Introduction to the Abelian Stark Conjectures

Outline Notes, Version of 9/12/04

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0 Prerequisites

- A first course in Algebraic Number Theory (for number fields): integers, ideals, absolute values, class groups, units, Dirichlet's Theorem, behaviour of primes in Galois extensions, basic theory of Cyclotomic fields. See *e.g.* [F-T], [La], [Wa]
- Acquaintance with main theorems of (abelian) class-field theory in terms of ideals: Mainly ray-class groups/fields and the Artin map. See *e.g.* [La] or (especially) appendix to [Wa]
- Basic understanding of representation theory of finite groups over a field $\mathbb{F}$ of characteristic 0, almost exclusively abelian groups, mostly $\mathbb{F} = \mathbb{C}$. For the latter, deeper understanding of characters, orthogonality relations, idempotents, eigenspaces (isotypic parts) of modules, connection with the ring/module theory of the group-ring etc. See *e.g.* [F-T], [Se]
- Basic familiarity with Riemann zeta-function and Dirichlet L-functions (definitions, Euler product and acquaintance with the functional equation will probably suffice. Some knowledge of equivalents for Dedekind zeta-function and Hecke L-functions helpful). See *e.g.* [F-T], [La], [Wa], article by Martinet in [Fr]
- General Algebra: Basic theory of rings and modules. Tensor product and exterior powers over commutative rings. Group-rings.
- Basic notions of complex analysis (analytic continuation, Dirichlet series, Gamma function)
- Knowledge of p-adic numbers and very basic p-adic analysis. See *e.g.* [Wa, Ch. 5], [Ko]

1 Motivation: L-functions of Cyclotomic Fields

1.1 Some Definitions

Let $\zeta_f := \exp(2\pi i/f)$ for $f \in \mathbb{Z}_{\geq 1}$

Set $K_f := \mathbb{Q}(\zeta_f)$ and $G_f := \mathrm{Gal}(K_f/\mathbb{Q})$.

$$\begin{array}{ccc} G_f & \cong & (\mathbb{Z}/f\mathbb{Z})^\times \\ \sigma_a & \longleftrightarrow & \bar{a} \end{array}$$

where $\sigma_a(\zeta_f) = \zeta_f^a$.

Identify character group $\widehat{G_f}$ with *Dirichlet characters modulo f i.e.*

$$(\chi : G_f \to \mathbb{C}^\times) \longleftrightarrow (\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times)$$

For $s \in \mathbb{C}$, $\Re(s) > 1$, set

$$\begin{aligned} L_f(s,\chi) &= \sum_{\substack{n\geq 1 \\ (n,f)=1}} \frac{\chi(\bar{n})}{n^s} \\ &= \prod_{p\nmid f} \left(1 - \frac{\chi(\bar{p})}{p^s}\right)^{-1} \end{aligned}$$

1.2 Primitivity

Note that $L_f(s,\chi)$ may be *'imprimitive' i.e.* there may exist f' properly dividing f and Dirichlet character χ' defined modulo f' such that

$$\chi'(\bar{a} \bmod f') = \chi(\bar{a} \bmod f) \tag{1}$$

whenever $(a,f) = 1$.

In any case, there exists a unique minimal $f'|f$ (the *conductor of* χ denoted f_χ) and character χ' defined modulo f_χ satisfying (1) (called the *primitive character associated to* χ, denoted $\hat{\chi}$).

The usual primitive Dirichlet L-function $L(s,\hat{\chi})$ is just

$$\prod_{p\nmid f_\chi} \left(1 - \frac{\hat{\chi}(\bar{p})}{p^s}\right)^{-1} = L_{f_\chi}(s,\hat{\chi})$$

and so, since f_χ divides f, we have

$$L_f(s,\chi) = \prod_{\substack{p\nmid f_\chi \\ p|f}} \left(1 - \frac{\hat{\chi}(\bar{p})}{p^s}\right) L(s,\hat{\chi}) \tag{2}$$

For example

$$L_f(s,\chi_0) = \prod_{p|f}\left(1-\frac{1}{p^s}\right)L(s,\hat\chi_0) = \prod_{p|f}\left(1-\frac{1}{p^s}\right)\zeta(s)$$

1.3 Analytic Facts about $L(s,\hat\chi)$ and $L_f(s,\chi)$

For the following analytic facts about primitive Dirichlet L-functions, see *e.g.* [Wa, Ch. 4]:

(PDL1) $L(s,\hat\chi)$ has a continuation to $\mathbb{C}$ that is analytic at all $s\in\mathbb{C}$ (except for $L(s,\hat\chi_0) = \zeta(s)$ at $s=1$)

(PDL2) $\mathrm{ord}_{s=1}L(s,\hat\chi) = 0$ (except that $\mathrm{ord}_{s=1}L(s,\hat\chi_0) = \mathrm{ord}_{s=1}\zeta(s) = -1$)

(PDL3) $L(s,\hat\chi)$ is related to $L(1-s,\hat\chi^{-1})$ by a functional equation (also involving $\Gamma(s)$, Gauss sums…)

For any Dirichlet character χ modulo f we define

$$r_\infty(\chi) := \begin{cases} 1 & \text{if } \chi(-\bar 1) = 1 \text{ (say `}\chi\text{ is \textit{even}') and } \chi\neq\chi_0 \\ 0 & \text{if } \chi(-\bar 1) = -1 \text{ (say `}\chi\text{ is \textit{odd}') or if } \chi=\chi_0 \end{cases}$$

Then the precise form of the functional equation plus **(PDL2)** gives :

(PDL4) $\mathrm{ord}_{s=0}L(s,\hat\chi) = r_\infty(\hat\chi)$

Returning to $L_f(s,\chi)$ for $\chi\in\widehat{G_f}$: Eqs. (2)+**(PDL1)**+**(PDL4)** give:

Theorem 1.1 *Suppose that χ is a Dirichlet character modulo f then $L_f(s,\chi)$ has a meromorphic continuation to $\mathbb{C}$ which is actually holomorphic except that* $\mathrm{ord}_{s=1}L_f(s,\chi_0) = -1$. *Moreover*

$$\mathrm{ord}_{s=0}L_f(s,\chi) = r_\infty(\chi) + \#\{p : p|f,\ p\nmid f_\chi,\ \hat\chi(\bar p) = 1\} \tag{3}$$

Note: If $\chi=\chi_0$ then R.H.S. of (3) becomes simply $\#\{p : p|f\}$.

1.4 Leading Terms at $s=0$

Theorem 1.2 *For any χ*

$$\begin{aligned} L_f(0,\chi) &= -\sum_{\substack{a=1\\(a,f)=1}}^{f}\left(\frac{a}{f}-\frac{1}{2}\right)\chi(\bar a) \\ &= -\sum_{\sigma_a\in G_f}\left(\left\langle\frac{a}{f}\right\rangle-\frac{1}{2}\right)\chi(\sigma_a) \end{aligned}$$

Proof: see [Wa, Thm. 4.2] (assumption that χ primitive is not used). □

One can check directly R.H.S.=0 whenever χ is even and $\chi\neq\chi_0$ (or $\chi=\chi_0$ but $f>1$), agreeing with (3).

(Harder: Check directly R.H.S.=0 if $\exists\, p$ s.t. $p|f,\ p\nmid f_\chi,\ \hat\chi(\bar p)=1$.)

Theorem 1.3 *If χ is even and $f>1$ (so that $L_f(0,\chi)=0$) then*

$$\begin{aligned} L_f'(0,\chi) &= -\frac{1}{2}\sum_{\substack{a=1\\(a,f)=1}}^{f}\log|1-\zeta_f^a|\chi(\bar a)\\ &= -\frac{1}{2}\sum_{\sigma_a\in G_f}\log|\sigma_a(1-\zeta_f)|\chi(\sigma_a)\end{aligned}$$

Proof: If χ is primitive: use [Wa, Thm. 4.9] for $L_f(1,\chi^{-1})$, then the functional equation (and *ibid.*, Lemmas 4.7, 4.8). General case can be deduced from this using 'norm relations' for cyclotomic units, or proved directly, see [Ha, Lec. 3] or refs. on [Tate, p. 79]. □

Basic Aim of the Stark Conjectures: *Formulate (and prove?) 'qualitative analogues' of Theorems 1.2 and 1.3 when:*

$K_f/\mathbb{Q}$	$\rightsquigarrow$	Galois extension of number fields K/k with group G
$\chi\in\widehat{G_f}$	$\rightsquigarrow$	group character of G *i.e.* character of cx. rep. of G
leading term of $L_f(s,\chi)$ at $s=0$	$\rightsquigarrow$	leading term of Artin L-function $L_{K/k,S}(s,\chi)$ at $s=0$

Extensions and Variations: *Replace $s=0$ by $s=1$; Replace number fields by function fields; Replace L-functions by p-adic L-functions;...*

Connections with: *Explicit Class Field Theory and Hilbert's 12th Problem; Stickelberger's Theorem and Generalisations; Additive and Multiplicative Galois-Module Structure; Euler Systems; K-Theory of K;...*

We shall consider almost exclusively the *abelian* case, *i.e.* Gal(K/k) is *abelian.*

2 The Function $\Theta_{K/k,S}(s)$

2.1 Motivation and Definition

K/k Galois extension of number fields, $G:=\mathrm{Gal}(K/k)$ *abelian*

$\mathbb{C}G$ = group-ring

$\chi\in\widehat{G}$ extends $\mathbb{C}$-linearly to ring hom. $\chi:\mathbb{C}G\to\mathbb{C}$

Note: in case $K/k=K_f/\mathbb{Q}$ we can rewrite:

$$\text{R.H.S. in Theorem 1.2} = \chi\left(-\sum_{\sigma_a\in G_f}\left(\left\langle\frac{a}{f}\right\rangle-\frac{1}{2}\right)\sigma_a\right)$$

$$\text{R.H.S. in Theorem 1.3} \quad = \quad \chi\left(-\frac{1}{2}\sum_{\sigma_a\in G_f}\log|\sigma_a(1-\zeta_f)|\sigma_a\right)$$

and parentheses contain elements of $\mathbb{C}G$ *which are independent of* χ

Idea: In Theorem 1.2 use *partial* ζ*-functions* to express L.H.S. as $\chi(\theta)$ for some $\theta\in\mathbb{C}G_f$ independent of χ. Then Theorem is a determination of θ.

(Similarly for Theorem 1.3, but only even characters involved…)

To do this in the general case K/k:

$$S_{\text{ram}} = S_{\text{ram}}(K/k) := \{\text{(non-zero) prime ideals } \mathfrak{p}\lhd\mathcal{O}_k\ :\ e_{\mathfrak{p}}(K/k)>1\} = \{\mathfrak{p}\ :\ \mathfrak{p}|\mathfrak{f}(K/k)\}$$

Fix S finite set of *places* of k containing S_{ram} as well as $S_\infty = S_\infty(k)$, the set of infnite (archimedean) places of k.

Define

$$I_S := \{\text{fractional ideals } \mathfrak{a} \text{ of } k \text{ supported away from } S\}$$

Thus $I_S\subset I_{\mathfrak{f}(K/k)}$ and Artin map restricts to a *surjection* $\sigma_{K/k}: I_S\to G$, $\mathfrak{a}\mapsto\sigma_{\mathfrak{a}}=\sigma_{\mathfrak{a},K/k}$

For $\Re(s)>1$ get an absolutely convergent sum in $\mathbb{C}G$

$$\Theta_{K/k,S}(s) \quad := \quad \sum_{\substack{\mathfrak{a}\in I_S\\ \mathfrak{a}\text{ integral}}} N\mathfrak{a}^{-s}\sigma_{\mathfrak{a}}^{-1}$$

So $\Theta_{K/k,S}(s):\{s:\Re(s)>1\}\to\mathbb{C}G$ is analytic.

The definition gives the following Basic Facts (currently only for $\Re(s)>1$):

(Θ1) In terms of partial zeta-functions:

$$\Theta_{K/k,S}(s) = \sum_{g\in G}\zeta_{K/k,S}(s,g)g^{-1}$$

where $\zeta_{K/k,S}(\cdot,g):\{s:\Re(s)>1\}\to\mathbb{C}$ is the *partial zeta-function* defined by

$$\zeta_{K/k,S}(s,g) := \sum_{\substack{\mathfrak{a}\in I_S,\sigma_{\mathfrak{a}}=g\\ \mathfrak{a}\text{ integral}}} N\mathfrak{a}^{-s}$$

(Θ2) Euler product in $\mathbb{C}G$: Unique factorisation of fractional ideals gives

$$\Theta_{K/k,S}(s) = \prod_{\mathfrak{p}\notin S}\sum_{n=0}^{\infty}(N\mathfrak{p}^n)^{-s}\sigma_{\mathfrak{p}^n}^{-1} = \prod_{\mathfrak{p}\notin S}\left(1-\frac{1}{N\mathfrak{p}^s}\sigma_{\mathfrak{p}}^{-1}\right)^{-1}$$

(Θ3) In terms of L-functions: Apply χ^{-1} for any $\chi \in \widehat{G}$ (also considered as a hom. $\chi : I_S \to G \to \mathbb{C}^\times$) to get

$$\chi^{-1}\left(\Theta_{K/k,S}(s)\right) = \prod_{\mathfrak{p}\notin S}\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} =: L_{K/k,S}(s,\chi)$$

(thus $L_{K/k,S}(s,\chi)$ is a not-necessarily-primitive Hecke L-function).

In other words, setting $e_\chi = \sum_{g\in G}\chi(g)g^{-1}$ (idempotent of $\mathbb{C}G$) we have, for $\Re(s) > 1$:

$$\begin{aligned} \Theta_{K/k,S}(s) &= \sum_{\chi\in\widehat{G}}\chi^{-1}(\Theta_{K/k,S}(s))e_{\chi^{-1}} \\ &= \sum_{\chi\in\widehat{G}} L_{K/k,S}(s,\chi)e_{\chi^{-1}} \end{aligned} \tag{4}$$

2.2 Example: $K/k = K_f/\mathbb{Q}$, $G = G_f$

Let

$$S_f := \{p\mathbb{Z} \,:\, p|f\} \cup \{\infty\} \supset S_{\mathrm{ram}} \cup S_\infty$$

(inclusion is an equality unless $f \equiv 2 \pmod 4$).

If $n \geq 1$ then the integral ideal $n\mathbb{Z}$ lies in I_{S_f} iff $(n,f) = 1$ in which case

$$\sigma_{n\mathbb{Z},K_f/\mathbb{Q}} = \sigma_n \quad \text{in } G_f$$

(explicit Class Field Theory over $\mathbb{Q}$). So, in terms of zeta-functions:

$$\Theta_{K_f/\mathbb{Q},S_f}(s) = \sum_{\sigma_a\in G_f}\zeta_{K_f/\mathbb{Q},S_f}(s,\sigma_a)\sigma_a^{-1}$$

where

$$\zeta_{K_f/\mathbb{Q},S_f}(s,\sigma_a) = \sum_{\substack{n\geq 1\\ (n,f)=1,\sigma_n=\sigma_a}}\frac{1}{n^s} = \sum_{\substack{n\geq 1\\ n\equiv a \pmod f}}\frac{1}{n^s}$$

($f^{-s}\times\zeta(s,\langle a/f\rangle)$, Hurwitz ζ-function)

Also, for any $\chi \in \widehat{G_f}$, fact (Θ3) gives

$$\chi^{-1}(\Theta_{K_f/\mathbb{Q},S_f}(s)) = L_{K_f/\mathbb{Q},S_f}(s,\chi) = \prod_{p\nmid f}\left(1 - \frac{\chi(\bar{p})}{p^s}\right)^{-1} = L_f(s,\chi)$$

for $\Re(s) > 1$ thus in terms of L-functions:

$$\Theta_{K_f/\mathbb{Q},S_f}(s) = \sum_{\chi\in\widehat{G_f}} L_f(s,\chi)e_{\chi^{-1}}$$

Thus Theorem 1.1 gives a meromorphic continuation $\Theta_{K_f/\mathbb{Q},S_f} : \mathbb{C} \to \mathbb{C}G_f$ and

Theorem 2.1

$$\Theta_{K_f/\mathbb{Q},S_f}(0) = -\sum_{\sigma_a \in G_f} \left(\left\langle \frac{a}{f} \right\rangle - \frac{1}{2} \right) \sigma_a^{-1}$$

Proof Suffices to show $\chi^{-1}(L.H.S.) = \chi^{-1}(R.H.S.)$ for all $\chi \in G_f$. But

$$\begin{aligned}\chi^{-1}(\Theta_{K_f/\mathbb{Q},S_f}(0)) &= L_{K_f/\mathbb{Q},S_f}(0,\chi) \\ &= L_f(0,\chi) \\ &= \chi^{-1}\left(-\sum_{\sigma_a \in G_f} \left(\left\langle \frac{a}{f} \right\rangle - \frac{1}{2} \right) \sigma_a^{-1} \right)\end{aligned}$$

by Theorem 1.2. □

2.3 Analytic Facts about $L_{K/k,S}(s,\chi)$ and $\Theta_{K/k,S}$

We need meromorphic continuation of these functions and behaviour at $s = 0$ in the general case K/k, G, S.

Let $\mathfrak{m} = \mathfrak{f}\mathfrak{z} = \mathfrak{m}(K/k) = \mathfrak{f}(K/k)\mathfrak{z}(K/k)$ be a cycle and use the Artin map to identify $\chi \in \widehat{G}$ with a character of the *ray-class group* of k modulo $\mathfrak{m}$ *i.e.* $\chi : \mathrm{Cl}_{\mathfrak{m}}(k) \to \mathbb{C}^\times$. (See 'Basic Facts and Notations')

Just as for $L_f(s,\chi)$, we reduce the treatment of $L_{K/k,S}(s,\chi)$ to the primitive case:

$\exists$ unique minimal cycle $\mathfrak{m}(\chi) = \mathfrak{f}(\chi)\mathfrak{z}(\chi)$ dividing $\mathfrak{m}$ (the *conductor of* χ) and character $\hat{\chi} : \mathrm{Cl}_{\mathfrak{m}(\chi)}(k) \to \mathbb{C}^\times$ (the *primitive character associated with* χ) s.t. χ factors through $\hat{\chi}$ via the hom. $\mathrm{Cl}_{\mathfrak{m}}(k) \to \mathrm{Cl}_{\mathfrak{m}(\chi)}(k)$.

Note: Let $K_\chi := K^{\ker \chi}$ so χ factors through $\mathrm{Gal}(K_\chi/k)$. Then $\mathfrak{m}(\chi) = \mathfrak{m}(K_\chi/k)$...

Thus $\hat{\chi}$ defines hom. $\hat{\chi} : I_{\mathfrak{f}(\chi)} \to \mathbb{C}^\times$ and the usual *primitive* Hecke L-function is

$$L(s,\hat{\chi}) = \prod_{\mathfrak{p} \nmid \mathfrak{f}(\chi)} \left(1 - \frac{\hat{\chi}(\mathfrak{p})}{N\mathfrak{p}^s} \right)^{-1}$$

On the other hand $I_{\mathfrak{f}(\chi)} \supset I_{\mathfrak{f}} \supset I_S$ and $\hat{\chi}|_{I_S} = \chi$ so, by defn. in (Θ3):

$$L_{K/k,S}(s,\chi) = \prod_{\substack{\mathfrak{p} \in S \\ \mathfrak{p} \nmid \mathfrak{f}(\chi)}} \left(1 - \frac{\hat{\chi}(\mathfrak{p})}{N\mathfrak{p}^s} \right) L(s,\hat{\chi}) \tag{5}$$

The following 'well-known' facts about primitive Hecke L-functions generalise (PDL1)–(PDL4):

(PHL1) $L(s,\hat{\chi})$ has a continuation to $\mathbb{C}$ that is analytic at all $s \in \mathbb{C}$ (except for $L(s,\hat{\chi}_0) = \zeta_k(s)$ at $s = 1$)

(PHL2) $\mathrm{ord}_{s=1}L(s,\hat{\chi}) = 0$ (except that $\mathrm{ord}_{s=1}L(s,\hat{\chi}_0) = \mathrm{ord}_{s=1}\zeta_k(s) = -1$)

(PHL3) $L(s,\hat{\chi})$ is related to $L(1-s,\hat{\chi}^{-1})$ by a functional equation. (This involves a Γ-type factor for each $v \in S_\infty(k)$, Gauss sums...) For more details see [Tate, §1.3] or [Ha, Lecture 2]).

(PHL4) The precise form of the functional equation plus **(PHL2)** gives :

$$\mathrm{ord}_{s=0}L(s,\hat{\chi}) = r_\infty(\hat{\chi})$$

where

$$r_\infty(\chi) := \begin{cases} \#\{v \text{ real place of } k \,:\, v \nmid \mathfrak{z}(\chi)\} & \text{if } \chi \neq \chi_0 \\ \#\{v \text{ real place of } k \,:\, v \nmid \mathfrak{z}(\chi_0) = \varnothing\} - 1 = r_1(k) - 1 & \text{if } \chi = \chi_0 \end{cases}$$

Returning to $\chi \in \widehat{G}$*:* For *any* finite set T of places of k, let

$$r_T(\chi) := \begin{cases} \#\{v \in T \,:\, \chi(D_v(K/k)) = \{1\}\} & \text{if } \chi \neq \chi_0 \\ \#\{v \in T \,:\, \chi(D_v(K/k)) = \{1\}\} - 1 = \#T - 1 & \text{if } \chi = \chi_0 \end{cases}$$

Theorem 2.2 *If S contains $S_{\mathrm{ram}} \cup S_\infty$ and $\chi \in \widehat{G}$ then $L_{K/k,S}(s,\chi)$ has a meromorphic continuation to $\mathbb{C}$ (holomorphic except for $\chi = \chi_0$ at $s = 1$). Moreover*

$$\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi) = r_S(\chi)$$

Proof Eqs. (5)+**(PHL1)**+**(PHL4)** give the meromorphic continuation of $L_{K/k,S}(s,\chi)$ and the formula

$$\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi) = r_\infty(\hat{\chi}) + \#\{\mathfrak{p} \in S \,:\, \mathfrak{p} \nmid \mathfrak{f}(\chi) \text{ and } \hat{\chi}(\mathfrak{p}) = 1\}$$

But a real place v doesn't divide $\mathfrak{z}(\chi)$ iff it remains real in K_χ iff $\chi(D_v(K/k)) = \{1\}$. Similarly, $\mathfrak{p} \nmid \mathfrak{f}(\chi)$ satisfies $\hat{\chi}(\mathfrak{p}) = 1$ iff $\hat{\chi}(D_v(K_\chi/k)) = \{1\}$ iff $\chi(D_v(K/k)) = \{1\}$. The result follows. □

¿From equation (4) we get

Corollary 2.1 $\Theta_{K/k,S}(s)$ *can be continued to a function $\mathbb{C} \to \mathbb{C}G$ that is analytic except at $s = 1$.* □

Consider the following hypothesis on a set $S \supset S_{\mathrm{ram}} \cup S_\infty$ and an integer $r \geq 0$:

$$H(S,r): \quad \begin{cases} (i) & S \text{ contains at least } r \text{ places that split (completely) in } K, \text{ and} \\ (ii) & \#S \geq r+1 \end{cases}$$

To say that v 'splits completely' is equivalent to $D_v(K/k) = \{1\}$ so $H(S,r) \Rightarrow r_S(\chi) \geq r \;\forall \chi \in \widehat{G}$, so

Corollary 2.2 *Suppose $H(S,r)$ holds, then there exists $\Theta^{(r)}_{K/k,S}(0) \in \mathbb{R}G$ (unique) s.t.*

$$\Theta_{K/k,S}(s) = \Theta^{(r)}_{K/k,S}(0)s^r + o(s^r) \quad \textit{in } \mathbb{C}G \textit{ as } s \to 0$$

*(*i.e. *$H(S,r) \Rightarrow \Theta_{K/k,S}(s)$ has at least an rth order zero at $s=0$).*

Proof Since $r_S(\chi) \geq r\ \forall \chi \in \widehat{G}$, Equation (4) and the Theorem imply the existence of $\Theta^{(r)}_{K/k,S}(0)$ in $\mathbb{C}G$. Now the definition shows that $\Theta_{K/k,S}$ is real valued on $\mathbb{R}_{>1}$ hence on the whole of $\mathbb{R}$ (since meromorphic on $\mathbb{C}$) from which it follows that $\Theta^{(r)}_{K/k,S}(0)$ actually lies in $\mathbb{R}G$. □

Notes:

(i) $\Theta^{(r)}_{K/k,S}(0) = \frac{1}{r!}\frac{d^r}{ds^r}|_{s=0}\Theta_{K/k,S}$. (*Warning:* Note the implied factor of $\frac{1}{r!}$ in this notation.)

(ii) $\Theta_{K/k,S}(s)$ *can* have an rth order zero at $s=0$ without $H(S,r)$…

3 Stark Conjectures at $s = 0$

K/k finite abelian extension of number fields

$G = \mathrm{Gal}(K/k)$, S a finite set of places of k, $S \supset S_\infty \cup S_{\mathrm{ram}}$

Idea for an rth order Stark Conjecture at $s=0$ ($r \geq 0$):

Assume $H(S,r)$ and make conjecture about the form of $\Theta^{(r)}_{K/k,S}(0)$

3.1 Basic Zeroth Order Conjecture

$\#S \geq 1$ so $H(S,0)$ is always satisfied. So what is $\Theta^{(0)}_{K/k,S}(0) = \Theta_{K/k,S}(0)$?

Theorem 2.1 answers this for $K/k = K_f/\mathbb{Q}$, $S = S_f$. One generalisation is

Theorem 3.1 (Siegel-Klingen, Shintani)

$$\Theta_{K/k,S}(0) \in \mathbb{Q}G$$

□

Thus the 'Basic Zeroth Order Stark Conjecture at $s=0$' is a theorem!

Note: Theorem 3.1 is non-trivial$\Rightarrow \Theta_{K/k,S}(0) \neq 0 \Rightarrow H(S,1)$ fails $\Rightarrow$

(i) S contains no place splitting in K, or

(ii) $S = \{v\}$ for some v splitting in K.

Case (i) :

then S_∞ contains no place splitting in K so *k is totally real, K is totally complex.* In this

case the element $\Theta_{K/k,S}(0) \in \mathbb{Q}G$ still contains a lot of interesting information.
Case (ii) (less interesting, because $\chi(\Theta_{K/k,S}(0)) = 0$ for $\chi \neq \chi_0$):
then $S_\infty = \{v\}$, $S_{\rm ram} = \varnothing \Rightarrow k = \mathbb{Q} = K$ or $k = \mathbb{Q}(\sqrt{-D}), K \subset H_k$

3.2 Basic First Order Conjecture

Assume that $H(S,1)$ holds, *i.e.*

$$\exists v \in S \text{ splitting in } K \text{ and } |S| \geq 2$$

Motivating Example: $K/k = K_f^+/\mathbb{Q}$ for $f \in \mathbb{Z},\ f > 1$ where

$$K_f^+ := \mathbb{Q}(\zeta_f + \zeta_f^{-1}) = \mathbb{Q}(\cos(2\pi/f))$$

Take $S = S_f = \{p\mathbb{Z} : p|f\} \cup \{\infty\}$ as before (minimal unless $f \equiv 2 \pmod 4$ or $f =\leq 4$)
K_f^+ is the maximal real subfield of K_f so that the real place $v = \infty : \mathbb{Q} \hookrightarrow \mathbb{R} \subset \mathbb{C}$ splits in K_f^+.
Since also $f > 1$, hypothesis $H(S,1)$ is satisfied.
The restriction map $G_f \to G_f^+ := \mathrm{Gal}(K_f^+/\mathbb{Q})$ yields an isomorphism

$$\begin{array}{ccc} G_f^+ & \cong & (\mathbb{Z}/f\mathbb{Z})^\times/\{\pm\bar{1}\} \\ \sigma_a|_{K_f^+} & \longleftrightarrow & [\bar{a}] \end{array}$$

Hence

$$\widehat{G_f^+} \longleftrightarrow \{\textit{even } \text{Dirichlet characters } \chi \bmod f\}$$

and for any such character

$$\chi^{-1}(\Theta_{K_f^+/\mathbb{Q},S_f}(s)) = L_{K_f^+/\mathbb{Q},S_f}(s,\chi) = \prod_{p\nmid f}\left(1 - \frac{\chi(\bar{p})}{p^s}\right)^{-1} = L_f(s,\chi)$$

Thus $\chi^{-1}(\Theta^{(1)}_{K_f^+/\mathbb{Q},S_f}(0)) = L'_f(0,\chi)$ and we get the following analogue of Theorem 2.1:

Theorem 3.2

$$\begin{aligned} \Theta^{(1)}_{K_f^+/\mathbb{Q},S_f}(0) &= -\frac{1}{2}\sum_{\sigma_a \in G_f} \log|\sigma_a(1-\zeta_f)|[\sigma_a]^{-1} \\ &= -\frac{1}{2}\sum_{g \in G_f^+} \log|g((1-\zeta_f)(1-\zeta_f^{-1}))|g^{-1} \end{aligned}$$

(Last equality for $f > 2$ only.)

PROOF For first eq. suffices to show $\chi^{-1}(L.H.S.) = \chi^{-1}(R.H.S.)$ for all $\chi \in \widehat{G_f^+}$. This follows as for Thm. 2.1 (using Thm. 1.3 in place of Thm. 1.2). If $f > 2$, second eq. follows by pairing σ_a with σ_{-a}. □

Important Remark: $(1-\zeta_f)(1-\zeta_f^{-1})$ is element of $K_f^{+,\times}$. Moreover it's a unit away from primes of K above those in S_f.

More precisely:
if $f = p^l$, p prime, then it's a unit away from the (unique) prime $\mathfrak{P}$ of K_f^+ above p (which it generates).

If $f \neq p^l$ it's a genuine *'cyclotomic unit'* in $E(K_f^+)$. (See *e.g.* [Wa, Prop. 2.8].)

For any number field L and finite set T of places we define the (group of) *T-units of L* to be

$$\begin{aligned} U_T(L) &:= \{x \in L^\times \,:\, |x|_w = 1 \;\forall\, w \notin T\} \\ &= \{\text{"elements of } L^\times \text{ which are local units away from } T\text{"}\} \end{aligned}$$

where $|\cdot|_w$ denotes the (normalised) absolute value associated to a place w. (The second statement only makes sense if $T \supset S_\infty(L)$.)

Thus $U_\varnothing(L) = \mu(L)$ (roots of unity), $U_{S_\infty(L)}(L) = \mathcal{O}_L^\times = E(L)$ (unit group) and $U_T(L)$ is always f.g. (see below).

Often, but *not always*, one assumes $T \supset S_\infty(L)$: If $T = S_\infty(L) \,\dot\cup\, T'$ then $U_T(L) = \mathcal{O}_{L,T'}^\times$.

Return to K/k, S, G as above and set $S_K :=$ {places w of K above those in S} $\supset S_\infty(K)$

Then S_K is stable under action of G on places. Since $|gx|_w = |x|_{g^{-1}w}$, follows that $U_{S_K}(K)$ is also G-stable so a f.g. module over $\mathbb{Z}G$.

Henceforth we write $U_S(K)$ to mean $U_{S_K}(K)$.

Conjecture $C(K/k, S, 1)$: The Basic First Order Stark Conjecture at $s = 0$
Suppose K/k is abelian extension of number fields with Galois group G and S is a finite set of places of k containing $S_\infty \cup S_{\mathrm{ram}}$.

Suppose $H(S,1)$ is satisfied. Choose v in S splitting completely in K and a place w dividing v in K.

Then there exists $\varepsilon \in U_S(K)$ and $\alpha \in \mathbb{Q}$ such that

$$\Theta^{(1)}_{K/k,S}(0) = \alpha \sum_{g \in G} \log|g\varepsilon|_w g^{-1} \tag{6}$$

Remark. Our Cyclotomic Example: Take $K/k = K_f^+/\mathbb{Q}$, $S = S_f$ so $H(S,1)$ is satisfied as above. Take $v = \infty : \mathbb{Q} \hookrightarrow \mathbb{R} \subset \mathbb{C}$, $w : K_f^+ \hookrightarrow \mathbb{R} \subset \mathbb{C}$. If $f > 2$ then Theorem3.2 says that (6) is satisfied with $\alpha = -\frac{1}{2}$, $\varepsilon = (1-\zeta_f)(1-\zeta_f^{-1})$

Remark. Alternative Formulation of (6): In terms of partial zeta-functions:

$$\zeta'_{K/k,S}(0,g) = \alpha \log|g\varepsilon|_w \quad \text{for all } g \in G$$

Remark. Variation of w: Fix v and suppose α, ε satisfy Conj. for some $w|v$. Then $w'|v \Leftrightarrow w' = hw$ for some $h \in G \Rightarrow |g\varepsilon|_w = |gh\varepsilon|_{w'} \Rightarrow \alpha, h\varepsilon$ satisfy Conj. for w'.

Remark. Variation of v: If S contains another splitting place $v' \neq v$ then we shall see that Conjecture is trivial unless $S = \{v, v'\}$ in which case it follows from Analytic Class Number Formula for k (using either v or v').

3.3 Basic rth Order Conjecture

How can we generalise the basic first order conjecture when $H(S,r)$ holds? We need to know more about $U_S(K)$...

More generally, for L a number field and any T as above containing $S_\infty(L)$, we define a *logarithmic 'embedding' $\mathcal{L}_T$ of $U_T(L)$:*

$$\begin{array}{rcccl} \mathcal{L}_T & : & U_T(L) & \longrightarrow & \mathbb{R}T := \bigoplus_{w\in T} \mathbb{R}w \\ & & \varepsilon & \longmapsto & \sum_{w\in T} \log|\varepsilon|_w w \end{array}$$

Let $(\mathbb{R}T)_0$ be the kernel of the map $\mathbb{R}T \to \mathbb{R}$ sending $\sum_{w\in T} a_w w$ to $\sum_{w\in T} a_w$

Thus $(\mathbb{R}T)_0$ is a hyperplane in $\mathbb{R}T$ and the Product formula implies $\mathrm{im}(\mathcal{L}_T) \subset (\mathbb{R}T)_0$.

Theorem 3.3 (Dirichlet's Theorem for T-units) *Suppose that T contains $S_\infty(L)$. Then*
(i) $\ker(\mathcal{L}_T) = \mu(K)$

(ii) $\mathrm{im}(\mathcal{L}_T)$ *is a* lattice of full rank *in* $(\mathbb{R}T)_0$.

Proof (Sketch) It is easy to show (i) and that $\mathrm{im}(\mathcal{L}_T)$ is discrete, hence a lattice. So suffices to show $\mathrm{rk}_{\mathbb{Z}}(U_T(L)/\mu(K)) = \#T - 1$. For $T = S_\infty(L)$ this is classical Dirichlet. Now use induction on $\#(T \setminus S_\infty(L))$, the finiteness of $\mathrm{Cl}(L)$ *etc.*... □

Corollary 3.1 *$U_T(L)$ is isomorphic as an abelian group to $\mathbb{Z}^{\#T-1} \oplus \mu(L)$. Moreover $\mathcal{L}_T$ extends by $\mathbb{R}$-linearity to a map $\mathcal{L}_T : \mathbb{R} \otimes_{\mathbb{Z}} U_T(L) \to \mathbb{R}T$ sending $\sum_i \alpha_i \otimes \varepsilon_i$ to $\sum_i \alpha_i \mathcal{L}_T(\varepsilon_i)$ and giving an $\mathbb{R}$-isomorphism of $\mathbb{R} \otimes_{\mathbb{Z}} U_T(L)$ onto $(\mathbb{R}T)_0$.*

Return to the case K/k, S, G:

the G action on places makes $\mathbb{R}S_K$ into a natural $\mathbb{R}G$-module and

$$\mathbb{R}S_K = \bigoplus_{v\in S} \left(\bigoplus_{w|v} \mathbb{R}w \right) \cong \bigoplus_{v\in S} \mathbb{R}[G/D_v(K/k)] \tag{7}$$

Notation: If M is a $\mathbb{Z}G$-mod. and R a comm. ring, write RM for $R \otimes_{\mathbb{Z}} M$ considered as R-module and G-module so RG-module.

E.g. $\mathbb{R}U_S(K)$ is an $\mathbb{R}G$-module and $\mathcal{L}_{S_K} : \mathbb{R}U_S(K) \to \mathbb{R}S_K$ is $\mathbb{R}G$-linear restricting to $\mathbb{R}G$-iso. $\mathbb{R}U_S(K) \to (\mathbb{R}S_K)_0$. Thus we have an exact sequence of $\mathbb{R}G$-modules

$$0 \to \mathbb{R}U_S(K) \longrightarrow \mathbb{R}S_K \longrightarrow \mathbb{R} \to 0 \tag{8}$$

(with trivial G-action on $\mathbb{R}$).

Corollary 3.2 *For all* $\chi \in \widehat{G}$

$$\dim_{\mathbb{C}}(e_\chi \mathbb{C}U_S(K)) = r_S(\chi) \left[= \mathrm{ord}_{s=0} L_{K/k,S}(s,\chi) = \mathrm{ord}_{s=0} \chi^{-1}(\Theta_{K/k,S}(s))\right]$$

Proof Exercise: tensor (7) and (8) with $\mathbb{C}$ and apply a little character theory. □

In vague terms: *the bigger the 'rank' of $\mathbb{C}U_S(K)$, the higher the order of vanishing of $\Theta_{K/k,S}(s)$*

Now suppose $H(S,r)$ holds and choose an (ordered) set $V = \{v_1, \ldots, v_r\} \subset S$ *such that v_i splits in K for $i = 1, \ldots, r$.*

We write $S = V \dot{\cup} V'$ so that $\mathbb{R}S_K = \mathbb{R}V_K \oplus \mathbb{R}V'_K$ (as $\mathbb{R}G$-modules) and let π_V denote the projection from $\mathbb{R}S_K$ to $\mathbb{R}V_K$ with kernel $\mathbb{R}V'_K$.

We shall write $\mathcal{L}_S$ for the map $\mathcal{L}_{S_K}$ and $\mathcal{L}_{S,V}$ for the composite $\pi_V \circ \mathcal{L}_S$:

$$\mathcal{L}_{S,V} : \mathbb{R}U_S(K) \xrightarrow{\mathcal{L}_S} \mathbb{R}S_K \xrightarrow{\pi_V} \mathbb{R}V_K$$

Choose $W = \{w_1, \ldots, w_r\} \subset V_K$ such that $w_i | v_i$ for each i.

Then $\mathbb{R}V_K$ is $\mathbb{R}G$-free of rank r with the basis $\{w_1, \ldots, w_r\}$ so for any $x \in \mathbb{R}U_S(K)$ we can write uniquely

$$\mathcal{L}_{S,V}(x) = \sum_{i=1}^{r} \lambda_{w_i}(x) w_i$$

where $\lambda_{w_i}(x) \in \mathbb{R}G$

One easily checks that $\lambda_{w_i} : \mathbb{R}U_S(K) \to \mathbb{R}G$ is the unique $\mathbb{R}G$-linear map satisfying

$$\lambda_{w_i}(1 \otimes u) = \sum_{g \in G} \log |gu|_{w_i} g^{-1}$$

Taking rth exterior powers over the commutative ring $\mathbb{R}G$ gives an $\mathbb{R}G$-linear map

$$\textstyle\bigwedge^r_{\mathbb{R}G} \mathcal{L}_{S,V} : \bigwedge^r_{\mathbb{R}G} \mathbb{R}U_S(K) \longrightarrow \bigwedge^r_{\mathbb{R}G} \mathbb{R}V_K = \mathbb{R}G(w_1 \wedge \ldots \wedge w_r)$$

Since $w_1 \wedge \ldots \wedge w_r$ is a *free* generator we can define a unique $\mathbb{R}G$*-linear 'regulator'* $R_{K/k,W} : \bigwedge^r_{\mathbb{R}G} \mathbb{R}U_S(K) \to \mathbb{R}G$ by $\bigwedge^r_{\mathbb{R}G} \mathcal{L}_{S,V}(x) = R_{K/k,W}(x)(w_1 \wedge \ldots \wedge w_r)$.

Explicitly, every element of $\bigwedge_{\mathbb{R}G}^{r}\mathbb{R}U_S(K)$ is a finite sum of terms of form $x_1 \wedge \ldots \wedge x_r$ with $x_i \in \mathbb{R}U_S(K)$ and

$$R_{K/k,W}(x_1 \wedge \ldots \wedge x_r) = \det(\lambda_{w_i}(x_j))_{i,j=1}^{r}$$

Conjecture $C(K/k,S,r)$: The Basic rth Order Stark Conjecture at $s=0$
Suppose K/k is abelian extension of number fields with Galois group G and S a is finite set of places of k containing $S_\infty \cup S_{\text{ram}}$.

Suppose $H(S,r)$ is satisfied. Choose a set $V = \{v_1, \ldots v_r\} \subset S$ of places splitting completely in K and a set $W = \{w_1, \ldots w_r\} \subset V_K$ with $w_i|v_i \ \forall\, i$.

Then there exists $\eta \in \bigwedge_{\mathbb{Q}G}^{r} \mathbb{Q}U_S(K) \subset \bigwedge_{\mathbb{R}G}^{r} \mathbb{R}U_S(K)$ such that

$$\Theta_{K/k,S}^{(r)}(0) = R_{K/k,W}(\eta) \tag{9}$$

Remark. Variation of W: Fix V and suppose η satisfies the conjecture with the choice $W = \{w_1, \ldots, w_r\}$. Let $W' = \{w'_1, \ldots, w'_r\}$ be another choice. Then $w'_i = h_i w_i$ for some $h_i \in G$ for all i so $\lambda_{w'_i}(hx) = \lambda_{w_i}(x)$ and we find $R_{K/k,W}(x) = R_{K/k,W'}(h_1 \ldots h_r x)$. Thus $h_1 \ldots h_r \eta$ satisfies the conjecture with the choice W'.

Remark. Variation of V: Permuting the v_i (and hence the w_i) multiplies $R_{K/k}(\eta)$ by ± 1 so $\pm\eta$ still satisfies the conjecture.

This is the only variation possible unless S contains $r+1$ splitting places in which case the conjecture can be shown to hold for any choice of V.

When $r=1$, $V = \{v\}$ and $W = \{W\}$, we have $\eta \in \bigwedge_{\mathbb{Q}G}^{1}\mathbb{Q}U_S(K) = \mathbb{Q}U_S(K)$, so $\eta = \alpha \otimes \varepsilon$ for some $\alpha \in \mathbb{Q}$ and $\varepsilon \in U_S(K)$. Hence $R_{K/k,W}(\eta) = \lambda_w(\eta) = \alpha \sum_{g \in G} |g\varepsilon|_w g^{-1}$. *So for $r=1$, $C(K/k,S,r)$ indeed agrees with first order conjecture denoted $C(K/k,S,1)$ enunciated above.*

3.4 Uniqueness of η

In order to refine the conjecture (for example) we want to study the element $\eta \in \bigwedge_{\mathbb{Q}G}^{r} \mathbb{Q}U_S(K)$ satisfying (9). However, is not in general unique, even if it exists. We explain a way to render it unique by projecting onto a certain 'eigenspace' of $\bigwedge_{\mathbb{Q}G}^{r} \mathbb{Q}U_S(K)$ as a $\mathbb{Q}G$-module:

Still in the set-up of $C(K/k,S,r)$, we assume $H(S,r)$ is satisfied and V, W are chosen as above.

Then $r_S(\chi) \geq r$ for every $\chi \in \widehat{G}$ and we define

$$e_{S,r} := \sum_{r_S(\chi)=r} e_\chi \quad \text{and} \quad e_{S,>r} = \sum_{r_S(\chi)>r} e_\chi = 1 - e_{S,r}$$

Thus $e_{S,r}$ is an idempotent, and the unique element of $\mathbb{C}G$ satisfying

$$\chi(e_{S,r}) = \begin{cases} 1 & \text{if } r_S(\chi) = r \\ 0 & \text{if } r_S(\chi) > r \end{cases} \tag{10}$$

Set $s = |S| \geq r+1$ so that

$$S = \{v_1, \ldots, v_r\} \,\dot\cup\, \{v_{r+1}, \ldots, v_s\} = V \,\dot\cup\, V'$$

Write D_v for $D_v(K/k) \subset G$. (So $D_v = \{1\}$ if $v = v_1, \ldots, v_r$).

Let $N_{D_v} := \sum_{g \in D_v} g$ so that $\frac{1}{\# D_v} N_{D_v}$ is an idempotent of $\mathbb{Q}G$ and

$$\chi\left(\frac{1}{\# D_v} N_{D_v}\right) = \begin{cases} 1 & \text{if } \chi(D_v) = \{1\} \\ 0 & \text{if } \chi(D_v) \neq \{1\} \end{cases}$$

Lemma 3.1

$$e_{S,r} = \begin{cases} \prod_{v \in V'} \left(1 - \frac{1}{\# D_v} N_{D_v}\right) & \textit{if } s > r+1 \textit{ (so } r_S(\chi_0) > r\textit{)} \\ \prod_{v \in V'} \left(1 - \frac{1}{\# D_v} N_{D_v}\right) + e_{\chi_0} & \textit{if } s = r+1 \textit{ (so } r_S(\chi_0) = r\textit{)} \end{cases}$$

In particular, $e_{S,r}$ lies in $\mathbb{Q}G$, hence also $e_{S,>r}$.

Proof Exercise using (10) and definition of $r_S(\chi)$. □

Thus for any $\mathbb{Q}G$-module A we can define a $\mathbb{Q}G$-submodule $A^{[S,r]}$ by

$$A^{[S,r]} = e_{S,r}A = \{a \in A \,:\, e_{S,r}a = a\} = \ker(e_{S,>r}|A)$$

Remarks

• The map $a \mapsto e_{S,r}a$ projects A onto $A^{[S,r]}$

• $e_{S,r}\mathbb{Q}G = \mathbb{Q}G^{[S,r]}$ and $e_{S,>r}\mathbb{Q}G$ are rings with identities the idempotents $e_{S,r}$ and $e_{S,>r}$ respectively. Moreover $\mathbb{Q}G$ is a product of rings $\mathbb{Q}G^{[S,r]} \times e_{S,>r}\mathbb{Q}G$ and acts on $A^{[S,r]}$ via its projection on $\mathbb{Q}G^{[S,r]}$.

• If $f : A \to B$ is a $\mathbb{Q}G$-hom. then the restriction of f to $A^{[S,r]}$ defines a $\mathbb{Q}G$-homomorphism $f^{[S,r]} : A^{[S,r]} \to B^{[S,r]}$. In this way, $\_^{[S,r]}$ defines an exact functor from the category of $\mathbb{Q}G$-modules to the category of $\mathbb{Q}G^{[S,r]}$ modules.

• Suppose A is embedded in a $\mathbb{C}G$-module $\tilde{A}$. Then $A^{[S,r]}$ is the intersection of A with the sum of the χ-eigenspaces $e_\chi \tilde{A}$ for those χ s.t. $r_S(\chi) = r$. Alternatively,

$$a \in A^{[S,r]} \iff e_\chi a = 0 \text{ in } \tilde{A}, \text{ for all } \chi \text{ s.t. } r_S(\chi) > r$$

Proposition 3.1 $\Theta^{(r)}_{K/k,S}(0)$ *lies in* $\mathbb{R}G^{[S,r]}$

PROOF Suppose $r_S(\chi) > r$ for some $\chi \in \widehat{G}$ then (working in $\mathbb{C}G$):

$$e_\chi \Theta^{(r)}_{K/k,S}(0) = \chi(\Theta^{(r)}_{K/k,S}(0))e_\chi = L^{(r)}_{K/k,S}(0,\chi)e_\chi = 0$$

by Thm. 2.2. □

Proposition 3.2 *$\mathbb{Q}U_S(K)^{[S,r]}$ is isomorphic to $(\mathbb{Q}G^{[S,r]})^r$ over $\mathbb{Q}G$.*

PROOF Since $\mathbb{Q}G$-modules are defined up to iso. by their character which is invariant under $\mathbb{C} \otimes \_$, it suffices show that $\mathbb{C}U_S(K)^{[S,r]} \cong (\mathbb{C}G^{[S,r]})^r$ over $\mathbb{C}G$. But for any $\chi \in \widehat{G}$, Cor. 3.2 gives $\dim_{\mathbb{C}}(e_\chi \mathbb{C}U_S(K)^{[S,r]}) = r$ if $r_S(\chi) = r$, otherwise $= 0$. The same is obviously true of $\dim_{\mathbb{C}}(e_\chi(\mathbb{C}G^{[S,r]})^r)$ so the result follows. □

Remark Similar reasoning shows more generally that $\mathbb{Q}U_S(K)$ is isomorphic over $\mathbb{Q}G$ to $\ker(\mathbb{Q}S_K \to \mathbb{Q})$ because the isomorphism holds over $\mathbb{R}G$ after applying $\otimes_{\mathbb{Q}}\mathbb{R}$, by (8)

Definition (*Warning:* this is nonstandard!) We shall say that a solution $\eta \in \bigwedge^r_{\mathbb{Q}G}\mathbb{Q}U_S(K)$ of $C(K/k,S,r)$ is *canonical* iff it lies in $\left(\bigwedge^r_{\mathbb{Q}G}\mathbb{Q}U_S(K)\right)^{[S,r]} = \bigwedge^r_{\mathbb{Q}G}(\mathbb{Q}U_S(K)^{[S,r]})$.

Proposition 3.3 *If η is any solution of $C(K/k,S,r)$ then $e_{S,r}\eta$ is a canonical solution.*

PROOF $R_{K/k,W}(e_{S,r}\eta) = e_{S,r}R_{K/k,W}(\eta) = e_{S,r}\Theta^{(r)}_{K/k,S}(0) = \Theta^{(r)}_{K/k,S}(0)$ by Prop. 3.1 □

We shall next show that a canonical solution is unique. The $\mathbb{R}G$-injection $\mathcal{L}_S : \mathbb{R}U_S(K) \to \mathbb{R}S_K$ gives rise to

$$\mathcal{L}_S^{[S,r]} : \mathbb{R}U_S(K)^{[S,r]} \longrightarrow \mathbb{R}S_K^{[S,r]} = \mathbb{R}V_K^{[S,r]} \oplus \mathbb{R}V'^{[S,r]}_K$$

If $w \in V'_K$ divides $v \in V'$ then D_v fixes w and it follows from Lemma 3.1 that

$$\mathbb{R}V'^{[S,r]}_K = e_{S,r}\mathbb{R}V'_K = \begin{cases} 0 & \text{if } s > r+1 \\ \mathbb{R}\sum_{w|v_{r+1}} w & \text{if } s = r+1 \text{ (so } V' = \{v_{r+1}\}) \end{cases} \tag{11}$$

Now recall that $\mathcal{L}_{S,V} = \pi_V \circ \mathcal{L}_S : \mathbb{R}U_S(K) \longrightarrow \mathbb{R}V_K$, so

$$\mathcal{L}_{S,V}^{[S,r]} = \pi_V^{[S,r]} \circ \mathcal{L}_S^{[S,r]} : \mathbb{R}U_S(K)^{[S,r]} \longrightarrow \mathbb{R}V_K^{[S,r]}$$

Proposition 3.4

(i) If $s > r+1$ then $\mathcal{L}_{S,V}^{[S,r]} = \mathcal{L}_S^{[S,r]}$.

(ii) If $s > r+1$ and $\varepsilon \in U_S(K)$ is such that $1 \otimes \varepsilon \in \mathbb{Q}U_S(K)^{[S,r]}$ then in fact $\varepsilon \in U_V(K)$.

(iii) In any case $\mathcal{L}_{S,V}^{[S,r]} : \mathbb{R}U_S(K)^{[S,r]} \to \mathbb{R}V_K^{[S,r]}$ is an isomorphism.

Proof If $s > r+1$ then (11) shows that $\mathbb{R}S_K^{[S,r]} = \mathbb{R}V_K^{[S,r]}$ on which π_V is the identity, proving (i). For (ii), $1\otimes\varepsilon \in \mathbb{Q}U_S(K)^{[S,r]}$ implies $\mathcal{L}_S(1\otimes\varepsilon)$ lies in $\mathbb{R}S_K^{[S,r]} \subset \mathbb{R}V_K$ by the above. Hence $\varepsilon \in U_V(K)$. For (iii), Prop. 3.2 implies $\mathbb{R}U_S(K)^{[S,r]} \cong (\mathbb{R}G^{[S,r]})^r \cong \mathbb{R}V_K^{[S,r]}$ by the splitting assumption. So it suffices to prove that $\mathcal{L}_{S,V}^{[S,r]}$ is injective. But $\mathcal{L}_S$, hence $\mathcal{L}_S^{[S,r]}$, is injective and $\mathrm{im}(\mathcal{L}_S^{[S,r]}) \cap \ker \pi_V^{[S,r]} = (\mathbb{R}S_K)_0^{[S,r]} \cap \mathbb{R}{V'}_K^{[S,r]} = \{0\}$, by (11). □

Taking exterior powers in (iii), we find that $\bigwedge_{\mathbb{R}G}^r \mathcal{L}_{S,V}^{[S,r]}$ maps $\bigwedge_{\mathbb{R}G}^r(\mathbb{R}U_S(K)^{[S,r]}) = (\bigwedge_{\mathbb{R}G}^r \mathbb{R}U_S(K))^{[S,r]}$ isomorphically onto $\bigwedge_{\mathbb{R}G}^r(\mathbb{R}V_K^{[S,r]}) = (\bigwedge_{\mathbb{R}G}^r \mathbb{R}V_K)^{[S,r]} = \mathbb{R}G^{[S,r]}(w_1\wedge\ldots\wedge w_r)$ and so

Corollary 3.3 *$R_{K/k,W}$ restricts to an isomorphism $\bigwedge_{\mathbb{R}G}^r \mathbb{R}U_S(K)^{[S,r]} \to \mathbb{R}G^{[S,r]}$*

Consequently:

Corollary 3.4 *There exists at most one canonical solution of $C(K/k,S,r)$.*

Remark: To summarise: If η is any solution of $C(K/k,S,r)$, then $e_{S,r}\eta$ is the *unique* canonical solution.

It is a remarkable (?) fact that in the cases where $C(K/k,S,r)$ is proven, the 'naturally occurring' solution is also usually the canonical solution.

Remark: Since $\mathbb{Q}G$ and $\mathbb{Q}G^{[S,r]}$ are products of fields, it follows from Prop. 3.2 that every element $\eta \in (\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S(K))^{[S,r]} = \bigwedge_{\mathbb{Q}G^{[S,r]}}^r(\mathbb{Q}U_S(K)^{[S,r]})$ can be written as $x_1\wedge\ldots\wedge x_r$, with $x_i \in \mathbb{Q}U_S(K)^{[S,r]}\ \forall i$. We can write x_i as $\frac{1}{n_i}\otimes\varepsilon_i$ with $n_i\in\mathbb{Z}_{\geq 1}$ and $1\otimes\varepsilon_i \in \mathbb{Q}U_S(K)^{[S,r]}$ (which implies $\varepsilon_i \in U_V(K)$ if $s>r+1$, by Prop. 3.4 (ii)). In any case, putting $n = n_1\ldots n_r \in \mathbb{Z}_{\geq 1}$, we have

$$\eta = \frac{1}{n}(1\otimes\varepsilon_1)\wedge\ldots(1\otimes\varepsilon_r)$$

Of course, this expression is not unique (even though η is). Nevertheless, the condition (9) that η be a canonical solution of $C(K/k,S,r)$ can now be written explicitly as follows

$$\Theta_{K/k,S}^{(r)}(0) = \frac{1}{n}\det(\lambda_{w_i}(1\otimes\varepsilon_j))_{i,j=1}^r = \frac{1}{n}\det\left(\sum_{g\in G}\log|g\varepsilon_j|_{w_i}g^{-1}\right)_{i,j=1}^r \tag{12}$$

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