CORRIGENDUM TO THE PAPER "A BRUMER-STARK CONJECTURE FOR NON-ABELIAN GALOIS EXTENSIONS"

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Takenori Kataoka has found an error in the proof of Lemma 4.4 in the paper [1] "A Brumer-Stark conjecture for non-abelian Galois extensions". The proof assumes that every element of the commutator group $[\Gamma, \Gamma]$ of a finite group Γ is a commutator which is not true in general. Furthermore, he provided us with an explicit example contradicting the statement of the lemma. In order to fix the error, we change the definition of strong central extensions. With the new definition, the statement and the proof of Lemma 4.4 are now correct. However, significant changes must be made to Section 4 of the paper. We state below a corrected version of this section (note that we keep the same numbering to be consistent with the rest of the paper). With the new definition and the changes to Section 4, the rest of the paper is correct as it is with the exception of Proposition 6.9 which should be deleted.

Section 4 of the paper should be replaced by the following text.

4. Strong central extensions

Before we state our generalization of the abelian Brumer-Stark conjecture for Galois extensions, we introduce the notion of strong central extensions that will play a crucial role. For that, we stop assuming for a moment that G is the Galois group of the extension K/k and just consider G as a finite group. Let Γ and Δ be two other finite groups with Δ a normal subgroup of Γ. Assume that the following sequence is exact

$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1, \tag{1}$$

that is, Γ is a group extension of G by Δ. Recall that the extension is said to be central if Δ is a subgroup of the center of Γ. This implies, in particular, that Δ is an abelian group. If, furthermore, the sequence (1) is split, that is there exists an homomorphism $s : G \to \Gamma$ such that $s \circ \pi$ is the identity, then the extension is trivial, that is $\Gamma \simeq \Delta \times G$.

We say that Γ is a strong central extension of G by Δ if Δ contains no non-trivial commutator of Γ, that is, for all $\gamma_1, \gamma_2 \in \Gamma$ with $[\gamma_1, \gamma_2] \in \Delta$, we have $[\gamma_1, \gamma_2] = 1$. The choice of terminology is explained by the following lemma.

Lemma 4.1. *Let Γ be a strong central extension of G by Δ. Then Γ is a central extension of G by Δ.*

Proof. Let $\gamma \in \Gamma$ and $\delta \in \Delta$. Observe that

$$[\gamma, \delta] = (\gamma\delta\gamma^{-1})\delta^{-1} \in \Delta$$

since Δ is normal in Γ. Thus, $[\gamma, \delta] = 1$ and γ and δ commute. Therefore Δ is in the center of Γ and the extension is central. □

The trivial extension $\Delta \times G$ is always a strong central extension. As noted above, a strong central extension is trivial if and only if the exact sequence (1) is split. By the Schur-Zassenhaus theorem, this is the case when the orders of Δ and G are relatively prime. We give an example of a non-trivial strong central extension.

Example. Let Γ be the dicyclic group of order 12. It is the group generated by the two elements a and b with the following relations: $a^3 = b^4 = 1$ and $bab^{-1} = a^{-1}$. Let $\Delta := \langle b^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$; it is the center of Γ and one can verify that $\Gamma/\Delta \simeq S_3$, the symmetric group on 3 letters. We compute that the commutators are 1, a and a^2, and thus we have the strong central extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma \longrightarrow S_3 \longrightarrow 1.$$

However, the group Γ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_3$ since the latter group does not have any element of order 4.

The following lemma provides us with a very useful characterization of strong central extensions.

Lemma 4.4. *Consider the group extension* (1). *This extension is strong central if and only if, for every abelian subgroup H of G, the subgroup $\pi^{-1}(H)$ of Γ is abelian.*

Proof. Assume that the extension is strong central. Let H be an abelian subgroup of G. Let $\gamma_1, \gamma_2 \in \pi^{-1}(H)$, say $\pi(\gamma_1) = h_1$, $\pi(\gamma_2) = h_2$ with $h_1, h_2 \in H$. We compute

$$\pi([\gamma_1, \gamma_2]) = [h_1, h_2] = 1.$$

By hypothesis, this implies that $[\gamma_1, \gamma_2] = 1$ and therefore $\pi^{-1}(H)$ is abelian.

Reciprocally, we assume that, for any abelian subgroup H of G, the group $\pi^{-1}(H)$ is abelian. Let $\gamma_1, \gamma_2 \in \Gamma$ be such that $[\gamma_1, \gamma_2] \in \Delta$. Then $\pi([\gamma_1, \gamma_2]) = 1$ and $\pi(\gamma_1)$ and $\pi(\gamma_2)$ commute. The subgroup of G that they generate is abelian and, by hypothesis, it follows that γ_1 and γ_2 commute, that is $[\gamma_1, \gamma_2] = 1$. Therefore the extension Γ of G by Δ is strong central. □

We give another property of strong central extensions. For a finite group A, recall that m_A denote the lcm of the cardinalities of the conjugacy classes of A.

Lemma 4.5. *Let Γ be a strong central extension of G by Δ. Then we have $m_\Gamma = m_G$.*

Proof. Let $\gamma \in \Gamma$. Denote by C and Z respectively the conjugacy class and the centralizer of γ in Γ. We have

$$\begin{aligned} |C| &= (\Gamma : Z) = (\pi(\Gamma) : \pi(Z))(\mathrm{Ker}(\pi) : \mathrm{Ker}(\pi) \cap Z) = (G : \pi(Z))(\Delta : \Delta \cap Z) \\ &= (G : Z_0)(Z_0 : \pi(Z))(\Delta : \Delta \cap Z) = |C_0|(Z_0 : \pi(Z))(\Delta : \Delta \cap Z) \end{aligned}$$

where C_0 and Z_0 are respectively the conjugacy class and the centralizer of $\pi(\gamma)$ in G. Since Δ is in the center of Γ by Lemma 4.1, we have $\Delta \subset Z$ and $(\Delta : \Delta \cap Z) = 1$. Now, let $\rho_0 \in Z_0$ and let $\rho \in \pi^{-1}(\rho_0)$. We have $\pi([\rho, \gamma]) = [\rho_0, \pi(\gamma)] = 1$ and, therefore, $[\rho, \gamma] = 1$ and $\rho \in Z$. Thus, $\pi(Z) = Z_0$ and we have $|C| = |C_0|$. Since any conjugacy class of G is the image by π of a conjugacy class of Γ, we see that $m_\Gamma = m_G$ and the result is proved. □

We now come back to our previous setting and assume that G is the Galois group of an extension of number fields K/k. Let L be a finite extension of K. We say that L is a strong central extension of K/k if L/k is Galois and the group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is strong central where $\Delta := \mathrm{Gal}(L/K)$ and $\Gamma := \mathrm{Gal}(L/k)$.

Lemma 4.6. *Let F be an abelian extension of k. Then, KF is a strong central extension of K/k.*

Proof. It is clear that KF/k is a Galois extension. Let $L := KF$, $A := \mathrm{Gal}(L/F)$, $\Gamma := \mathrm{Gal}(L/k)$ and $\Delta := \mathrm{Gal}(L/K)$. Since F/k is abelian, we have $[\Gamma, \Gamma] \subset A$. On the other hand, since $L = KF$, we have $\Delta \cap A = \{1\}$ and thus the result is proved. □

We conclude this section with a lemma that shows that strong central extensions behave somewhat nicely.

Lemma 4.7. *Let L be a strong central extension of K/k.*

(1) Let L_0/K be a sub-extension of L/K. Then L_0 is a strong central extension of K/k.
(2) Let M be another strong central extension of K/k. Then LM is a strong central extension of K/k.

Proof. We prove *(1)*. We denote as above $\Gamma := \mathrm{Gal}(L/k)$, $G := \mathrm{Gal}(K/k)$ and $\Delta := \mathrm{Gal}(L/K)$. Furthermore, let $\Gamma_0 := \mathrm{Gal}(L_0/k)$, $\Delta_0 := \mathrm{Gal}(L_0/K)$ and $A := \mathrm{Gal}(L/L_0)$. Since $A \subset \Delta$, A is normal in Γ and therefore L_0/k is Galois. Denote by $s : \Gamma \to \Gamma_0$ the canonical surjection induced by the restriction to L_0, thus $s(\Delta) = \Delta_0$. Assume that $\sigma_1, \sigma_2 \in \Gamma_0$ are such that $[\sigma_1, \sigma_2] \in \Delta_0$. Let $\gamma_1, \gamma_2 \in \Gamma$ with $s(\gamma_1) = \sigma_1$ and $s(\gamma_2) = \sigma_2$. Then, $[\gamma_1, \gamma_2] \in s^{-1}(\Delta_0) = \Delta$, thus $[\gamma_1, \gamma_2] = 1$ and $[\sigma_1, \sigma_2] = 1$. We have proved that L_0 is a strong central extension of K/k.

We now prove *(2)*. We use the following notations: $G := \mathrm{Gal}(K/k)$, $\Gamma_1 := \mathrm{Gal}(L/k)$, $\Gamma_2 := \mathrm{Gal}(M/k)$, $\Delta_1 := \mathrm{Gal}(L/K)$, $\Delta_2 := \mathrm{Gal}(M/K)$, $F := LM$, $\Gamma := \mathrm{Gal}(F/k)$ and $\Delta := \mathrm{Gal}(F/K)$. Denote by $s_1 : \Gamma \to \Gamma_1$ and $s_2 : \Gamma \to \Gamma_2$ the canonical surjections induced by the restrictions to L and M respectively. Let $\gamma, \rho \in \Gamma$ be such that $[\gamma, \rho] \in \Delta$. Then, $s_1([\gamma, \rho]) = [s_1(\gamma), s_1(\rho)] \in s_1(\Delta) = \Delta_1$, thus $s_1([\gamma, \rho]) = 1$. In the same way, we find that $s_2([\gamma, \rho]) = 1$. Since $F = LM$, we have $\mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2) = \{1\}$, thus $[\gamma, \rho] = 1$ and F is a strong central extension of K/k. □

As mentioned in the introduction, the change of definition for strong central extensions does not affect other parts of the paper except for Proposition 6.9. With the modified definition, we were not able to prove a version of Proposition 6.9 with the same generality as the original version. Therefore, Proposition 6.9, and Lemma 6.10 which does not serve any more purpose, should be deleted.

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References

[1] G. Dejou, X.-F. Roblot, A Brumer-Stark conjecture for non-abelian Galois extensions, J. Number Theory 142 (2014) 51–88.
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