Computing values of p-adic L-functions of real quadratic fields

X.-F. Roblot

Institut Camille Jordan
Université Claude Bernard – Lyon 1

- p-adic zeta function of number fields first constructed by Serre using Hilbert modular forms (1972).
- Construction of p-adic L-functions by Deligne-Ribet using algebraic geometry (1980).
- Construction of p-adic L-functions by Cassou-Noguès (1979) and Barsky (1978) using Shintani's methods, reformulated in terms of p-adic measures by Katz (1981). Also nice constructions of Colmez (1988) and Barsky (2004).
- Method generalizes previous work (2004) with D. Solomon (values at $s = 1$, conductor relatively prime with p, p split in quadratic field).
- Method (almost) works for higher degree totally real number fields.
- Method used in joint work with A. Besser, P. Buckingham and R. de Jeu (and a current work with A. Weiss).

Hecke L-function. Let χ be a character of the ray class group $\mathrm{Cl}_{\mathfrak{f}}(E)$ modulo $\mathfrak{f}$ of a real quadratic field E. We define for $\Re e(s) > 1$

$$L(s, \chi) := \prod_{\mathfrak{q} \nmid \mathfrak{f}} \left(1 - \chi(\mathfrak{q}) \mathcal{N}\mathfrak{q}^{-s}\right)^{-1}$$

This function has an analytic continuation to $\mathbb{C}$ and its values at integers have *special arithmetical meaning*.

Problem. Construct a continuous function on $\mathbb{Z}_p$ interpolating the values at negative integers of this function.

For simplicity, assume p is odd and $p \mid \mathfrak{f}$.

And, for technical reason, assume the infinite part $\mathfrak{f}_\infty$ contains the two infinite places of E.

Partial zeta functions. Fix a class $\mathcal{C}$ in $\mathsf{Cl}_{\mathfrak{f}}(E)$, and define

$$\zeta(s,\mathcal{C}) := \sum_{\substack{\mathfrak{a}\in\mathcal{C}\\ \mathfrak{a}\subset\mathbb{Z}_E}} \mathcal{N}\mathfrak{a}^{-s}$$

Then

$$L(s,\chi) = \sum_{\mathcal{C}\in\mathsf{Cl}_{\mathfrak{f}}(E)} \chi(\mathcal{C})\zeta(s,\mathcal{C})$$

and (Klingen-Siegel)

$$\zeta(-k,\mathcal{C}) \in \mathbb{Q} \quad \text{for all } k \geq 0.$$

So better to interpolate the partial zeta functions!

A continous p-adic function $f : \mathbb{Z}_p^2 \to \mathbb{C}_p$ has a unique Mahler expansion

$$f(x_1, x_2) = \sum_{n_1, n_2 \geq 0} a_{n_1,n_2} \binom{x_1}{n_1} \binom{x_2}{n_2}$$

with $a_{n_1,n_2} \to_p 0$ as $n_1 + n_2 \to \infty$ and $\displaystyle\binom{x}{n} := \frac{x(x-1)\cdots(x-(n-1))}{n!}$.

A measure on $\mathbb{Z}_p^2$ is a linear form μ on the $\mathbb{C}_p$-vector space $\mathcal{C}(\mathbb{Z}_p^2, \mathbb{C}_p)$ of continuous functions such that there exists a constant $C > 0$ with

$$\Big|\underbrace{\int f\, d\mu}_{\mu(f)}\Big| \leq C \underbrace{\max_{(x_1,x_2)\in\mathbb{Z}_p^2} |f(x_1,x_2)|}_{\|f\|} \quad \text{for all } f \in \mathcal{C}(\mathbb{Z}_p^2, \mathbb{C}_p)$$

We have

$$\int f\, d\mu = \sum_{n_1,n_2\geq 0} a_{n_1,n_2} \int \binom{x_1}{n_1}\binom{x_2}{n_2}\, d\mu$$

and so we can associate to μ a power series with bounded coefficients

$$F_\mu(T_1,T_2) := \sum_{n_1,n_2\geq 0} \int \binom{x_1}{n_1}\binom{x_2}{n_2}\, d\mu \cdot T_1^{n_1} T_2^{n_2} = \int (1+T_1)^{x_1}(1+T_2)^{x_2}\, d\mu.$$

In the same way, we can associate a measure μ_F to such a power series F.

The Δ operator is $\Delta := (1+T_1)(1+T_2)\dfrac{\partial^2}{\partial T_1 \partial T_2}$ and

$$\Delta F_\mu(T_1,T_2) = \int x_1 x_2 (1+T_1)^{x_1}(1+T_2)^{x_2}\, d\mu$$

and therefore for all $k \geq 0$

$$\int (x_1 x_2)^k\, d\mu = \left[\Delta^k F_\mu(T_1,T_2)\right]_{T_1=T_2=0}$$

Method of interpolation. Given a sequence $(a_k)_{k\geq 0}$ of rational numbers. Find a power series F with bounded coefficients such that for all $k \geq 0$

$$a_k = \left[\Delta^k F_\mu(T_1, T_2)\right]_{T_1=T_2=0} = \int (x_1 x_2)^k \, d\mu_F$$

For $s \in \mathbb{Z}_p$, let ψ_s be a continuous p-adic function such that for $k \geq 0$

$$\psi_k(x) = x^k$$

then we can replace $(x_1 x_2)^k$ by $\psi_s(x_1 x_2)$ and get a p-adic function.

Unfortunately, one can only find a function $\psi_{s,m}$ such that $\psi_{k,m}(x) = x^k$ for $x \in \mathbb{Z}_p^\times$ and $k \geq 0$ is such that $k \equiv m \pmod{\phi(p)}$.

Theorem.

$$\mathcal{F}_m(s) := \int \psi_{s,m}(x_1 x_2) \, d\mu$$

is a continuous function of $s \in \mathbb{Z}_p$. Furthermore, if $\mathrm{Supp}(\mu) \subset (\mathbb{Z}_p^\times)^2$, then

$$\mathcal{F}_m(k) = a_k, \quad \forall k \geq 0 \text{ with } k \equiv m \pmod{\phi(p)}$$

Remove the pole. Let $\mathfrak{c} \neq \mathcal{O}_E$ be an integral ideal relatively prime to $\mathfrak{f}$, then

$$\zeta(s, \mathfrak{c}, \mathcal{C}) := \mathcal{N}\mathfrak{c}^{1-s}\zeta(s, \mathfrak{c}^{-1} \cdot \mathcal{C}) - \zeta(s, \mathcal{C})$$

has no pole at $s = 1$ and still takes values in $\mathbb{Q}$ at negative integers.

Furthermore

$$L(s, \chi) = \left(\mathcal{N}\mathfrak{c}^{1-s}\,\chi(\mathfrak{c}) - 1\right)^{-1} \sum_{\mathcal{C} \in \mathrm{Cl}_{\mathfrak{f}}(E)} \bar{\chi}(\mathcal{C})\,\zeta(s, \mathfrak{c}, \mathcal{C})$$

So we need to take $\mathfrak{c}$ such that $\chi(\mathfrak{c}) \neq 1$.

Switch to elements. Let $\mathcal{C} = [\mathfrak{a}^{-1}]$ ($\mathfrak{a}$ integral ideal). Then integral ideals $\mathfrak{b} = \alpha\mathfrak{a}^{-1}$ in $\mathcal{C}$ are in bijection with α such that $\alpha \in \mathfrak{a}$, $\alpha \equiv 1 \pmod{\mathfrak{f}_0}$, $\alpha \gg 0$ modulo the multiplicative action of $U_{\mathfrak{f}}(E)$, the group of units u such $u \gg 0$ and $u \equiv 1 \pmod{\mathfrak{f}_0}$. Call $R(\mathfrak{a})$ a set of representatives. So

$$\zeta(s, \mathcal{C}) = \sum_{\alpha \in R(\mathfrak{a})} \mathcal{N}(\alpha\mathfrak{a}^{-1})^{-s} = \mathcal{N}\mathfrak{a}^{s} \sum_{\alpha \in R(\mathfrak{a})} \mathcal{N}(\alpha)^{-s}$$

Recall that. $\zeta(s,\mathcal{C}) = \mathcal{N}\mathfrak{a}^s \sum_{\alpha \in R(\mathfrak{a})} \mathcal{N}(\alpha)^{-s}$ with $\mathcal{C} = [\mathfrak{a}^{-1}]$

$$\text{so } \zeta(s,\mathfrak{c},\mathcal{C}) = \mathcal{N}\mathfrak{c}^{\,1-s}\,\zeta(s,\mathfrak{c}^{-1}\cdot\mathcal{C}) - \zeta(s,\mathcal{C})$$

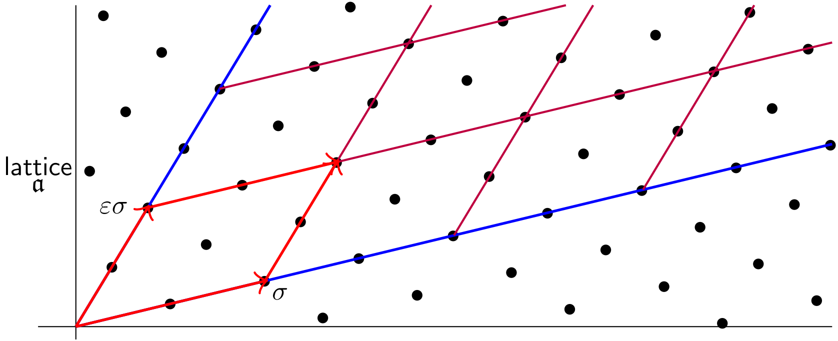
$$= \mathcal{N}\mathfrak{a}^s\Big(\mathcal{N}\mathfrak{c} \sum_{\alpha \in R(\mathfrak{a})\cap\mathfrak{c}} \mathcal{N}(\alpha)^{-s} - \sum_{\alpha \in R(\mathfrak{a})} \mathcal{N}(\alpha)^{-s}\Big)$$

Let $\mathcal{X}$ be the set of additive characters of $\mathcal{O}_E$ of annihilator $\mathfrak{c}$ then

$$\sum_{\xi \in \mathcal{X}} \xi(\theta) = \begin{cases} 0 & \text{if } \theta \notin \mathfrak{c} \\ \mathcal{N}\mathfrak{c} & \text{otherwise} \end{cases}$$

and

$$\zeta(s,\mathfrak{c},\mathcal{C}) = \mathcal{N}\mathfrak{a}^s \sum_{\substack{\xi \in \mathcal{X} \\ \xi \neq 1}} \underbrace{\sum_{\alpha \in R(\mathfrak{a})} \xi(\alpha)\mathcal{N}(\alpha)^{-s}}_{=:\ \zeta(s,\mathfrak{a},\xi)}$$



Let $\sigma \in \mathfrak{a}\mathfrak{f}_0$, $\sigma \notin \mathfrak{c}$ and $\sigma \gg 0$ and let ε a generator of $U_{\mathfrak{f}}(E)$.

Take $R(\mathfrak{a}) := \mathfrak{a} \cap (1 + \mathfrak{f}_0) \cap C(\sigma, \varepsilon)$, where

$$C(\sigma, \varepsilon\sigma) := \{s\sigma + t\varepsilon\sigma \text{ with } 0 < s \text{ and } 0 \leq t\}$$

Let $P(\sigma, \varepsilon) := \{s\sigma + t\varepsilon\sigma \text{ with } 0 < s \leq 1 \text{ and } 0 \leq t < 1\}$ then

$$R(\mathfrak{a}) = \bigcup_{n,m \geq 0} \Big\{ \underbrace{\big(\mathfrak{a} \cap (1 + \mathfrak{f}_0) \cap P(\sigma, \varepsilon)\big)}_{=:\, P(\mathfrak{a}, \sigma, \varepsilon)} + n\sigma + m\varepsilon\sigma \Big\}$$

We define

$$F(T_1, T_2, \mathfrak{a}, \xi) := \frac{\displaystyle\sum_{\alpha \in P(\mathfrak{a},\sigma,\varepsilon)} \xi(\alpha)(1+\mathbf{T})^{\alpha}}{\big(1-\xi(\sigma)(1+\mathbf{T})^{\sigma}\big)\big(1-\xi(\varepsilon\sigma)(1+\mathbf{T})^{\varepsilon\sigma}\big)}$$

where, for $\beta \in \mathcal{O}_E$

$$(1+\mathbf{T})^{\beta} := (1+T_1)^{\beta^{(1)}}(1+T_2)^{\beta^{(2)}} = \sum_{n_1, n_2 \geq 0} \binom{\beta^{(1)}}{n_1}\binom{\beta^{(2)}}{n_2} T_1^{n_1} T_2^{n_2}$$

Theorem. For all $k \geq 0$

$$\left[\Delta^k F(T_1, T_2, \mathfrak{a})\right]_{T_1 = T_2 = 0} = \zeta(-k, \mathfrak{a}, \xi)$$

Heuristic proof. Expand everything in terms of $(1+T_1)$ and $(1+T_2)$, apply Δ^k and take $T_1 = T_2 = 0$. We get

$$\text{《} \sum_{\substack{n,m \geq 0 \\ \alpha \in P(\mathfrak{a},\sigma,\varepsilon)}} \xi(\alpha + n\sigma + m\varepsilon\sigma)\mathcal{N}(\alpha + n\sigma + m\varepsilon\sigma)^k = \zeta(-k, \mathfrak{a}, \xi) \text{》}$$

Problem. When p is not split, the power series $(1+\mathbf{T})^{\beta}$ may have unbounded coefficients!

Change of variables. Let $\gamma \in \mathcal{O}_E$ such that $\mathcal{O}_E = \mathbb{Z} + \gamma\mathbb{Z}$. Define the operator $\mathcal{A}$ by

$$\mathcal{A}(T_1) = (1+T_1)(1+T_2) - 1 \text{ and } \mathcal{A}(T_2) = (1+T_1)^{\gamma^{(1)}}(1+T_2)^{\gamma^{(2)}} - 1$$

Then, for $\alpha = a + b\gamma \in \mathcal{O}_E^+$, we have

$$\mathcal{A}\big((1+T_1)^a(1+T_2)^b\big) = (1+T_1)^{a+b\gamma^{(1)}}(1+T_2)^{a+b\gamma^{(2)}} = (1+\mathbf{T})^{\alpha}$$

So $G(T_1, T_2, \mathfrak{a}, \xi) = \mathcal{A}^{-1}(F(T_1, T_2, \mathfrak{a}, \xi))$ has coefficients in $\mathbb{Z}[\xi]$.

Theorem. Let $\mu_{\mathfrak{a},\xi}$ the measure on $\mathbb{Z}_p^2$ associated to $G(T_1, T_2, \mathfrak{a}, \xi)$. Then for all $k \geq 0$

$$\zeta(-k, \mathfrak{a}, \xi) = \int \mathcal{N}(x_1 + x_2\gamma)^k \, d\mu_{\mathfrak{a},\xi}.$$

Interpolation. Write $G(T_1, T_2, \mathfrak{a}, \xi) = \sum_{n_1, n_2 \geq 0} g(\mathfrak{a}, \xi)_{n_1, n_2} T^{n_1} T^{n_2}$ and

$$(x_1, x_2) \mapsto \psi_{s,m}(\mathcal{N}(x_1 + x_2\gamma)) = \sum_{n_1, n_2 \geq 0} c(s, m)_{n_1, n_2} \binom{x_1}{n_1} \binom{x_2}{n_2}.$$

Define for all $s \in \mathbb{Z}_p$

$$\zeta_p^{(m)}(s, \mathfrak{a}, \xi) = \int \psi_{s,m}(\mathcal{N}(x_1 + x_2\gamma)) \, d\mu_{\mathfrak{a},\xi}$$
$$= \sum_{n_1, n_2 \geq 0} g(\mathfrak{a}, \xi)_{n_1, n_2} \, c(s, m)_{n_1, n_2}$$

Then $\zeta_p^{(m)}(s, \mathfrak{a}, \xi)$ is a continuous function on $\mathbb{Z}_p$ interpolating $\zeta(s, \mathfrak{a}, \xi)$ at negative integers k congruent to m modulo $\phi(p)$.

The natural choice is $m = -1$ for which the corresponding p-adic zeta function has a simple pole at $s = 1$.

Construction of $\psi_{s,m}(x)$. $\mathbb{Z}_p^\times$ has the natural decomposition

$$\begin{array}{ccc} \mathbb{Z}_p^\times & = & W_p \times (1 + p\mathbb{Z}_p) \\ x & \mapsto & \omega(x) \cdot \langle x \rangle \end{array}$$

so that $x \equiv \omega(x) \pmod{p\mathbb{Z}_p}$ and $\langle x \rangle \in 1 + p\mathbb{Z}_p$.

Power of principal units. For $s \in \mathbb{Z}_p$ and $\langle x \rangle = 1 + py$, we have

$$\langle x \rangle^s = \sum_{n \geq 0} \binom{s}{n} p^n y^n \in 1 + p\mathbb{Z}_p$$

Therefore the function

$$\psi_{s,m}(x) = \begin{cases} \omega(x)^m \langle x \rangle^s & \text{if } x \in \mathbb{Z}_p^\times \\ 0 & \text{if } x \in p\mathbb{Z}_p \end{cases}$$

interpolates x^k on $\mathbb{Z}_p^\times$ for $k \geq 0$, $k \equiv m \pmod{\phi(p)}$.

Computation of the measures. Assume $p \neq 2$, then it takes

$$\tilde{O}(f\, R_E\, M^6\, p^4\, c^2) \text{ operations} \quad \text{and} \quad \tilde{O}(M^2\, p^2) \text{ memory}$$

to compute the measure $\mu_{\mathfrak{a},\xi}$ to a precision p^M with $c = \mathcal{N}\mathfrak{c}$, $f\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$ and R_E the regulator of E.

Computation of values. Once the measure $\mu_{\mathfrak{a},\xi}$ has been computed, it takes

$$\tilde{O}(M^4\, p^3) \text{ operations}$$

to compute $\zeta_p(s, \mathfrak{a}, \xi)$, for some $s \in \mathbb{Z}_p$, to a precision of p^M.

It is possible to compute other expressions of the functions $\zeta_p(s, \mathfrak{a}, \xi)$, and thus of p-adic L-functions, using this method.

Mahler expansion. One can also compute the coefficients a_n of

$$\zeta_p(s, \mathfrak{a}, \xi) = \sum_{n \geq 0} a_n \binom{s}{n} \quad \text{with } a_n \in \mathbb{Q}_p$$

with

$$a_n = \int_{\mathcal{U}} \omega(\mathcal{N}(x_1 + x_2\gamma))^{-1} \left(\langle \mathcal{N}(x_1 + x_2\gamma) \rangle - 1\right)^n \, d\mu_{\mathfrak{a},\xi}$$

Once the measure $\mu_{\mathfrak{a},\xi}$ has been computed (to a precision p^M), it takes

$$\tilde{O}(N\, M^4\, p^3) \text{ operations}$$

to compute the first N coefficients a_n to a precision p^M $(N \leq M)$.

And then it takes only $\tilde{O}(M^3)$ operations to compute $\zeta_p(s, \mathfrak{a}, \xi)$ to a precision of p^M, for some $s \in \mathbb{Z}_p$.

Analytic function. One can also compute the coefficients c_n of

$$\zeta_p(s, \mathfrak{a}, \xi) = \sum_{n \geq 0} c_n\, s^n \quad \text{with } c_n \in \mathbb{Q}_p$$

with

$$c_n = \frac{1}{n!} \int_{\mathcal{U}} \omega(\mathcal{N}(x_1 + x_2\gamma))^{-1} \log_p \left(\langle \mathcal{N}(x_1 + x_2\gamma) \rangle\right)^n \, d\mu_{\mathfrak{a},\xi}$$

or in a simpler way

$$a_0 + a_1 \binom{X}{1} + a_2 \binom{X}{2} + \cdots = c_0 + c_1 X + c_2 X^2 + \cdots$$

Once the measure $\mu_{\mathfrak{a}}$ has been computed (to a precision p^M), it takes

$$\tilde{O}(N\, M^4\, p^3) \text{ operations}$$

to compute the first N coefficients c_n to a precision p^M $(N \leq M)$.

It is better not to use this expression to compute values of $\zeta_p(s, \mathfrak{a}, \xi)$.

Iwasawa function. Let u be a topologic generator of $1+p\mathbb{Z}_p$, then there exists

$$F_p(T,\mathfrak{a},\xi) = f_0 + f_1 T + f_2 T^2 + \cdots \in \mathbb{Q}_p[[T]]$$

with

$$\zeta_p(s,\mathfrak{a},\xi) = F_p(u^s - 1,\mathfrak{a},\xi)$$

We have

$$f_n = \int_{\mathcal{U}} \mathcal{N}(x_1 + x_2\gamma)^{-1} \binom{\log_p \langle \mathcal{N}(x_1 + x_2\gamma)\rangle / \log_p u}{n} \, d\mu_{\mathfrak{a},\xi}$$

But it is not really clear how much it costs to compute the f_n...

Compute values of $\zeta_{E,p}(5)$ for $E = \mathbb{Q}(\sqrt{3})$ and $p = 3$, 11 and 23

```
gp > data = init_data(12, 3, 10);
time = 4 ms.
gp > twz = init_twistzeta(data);
time = 604 ms.
gp > zetap_E(data, 5, twz)
time = 28 ms.
%3 = 2*3^-1 + 1 + 3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + O(3^9)
gp > data = init_data(12, 11, 7);
time = 0 ms.
gp > twz = init_twistzeta(data);
time = 3mn, 29,221 ms.
gp > zetap_E(data, 5, twz)
time = 1,417 ms.
%6 = 4*11^-1 + 3*11 + 7*11^2 + 9*11^3 + 10*11^4 + 4*11^5 + O(11^6)
gp > data = init_data(12, 23, 5);
time = 4 ms.
gp > twz = init_twistzeta(data);
time = 18mn, 45,670 ms.
gp > zetap_E(data, 5, twz)
time = 6,441 ms.
%9 = 17*23^-1 + 21 + 4*23 + 19*23^2 + 7*23^3 + O(23^4)
```

Write the Mahler expansion $\psi_{s,m}(x) = \sum_{n \geq 0} z_n \binom{x}{n}$ with $z_n \to_p 0$.

Problem. We need to estimate $v_p(z_n)$.

Locally analytic functions. A $\mathbb{Z}_p$-continuous function f is analytic of order $h \geq 0$ if for all $a \in \mathbb{Z}_p$

$$f(x) = f_{a,0} + f_{a,1}(x-a) + f_{a,2}(x-a)^2 + \cdots \quad \text{for } |x-a| \leq p^{-h}$$

Theorem. Let $f(x) = \sum_{n \geq 0} a_n \binom{x}{n}$ then $v_p(a_n) \geq v_p\big(\lfloor n/p^h \rfloor !\big) + C(f)$

Application. Let $a \in \mathbb{Z}_p^\times$. For $x \in a + p\mathbb{Z}_p$

$$\langle x \rangle^s = \langle a \rangle \sum_{n \geq 0} \binom{s}{n} \left(\frac{x-a}{a}\right)^n$$

so $\psi_{s,m}$ is analytic of order 1 and $v_p(z_n) \gtrsim n/p^2$.

But, we can do better easily.

Close functions. Let f and g be two continuous functions such that

$$v_p\big(f(x) - g(x)\big) \geq M \quad \text{for all } x \in \mathbb{Z}_p$$

Then $v_p(a_n - b_n) \geq M$ for all $n \geq 0$ where $g(x) = \sum_{n \geq 0} b_n \binom{x}{n}$.

Application. Let $M > 0$ and let $t \in \mathbb{Z}_{\geq 0}$ such that

$$t \equiv s \pmod{p^M}, \quad t \equiv m \pmod{p-1} \quad \text{and} \quad t > M$$

Then for all $x \in \mathbb{Z}_p$, we have

$$v_p(\psi_{s,m}(x) - x^t) \geq M$$

Therefore $v_p(z_n) \gtrsim n/p$.