]$ let $C(\vartheta)$ be the conjugacy class of the matrix $\mathrm{diag}(\mathrm{e}^{i\vartheta}, \mathrm{e}^{-i\vartheta})$ in $\mathrm{SU}(2)$. A direct calculation (using for instance Euler angles) shows that $\lambda_2\left(\bigcup_{\vartheta \in J} C(\vartheta)\right) > 0$ for each $J \subset [0, \pi]$ with nonzero Lebesgue measure. Set $E_1 := \bigcup_{\vartheta \in (0,1)} C(\vartheta)$, $E_2 := \bigcup_{\vartheta \in (2,\pi)} C(\vartheta)$, $\mathcal{E} := \{E_1, E_2\}$, $I_{E_1} := (-1, 1)$, and $I_{E_2} := (-3, -2) \cup (2, 3)$. Then $\mathcal{H}_{\rm ac}(H_{\chi_S}) \neq \{0\}$ if $S := \bigcup_{E \in \mathcal{E}} E \times I_E$.*

Example 16. *Let X be a central group, let $z \in Z(X) \setminus \mathscr{B}(X)$, and set $\mu := \delta_z + \delta_{z^{-1}} + \mu_1$ for some $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset \mathscr{B}(X)$. Then μ satisfies the hypotheses of Proposition 13, and we can choose $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ such that $\Phi(z) = \frac{1}{2}\Phi(z^2) \neq 0$ (note in particular that $z \notin \mathscr{B}(X)$ iff $z^2 \notin \mathscr{B}(X)$ and that $\Phi\mu_1 = 0$). Thus $\mathcal{H}_{\rm s}(H_\mu) \subset \ker(H_{\Phi\mu})$. But $f \in \mathcal{H}$ belongs to $\ker(H_{\Phi\mu}) = \ker\left(H_{\Phi(\delta_z + \delta_{z^{-1}})}\right)$ iff $f(z^{-1}x) = f(zx)$ for* a.e. *$x \in X$. This periodicity w.r.t. the non-compact element z^2 implies that the L^2-function f should vanish* a.e. *and thus that $\mathcal{H}_{\rm ac}(H_\mu) = \mathcal{H}$.*

If X is abelian all the commutation relations in Definition 10 are satisfied. Moreover one can use the Fourier transform $\mathscr{F}$ to map unitarily H_μ on the operator M_m of multiplication with $m = \mathscr{F}(\mu)$ on the dual group $\widehat{X}$ of X. So one gets from from Theorem 11 a general lemma on muliplication operators. We recall some definitions before stating it.

Definition 17. *The function $m : \widehat{X} \to \mathbb{C}$ is* differentiable at $\xi \in \widehat{X}$ along the one-parameter subgroup $\varphi \in \mathrm{Hom}(\mathbb{R}, \widehat{X})$ *if the function $\mathbb{R} \ni t \mapsto m(\xi + \varphi(t)) \in \mathbb{C}$ is differentiable at $t = 0$. In such a case we write $(d_\varphi m)(\xi)$ for $\frac{\mathrm{d}}{\mathrm{d}t} m(\xi + \varphi(t))\big|_{t=0}$. Higher order derivatives, when existing, are denoted by $d_\varphi^k m$, $k \in \mathbb{N}$.*

We say that the one-parameter subgroup $\varphi : \mathbb{R} \to \widehat{X}$ is in $\mathrm{Hom}^1_m(\mathbb{R}, \widehat{X})$ if m is twice differentiable w.r.t. φ and $d_\varphi m, d_\varphi^2 m \in \mathscr{F}(\mathsf{M}(X))$. If, in addition, m is three times differentiable w.r.t. φ and $d_\varphi^3 m \in \mathscr{F}(\mathsf{M}(X))$ too, we say that φ belongs to $\mathrm{Hom}^2_m(\mathbb{R}, \widehat{X})$.

Lemma 18 (Corollary 4.7 of [27])**.** *Let X be a locally compact abelian group and let m_0, m_1 be real functions with $\mathscr{F}^{-1}(m_0), \mathscr{F}^{-1}(m_1) \in \mathsf{M}(X)$ and $\mathrm{supp}(\mathscr{F}^{-1}(m_1)) \subset \mathscr{B}(X)$. Then*

$$\mathcal{H}_{\rm p}(M_{m_0+m_1}) \subset \bigcap_{\varphi \in \mathrm{Hom}^1_{m_0}(\mathbb{R}, \widehat{X})} \ker(M_{d_\varphi m_0})$$

and

$$\mathcal{H}_{\rm s}(M_{m_0+m_1}) \subset \bigcap_{\varphi \in \mathrm{Hom}^2_{m_0}(\mathbb{R}, \widehat{X})} \ker(M_{d_\varphi m_0}).$$

We end up this section by considering a class of semidirect products. Let N, G be two discrete groups with G abelian (for which we use additive notations), and let $\tau : G \to \mathrm{Aut}(N)$ be a group morphism. Let $X := N \times_\tau G$ be the τ-semidirect produt of N by G. The multiplication in X is defined by

$$(n, g)(m, h) := (n\tau_g(m), g + h),$$

so that

$$(n, g)^{-1} = (\tau_{-g}(n^{-1}), -g).$$

In this situation it is shown in [27] that many convolution operators H_a, with $a : X \to \mathbb{C}$ of finite support, have a non-trivial absolutely continuous component. For instance, we have the following for a type of wreath products.

Example 19. *Take G a discrete abelian group and put $N := R^J$, where R is an arbitrary discrete group and J is a finite set on which G acts by $(g, j) \mapsto g(j)$. Then $\tau_g\big(\{r_j\}_{j\in J}\big) := \{r_{g(j)}\}_{j\in J}$ defines an action of G on R^J, thus we can construct the semidirect product $R^J \times_\tau G$. If $G_0 = -G_0 \subset G$ and $R_0 = R_0^{-1} \subset R$ are finite subsets with $G_0 \cap [G \setminus \mathscr{B}(G)] \neq \varnothing$, then $N_0 := R_0^J$ satisfies all the conditions of [27, Sec. 4.4]. Thus $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) \neq \{0\}$ if $S := N_0 \times G_0$.*

Virtually the methods of [27] could also be applied to non-split group extensions.

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Metoda operatorului slab conjugat: Extensii si aplicatii la operatori definiti pe grafuri si grupuri

Rezumat

In aceasta lucrare sunt prezentate cateva extensii recente ale metodei operatorului conjugat slab. Sunt ilustrate aceste dezvoltari prin exemple ale aplicarii operatorilor pe grafuri si grupuri.