

# Weber–Schafheitlin-type integrals with exponent 1

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Explicit formulae for Weber–Schafheitlin-type integrals with exponent 1 are derived. The results of these integrals are distributions on  $\mathbb{R}_+$ .

**Keywords:** Weber–Schafheitlin integrals; Bessel functions; hypergeometric function

## 1. Introduction

Let  $J_\mu$  denote the Bessel function of the first kind and of order  $\mu$ . An integral of the form

$$\int_0^\infty \kappa^\rho J_\mu(s\kappa) J_\nu(\kappa) d\kappa \tag{1}$$

for suitable  $\mu, \nu, \rho \in \mathbb{R}$  and  $s \in \mathbb{R}_+ = (0, \infty)$  is called a Weber–Schafheitlin integral [10, Chapter 13.4]. If  $\rho$  is strictly less than 1, the result of this integration is known and can be found in many textbooks or handbooks; see for example [3,10,12]. However, the critical case  $\rho = 1$  turns out to be of considerable interest in the scattering theory (forthcoming paper, *The Aharonov–Bohm wave operators revisited*, by J. Kellendonk and S. Richard). Therefore, we provide in this paper the result of Equation (1) for  $\rho = 1$  as well as the result of the related integral

$$\int_0^\infty \kappa H_\mu^{(1)}(s\kappa) J_\nu(\kappa) d\kappa \tag{2}$$

where  $H_\mu^{(1)}$  is the Hankel function of the first kind and of order  $\mu$ . We emphasize that both results are not functions of the variable  $s$  but distributions on  $\mathbb{R}_+$ . We also mention that in [2] the special cases  $\nu = \pm\mu$  of Equation (1) and  $\nu = \mu$  of Equation (2) have already been explicitly calculated.

It is interesting to note that for values of  $\rho$  strictly smaller than 1, integral (1) can also be seen as a very special case of a more general family of expressions analysed by Srivastava, Miller and

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their collaborators. For example, in [9] the integrals

$$\int_0^\infty t^{\rho-1} \left( \prod_{j=1}^n J_{\nu_j}(x_j t) \right) dt$$

are evaluated in terms of Lauricella hypergeometric functions. In [4–7], the Mellin transform of the product of two generalized hypergeometric functions is investigated, and again Equation (1) can be obtained by a very special choice of the parameters. However, the extreme values corresponding to  $\rho = 1$  are not considered in these references. Such an investigation would certainly be of interest, and our result can be seen as a first step in this direction (see also [8] for related work).

## 2. The derivation of integral (2)

Let us start by recalling that for  $z \in \mathbb{C}$  satisfying  $-\pi/2 < \arg(z) \leq \pi$  one has [1, Equation 9.6.4]

$$H_\mu^{(1)}(z) = \frac{2}{i\pi} e^{-i\pi\mu/2} K_\mu(-iz),$$

where  $K_\mu$  is the modified Bessel function of the second kind and of order  $\mu$ . Moreover, for  $\Re(z) > 0$  and  $\nu + 2 > |\mu|$  the following result holds [10, Section 13.45]:

$$\begin{aligned} \int_0^\infty \kappa K_\mu(z\kappa) J_\nu(\kappa) d\kappa &= \frac{\Gamma((\nu + \mu)/2 + 1)\Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} z^{-2-\nu} \\ &\times {}_2F_1\left(\frac{\nu + \mu}{2} + 1, \frac{\nu - \mu}{2} + 1; \nu + 1; -z^{-2}\right), \end{aligned}$$

where  ${}_2F_1$  is the Gauss hypergeometric function [1, Chapter 15]. Thus, by setting  $z = s + i\varepsilon$  with  $s \in \mathbb{R}_+$  and  $\varepsilon > 0$  one obtains

$$\begin{aligned} I_{\mu,\nu}(s + i\varepsilon) &:= \int_0^\infty \kappa H_\mu^{(1)}((s + i\varepsilon)\kappa) J_\nu(\kappa) d\kappa \\ &= \frac{2}{i\pi} e^{-i\pi\mu/2} \frac{\Gamma((\nu + \mu)/2 + 1)\Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} (-is + \varepsilon)^{-2-\nu} \\ &\times {}_2F_1\left(\frac{\nu + \mu}{2} + 1, \frac{\nu - \mu}{2} + 1; \nu + 1; (s + i\varepsilon)^{-2}\right). \end{aligned}$$

Taking into account [1, Equality 15.3.3] one can isolate from the  ${}_2F_1$ -function a factor that is singular for  $s = 1$  when  $\varepsilon$  goes to 0:

$$\begin{aligned} &{}_2F_1\left(\frac{\nu + \mu}{2} + 1, \frac{\nu - \mu}{2} + 1; \nu + 1; (s + i\varepsilon)^{-2}\right) \\ &= \frac{1}{1 - (s + i\varepsilon)^{-2}} {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; (s + i\varepsilon)^{-2}\right). \end{aligned}$$

Furthermore, by inserting the equalities

$$\frac{1}{1 - (s + i\varepsilon)^{-2}} = -(s + i\varepsilon)^2 \frac{1}{s((1 + \varepsilon^2)/s - s) - 2i\varepsilon}$$

and

$$(-is + \varepsilon)^{-2-\nu} = -e^{i\pi\nu/2}(s + i\varepsilon)^{-2-\nu},$$

in these expressions, one finally obtains that  $I_{\mu,\nu}(s + i\varepsilon)$  is equal to

$$\frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \frac{1}{s} \frac{(s + i\varepsilon)^{-\nu}}{((1 + \varepsilon^2)/s - s) - 2i\varepsilon} \frac{\Gamma((\nu + \mu)/2 + 1)\Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; (s + i\varepsilon)^{-2}\right). \tag{3}$$

We are now ready to study the  $\varepsilon$ -behaviour of each of the above terms. For the particular choice of the three parameters  $(\nu + \mu)/2$ ,  $(\nu - \mu)/2$  and  $\nu + 1$ , the map

$$z \mapsto {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; z\right),$$

which is holomorphic in the cut complex plane  $\mathbb{C} \setminus [1, \infty)$ , extends continuously to  $[1, \infty)$ . The limits from above and below yield generally two different continuous functions and, by convention, the hypergeometric function on  $[1, \infty)$  is the limit obtained from below. Since  $\Im((s + i\varepsilon)^{-2}) < 0$ , the  ${}_2F_1$ -factor in Equation (3) converges to  ${}_2F_1((\nu + \mu)/2, (\nu - \mu)/2; \nu + 1; s^{-2})$  as  $\varepsilon \rightarrow 0$ , uniformly in  $s$  on any compact subset of  $\mathbb{R}_+$ .

For the other factors, let us observe that  $(s + i\varepsilon)^{-\nu}$  converges to  $s^{-\nu}$  as  $\varepsilon \rightarrow 0$  uniformly in  $s$  on any compact subset of  $\mathbb{R}_+$ . Furthermore, it is known that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{((1 + \varepsilon^2)/s - s) - 2i\varepsilon} = \text{Pv}\left(\frac{1}{1/s - s}\right) + i\frac{\pi}{2}\delta(s - 1), \tag{4}$$

where the convergence has to be understood in the sense of distributions on  $\mathbb{R}_+$ . In the last expression,  $\delta$  is the Dirac measure centred at 0 and  $\text{Pv}$  denotes the principal value integral. By collecting all these results one can prove the following proposition.

**PROPOSITION 1** For any  $\mu, \nu \in \mathbb{R}$  satisfying  $\nu + 2 > |\mu|$  and  $s \in \mathbb{R}_+$  one has

$$\int_0^\infty \kappa H_\mu^{(1)}(s\kappa) J_\nu(\kappa) d\kappa = e^{i\pi(\nu-\mu)/2} \delta(s - 1) + \frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \text{Pv}\left(\frac{1}{1/s - s}\right) \times \frac{s^{-\nu} \Gamma((\nu + \mu)/2 + 1)\Gamma((\nu - \mu)/2 + 1)}{s \Gamma(\nu + 1)} \times {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; s^{-2}\right) \tag{5}$$

as an equality between two distributions on  $\mathbb{R}_+$ .

*Remark 1* *A priori*, the second term in the r.h.s. is not well defined, since it is the product of the distribution  $\text{Pv}(\frac{1}{1/s-s})$  with a function which is not smooth, or at least differentiable at  $s = 1$ . However, by using the development of the hypergeometric function in a neighbourhood of

1 [1, Equation 15.3.11] it is easily observed that

$$\frac{\Gamma((\nu + \mu)/2 + 1) \Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; s^{-2}\right) = 1 + (s - 1) h(s) \tag{6}$$

with a function  $h$  that belongs to  $L^1_{\text{loc}}(\mathbb{R}_+)$ . Thus, for any  $\alpha \in \mathbb{R}$ , the second term in the r.h.s. of Equation (5) is equal to

$$\begin{aligned} & \frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \left[ s^\alpha \text{Pv} \left( \frac{1}{1/s - s} \right) + \frac{s^\alpha}{1/s - s} \left( \frac{s^{-\nu-\alpha} \Gamma((\nu + \mu)/2 + 1) \Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} \right. \right. \\ & \quad \left. \left. \times {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; s^{-2}\right) - 1 \right) \right], \end{aligned} \tag{7}$$

with the second term in  $L^1_{\text{loc}}(\mathbb{R}_+)$ . Clearly, this distribution is now well defined. The parameter  $\alpha$  has been added because it may be useful in certain applications.

*Remark 2* To describe the singularity at  $s = 1$ , the decomposition (7) of the second term of the r.h.s. of Equation (5) is certainly valuable. However, it seems to us that this decomposition is less useful if one needs to control the behaviour at  $s = 0$ .

*Remark 3* In the special case  $\mu = \pm\nu$ , the hypergeometric function is equal to 1, and thus the r.h.s. simplifies drastically.

*Proof of Proposition 1* (a) For any  $\varepsilon > 0$ , let us define the function

$$p_\varepsilon : \mathbb{R}_+ \ni s \mapsto \frac{1}{((1 + \varepsilon^2)/s - s) - 2i\varepsilon} \in \mathbb{C}$$

and the function  $\mathbb{R}_+ \ni s \rightarrow q_\varepsilon(s) \in \mathbb{C}$  by

$$\begin{aligned} q_\varepsilon(s) &:= \frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \frac{(s + i\varepsilon)^{-\nu} \Gamma((\nu + \mu)/2 + 1) \Gamma((\nu - \mu)/2 + 1)}{s \Gamma(\nu + 1)} \\ & \quad \times {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; (s + i\varepsilon)^{-2}\right). \end{aligned}$$

Clearly, one gets from Equation (3) that  $p_\varepsilon(s)q_\varepsilon(s) = I_{\mu,\nu}(s + i\varepsilon)$  and from the above remarks that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} q_\varepsilon(s) &= \frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \frac{s^{-\nu} \Gamma((\nu + \mu)/2 + 1) \Gamma((\nu - \mu)/2 + 1)}{s \Gamma(\nu + 1)} \\ & \quad \times {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; s^{-2}\right) =: q_0(s), \end{aligned}$$

the convergence being uniform in  $s$  on any compact subset of  $\mathbb{R}_+$ . Furthermore, it follows from [1, Equation 15.1.20] that  $q_0(1) = (2/i\pi)e^{i\pi(\nu-\mu)/2}$ .

(b) Let  $g \in C^\infty_c(\mathbb{R}_+)$ , i.e.  $g$  is a smooth function with compact support in  $\mathbb{R}_+$ , and set

$$M_g := \sup_{\varepsilon \in [0, 1]} \int_{\mathbb{R}_+} |q_\varepsilon(s)g(s)| ds.$$

For any  $\eta > 0$ , one can then choose a compact subset  $K_\eta$  of  $[0, \infty)$  such that

$$\sup_{\varepsilon \in [0, 1]} \sup_{s \in \mathbb{R}_+ \setminus K_\eta} |p_\varepsilon(s)| \leq \frac{\eta}{6M_g},$$

which implies that  $\int_{\mathbb{R}_+ \setminus K_\eta} |q_\varepsilon(s) p_{\varepsilon'}(s) g(s)| ds \leq \eta/6$  for all  $\varepsilon, \varepsilon' \in [0, 1]$ .

(c) Finally, one has

$$\begin{aligned} \int_{\mathbb{R}_+} q_\varepsilon(s) p_\varepsilon(s) g(s) \, ds &= \int_{\mathbb{R}_+} q_0(s) p_\varepsilon(s) g(s) \, ds + \int_{\mathbb{R}_+} (q_\varepsilon(s) - q_0(s)) p_\varepsilon(s) g(s) \, ds \\ &= \int_{\mathbb{R}_+} p_\varepsilon(s) q_0(s) g(s) \, ds + \int_{K_\eta} (q_\varepsilon(s) - q_0(s)) p_\varepsilon(s) g(s) \, ds \\ &\quad + \int_{\mathbb{R}_+ \setminus K_\eta} (q_\varepsilon(s) - q_0(s)) p_\varepsilon(s) g(s) \, ds. \end{aligned}$$

For  $\varepsilon \rightarrow 0$ , the first term on the r.h.s. converges to

$$\begin{aligned} e^{i\pi(v-\mu)/2} g(1) + \frac{2}{i\pi} e^{i\pi(v-\mu)/2} \int_{\mathbb{R}_+} \text{Pv} \left( \frac{1}{1/s - s} \right) \frac{s^{-v} \Gamma((v + \mu)/2 + 1) \Gamma((v - \mu)/2 + 1)}{\Gamma(v + 1)} \\ \times {}_2F_1 \left( \frac{v + \mu}{2}, \frac{v - \mu}{2}; v + 1; s^{-2} \right) g(s) \, ds. \end{aligned}$$

Indeed, the convergence of Equation (4) holds in the sense of distributions not only on smooth functions on  $\mathbb{R}_+$  with compact support, but also on the product  $q_0 g$  of the non-smooth function  $q_0$  with the smooth function  $g$ . This can easily be obtained by using the development given in Equation (6). Furthermore, one has

$$\left| \int_{K_\eta} (q_\varepsilon(s) - q_0(s)) p_\varepsilon(s) g(s) \, ds \right| \leq \sup_{s \in K_\eta \cap \text{supp } g} |q_\varepsilon(s) - q_0(s)| \int_{K_\eta} |p_\varepsilon(s) g(s)| \, ds,$$

which is less than  $\eta/3$  for  $\varepsilon$  small enough since  $q_\varepsilon - q_0$  converges uniformly to 0 on any compact subset of  $\mathbb{R}_+$ . And finally, from the choice of  $K_\eta$  one has

$$\left| \int_{\mathbb{R}_+ \setminus K_\eta} (q_\varepsilon(s) - q_0(s)) p_\varepsilon(s) g(s) \, ds \right| \leq \eta/3.$$

Since  $\eta$  is arbitrary, one has thus obtained that the map  $s \mapsto p_\varepsilon(s) q_\varepsilon(s)$  converges in the sense of distributions on  $\mathbb{R}_+$  to the distribution given by the r.h.s. term of the statement of the proposition. ■

*Remark 4* In the previous proof, the Lebesgue measure on  $\mathbb{R}_+$  has been used for the evaluation of the distribution on a smooth function with compact support in  $\mathbb{R}_+$ . Let us notice that the same result holds if the measure  $ds/s$  is chosen instead of the Lebesgue measure.

### 3. The derivation of integral (1)

In the next proposition, the function  $H_\mu^{(1)}$  of the previous statement is replaced by the Bessel function  $J_\mu$ . Since  $H_\mu^{(1)} = J_\mu + iY_\mu$  with  $J_\mu$  and  $Y_\mu$  real on  $\mathbb{R}_+$ , taking the real part of both sides of Equation (5) would lead to the result. However, since the real and the imaginary parts of the Gauss hypergeometric function are not very explicit, we prefer to sketch an independent proof.

PROPOSITION 2 For any  $\mu, \nu \in \mathbb{R}$  satisfying  $\nu + 2 > |\mu|$  and  $\mu + 2 > |\nu|$ , and  $s \in \mathbb{R}_+$  one has

$$\int_0^\infty \kappa J_\mu(s\kappa) J_\nu(\kappa) \, d\kappa = \cos(\pi(\nu - \mu)/2)\delta(s - 1) + \frac{2}{\pi} \sin(\pi(\nu - \mu)/2) \text{Pv} \left( \frac{1}{1/s - s} \right) \\ \times \begin{cases} \frac{s^\mu}{s} \frac{\Gamma((\mu + \nu)/2 + 1)\Gamma((\mu - \nu)/2 + 1)}{\Gamma(\mu + 1)} {}_2F_1 \left( \frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}; \mu + 1; s^2 \right) & \text{if } s \leq 1, \\ \frac{s^{-\nu}}{s} \frac{\Gamma((\nu + \mu)/2 + 1)\Gamma((\nu - \mu)/2 + 1)}{\Gamma(\nu + 1)} {}_2F_1 \left( \frac{\nu + \mu}{2}, \frac{\nu - \mu}{2}; \nu + 1; s^{-2} \right) & \text{if } s > 1, \end{cases}$$

as an equality between two distributions on  $\mathbb{R}_+$ .

Remark 5 We refer to Remark 1 for a discussion on the fact that the distribution corresponding to the second term on the r.h.s. is well defined.

Proof of Proposition 2 (a) Even if the proof is very similar to the previous one, two additional observations have to be taken into account: (1) the map  $z \mapsto {}_2F_1(\alpha, \beta; \gamma; z)$  is real when  $z$  is restricted to the interval  $[0, 1)$  and (2) there exists a simple relation between  $\Re(I_{\mu,\nu})$  and  $\Re(I_{\nu,\mu})$ . More precisely, for any  $s \in \mathbb{R}_+ \setminus \{1\}$  one has

$$\Re(I_{\mu,\nu}(s)) = \int_0^\infty \kappa J_\mu(s\kappa) J_\nu(\kappa) \, d\kappa \\ = s^{-2} \int_0^\infty \kappa J_\mu(\kappa) J_\nu(s^{-1}\kappa) \, d\kappa \\ = s^{-2} \Re(I_{\nu,\mu}(s^{-1})).$$

Thus the main trick of the proof is to use the previous expressions for  $s \in (1, \infty)$  since the contribution of the  ${}_2F_1$ -function in Equation (5) is then real, and to obtain similar formulae below for  $s \in (0, 1]$ . However, some care has to be taken because of the Dirac measure at 1 and of the principal value integral also centred at 1.

So, for any  $s \in (0, 1]$  and  $\varepsilon > 0$ , let us set  $z := s - i\varepsilon$ . Since the conditions  $-\pi/2 < \arg(z^{-1}) \leq \pi$ ,  $\Re(-iz^{-1}) > 0$  and  $\mu + 2 > |\nu|$  hold, one can obtain the analogue of Equation (3):

$$\frac{1}{z^2} \int_0^\infty \kappa H_\nu^{(1)}(z^{-1}\kappa) J_\mu(\kappa) \, d\kappa = -\frac{2}{i\pi} e^{-i\pi(\nu-\mu)/2} \frac{1}{((1 + \varepsilon^2)/s - s) + 2i\varepsilon} m_\varepsilon(s),$$

with  $m_\varepsilon : (0, 1] \rightarrow \mathbb{C}$ , the function defined by

$$m_\varepsilon(s) := \frac{(s - i\varepsilon)^\mu}{s} \frac{\Gamma((\mu + \nu)/2 + 1)\Gamma((\mu - \nu)/2 + 1)}{\Gamma(\mu + 1)} \\ \times {}_2F_1 \left( \frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}; \mu + 1; (s - i\varepsilon)^2 \right). \tag{8}$$

(b) We now collect both functions, for  $s \leq 1$  and  $s > 1$ . For that purpose, let  $l_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{C}$  be the function defined for  $s \in (0, 1]$  by

$$l_\varepsilon(s) = -\frac{2}{i\pi} e^{-i\pi(\nu-\mu)/2} \frac{1}{((1 + \varepsilon^2)/s - s) + 2i\varepsilon}$$

and for  $s \in (1, \infty)$  by

$$l_\varepsilon(s) = \frac{2}{i\pi} e^{i\pi(\nu-\mu)/2} \frac{1}{((1 + \varepsilon^2)/s - s) - 2i\varepsilon}.$$

By selecting only the real part of  $l_\varepsilon$ , one obtains that  $\Re(l_\varepsilon)$  converges in the sense of distributions on  $\mathbb{R}_+$  as  $\varepsilon \rightarrow 0$  to the distribution:

$$\cos(\pi(v - \mu)/2)\delta(s - 1) + \frac{2}{\pi} \sin(\pi(v - \mu)/2)\text{Pv} \left( \frac{1}{1/s - s} \right).$$

Furthermore, let  $m_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{C}$  be the function defined for  $s \in (0, 1]$  by (8) and for  $s \in (1, \infty)$  by

$$\frac{(s + i\varepsilon)^{-\nu} \Gamma((v + \mu)/2 + 1)\Gamma((v - \mu)/2 + 1)}{s \Gamma(v + 1)} {}_2F_1 \left( \frac{v + \mu}{2}, \frac{v - \mu}{2}; v + 1; (s + i\varepsilon)^{-2} \right).$$

As  $\varepsilon$  goes to 0, this function converges uniformly on any compact subset of  $\mathbb{R}_+$  to a continuous real function  $m_0$ . Furthermore, this function takes the value 1 for  $s = 1$ . Indeed, these properties of  $m_0$  follow from the facts that the  ${}_2F_1$ -functions are real on the  $[0, 1]$ , that the convergences from above or below this interval give the same values, and that the normalization factors have been suitably chosen. The function  $m_0$  is the one given after the curly bracket in the statement of the proposition.

(c) The remaining part of the proof can now be mimicked from part (c) of the proof of the previous proposition. Therefore, we simply refer to this paragraph and omit it from the present proof. ■

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