# Vortex-like finite-energy asymptotic profiles for isentropic compressible flows

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**Abstract**

Bidimensional incompressible viscous flows with well-localised vorticity are well-known to develop vortex structures. The purpose of the present paper is to recover the asymptotic profiles describing these phenomena for homogeneous finite-energy flows as asymptotic profiles for near-equilibrium isentropic compressible flows. This task is performed by extending the sharp description of the asymptotic behaviour of near-equilibrium compressible flows obtained by David Hoff and Kevin Zumbrun [8] to the case of finite-energy *vortex-like* solutions.

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## Introduction

The present paper is focused on the long-time asymptotic behaviour of viscous bidimensional flows. When no exterior force is applied the flow is expected to return to equilibrium, namely to a state of constant density and zero velocity. Our purpose is thus to determine asymptotic profiles for the return to equilibrium.

The motion of the considered flows may be described by the time evolution of the pair $(\rho, u)$, $\rho = \rho(t, x) > 0$ being the density field of the fluid and $u = u(t, x) \in \mathbf{R}^2$ the velocity field. The main purpose of the paper is to prove that for some initial data near equilibrium one recovers for isentropic compressible flows the same asymptotic profiles as in the constant-density case. Therefore let us begin introducing the constant-density profiles we are interested in.

When the density is constant, $\rho \equiv \rho_\star$, mass conservation and a force balance for Newtonian fluids lead to the Navier-Stokes evolution equations

$$\left.\begin{array}{rcl} \operatorname{div} u &=& 0 \\ \partial_t\,(\rho_\star\, u) \;+\; (u\,\cdot\nabla)\,(\rho_\star\, u) &=& \mu \triangle u \;-\; \nabla p \end{array}\right\} \tag{1}$$

where $\mu > 0$ is the shear Lamé viscosity coefficient and $p = p\,(t, x) \in \mathbf{R}$ is the pressure field of the fluid. In order to make the former equations compatible the pressure must be determined (up to a constant) by the elliptic equation

$$\triangle p \;\; = \;\; -\,\rho_\star\, \operatorname{div}\big(\,(u\,\cdot\nabla)\; u\big)\;. \tag{2}$$

In this bidimensional incompressible context, it may seem more natural and it is often more convenient to work with the curl of the velocity rather than with the velocity itself. The evolution of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ obeys

$$\partial_t\,\omega \;+\; u\,\cdot\,\nabla\,\omega \;\; = \;\; \nu \triangle \omega \tag{3}$$

where $\nu = \mu/\rho_\star$ is the kinematic viscosity and the velocity $u$ is recovered by the Biot-Savart law,

$$u(x) \;=\; \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega\,(y)\,dy \quad , \quad x \in \mathbf{R}^2\;, \tag{4}$$

with $(z_1, z_2)^\perp = (-z_2, z_1)$, which we also denote $u = K_{BS} \star \omega$, $K_{BS}$ being called the Biot-Savart kernel. Note that in terms of Fourier transforms the Biot-Savart law becomes

$$\widehat{u}\,(\eta) \;\; = \;\; \frac{\mathrm{i}\;\eta^\perp}{|\eta|^2}\;\widehat{\omega}\,(\eta)\;, \qquad \eta \in \mathbf{R}^2\;. \tag{5}$$

Concerning the widely-developed literature about the (homogeneous) Navier-Stokes equations, the reader is referred to some advanced entering gates such as the following books [2], [12], [13], [17], and to the more vorticity-focused review [1].

Flows with constant density and initially well-localised vorticity are well-known to develop vortex-like structures. In the compressible case we shall recover near equilibrium this kind of behaviour.

For instance it is proved in [7] that any solution $\omega$ of (3) with integrable initial datum $\omega_0$ satisfies in Lebesgue spaces

$$\lim_{t\to\infty}\; t^{1-\frac{1}{p}}\;\|\,\omega(t) - \alpha\,\omega^G(t)\,\|_p \;\; = \;\; 0\;, \tag{6}$$

$$\lim_{t\to\infty}\; t^{\frac{1}{2}-\frac{1}{q}}\;\|\,u(t) - \alpha\,u^G(t)\,\|_q \;\; = \;\; 0\;, \tag{7}$$

for any $1 \le p \le \infty$ and any $2 < q \le \infty$, where

$$\omega^G(t,x) \;=\; \frac{1}{t}\;G\!\left(\frac{x}{\sqrt{\nu\,t}}\right)\,, \quad u^G(t,x) \;=\; \sqrt{\frac{\nu}{t}}\;v^G\!\left(\frac{x}{\sqrt{\nu\,t}}\right) \tag{8}$$

with profiles

$$G(\xi) \;=\; \frac{1}{4\pi}\;e^{-|\xi|^2/4}\;, \qquad v^G(\xi) \;=\; \frac{1}{2\pi}\;\frac{\xi^\perp}{|\xi|^2}\;\big(1 - e^{-|\xi|^2/4}\big)\;, \tag{9}$$

and $\alpha \in \mathbf{R}$ is such that the initial velocity circulations at infinity coincide at the initial time $t = 0$,

$$\nu\,\alpha \quad = \quad \int_{\mathbf{R}^2} \omega_0\,(x)\;dx\;. \tag{10}$$

Actually, for any $\alpha \neq 0$, the vorticity $\alpha\,\omega^G$ is a (self-similar) solution of (3) with initial datum a Dirac mass — centred at the origin and of weight $\alpha/\nu$ ; the corresponding flow is called Oseen vortex. Thus when the circulation is non zero equalities (6) and (7) show that the flow behaves asymptotically as a single vortex, whereas when $\alpha = 0$ it returns to equilibrium faster than a single vortex does.

However finite energy flows have zero circulation. Indeed, as is easily derived from (5), to obtain an integrable vorticity and a square-integrable velocity one must ask for the vorticity to be of zero mean. To consider finite energy flows we must thus investigate profiles decaying faster. Yet it is well-known, see [6] (combined with [7, Proposition 1.5]) for instance, that if the initial vorticity $\omega_0$ is such that $(1 + |\cdot|)^{3/2}\,\omega_0$ is square-integrable and $\int_{\mathbf{R}^2} \omega_0 = 0$ then for any index $1 \leq p \leq \infty$

$$\lim_{t\to\infty}\; t^{\frac{3}{2}-\frac{1}{p}}\;\|\,\omega(t) - \omega^{\beta_1,\beta_2}(t)\,\|_p \quad = \quad 0 \tag{11}$$

where

$$\omega^{\beta_1,\beta_2}\,(t,x) \quad = \quad \beta_1\,\omega^{F_1}\,(t,x) \;+\; \beta_2\,\omega^{F_2}\,(t,x) \tag{12}$$

with for $i = 1, 2$

$$\omega^{F_i}\,(t,x) \quad = \quad \frac{1}{\sqrt{\nu}\,t^{3/2}}\;F_i\left(\frac{x}{\sqrt{\nu\,t}}\right) \tag{13}$$

$$F_i\,(\xi) \quad = \quad \partial_i\,G\,(\xi) \;\; = \;\; -\,\frac{\xi_i}{2}\;G\,(\xi) \tag{14}$$

and $\beta_i$ is such that

$$\nu\,\beta_i \quad = \quad -\int_{\mathbf{R}^2} x_i\;\omega_0(x)\,dx\;. \tag{15}$$

Observe that $\omega^{\beta_1,\beta_2}$ is not a solution of equation (3) but only of its linearisation around equilibrium, a heat equation. However equality (11) is easily seen to apply also to some flows with finite measures as initial vorticities, such as those of initial vorticity

$$\frac{1}{2\,\nu}\,\Big(\delta_{(-\beta_1,0)} \;-\; \delta_{(\beta_1,0)}\Big) \;\;+\;\; \frac{1}{2\,\nu}\,\Big(\delta_{(0,-\beta_2)} \;-\; \delta_{(0,\beta_2)}\Big)\;. \tag{16}$$

Moreover, defining the corresponding velocities

$$u^{\beta_1,\beta_2} \quad = \quad K_{BS} \star \omega^{\beta_1,\beta_2} \;\; = \;\; \beta_1\,u^{F_1} \;+\; \beta_2\,u^{F_2} \tag{17}$$

and for $i = 1, 2$

$$\begin{aligned} u^{F_i}\,(t,x) \quad &= \quad (K_{BS} \star \omega^{F_i}(t))\,(x) \;\; = \;\; \frac{1}{t}\;v^{F_i}\left(\frac{x}{\sqrt{\nu\,t}}\right) \\ v^{F_i}\,(\xi) \quad &= \quad K_{BS} \star F_i\,(\xi) \;\; = \;\; \partial_i\,v^G\,(\xi)\;, \end{aligned}$$

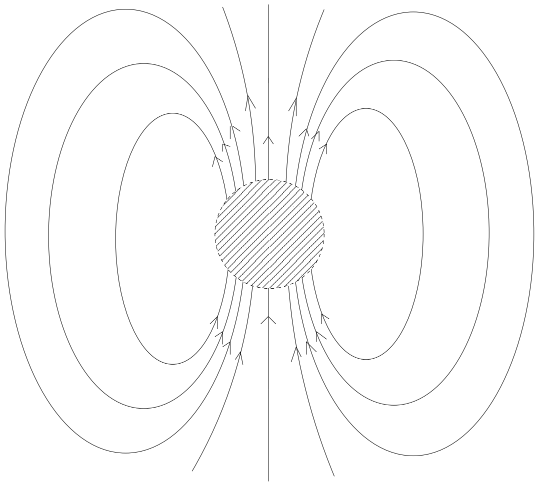

Figure 1: Shape at infinity of streamlines of $u^{F_1}$

one does observe a dipole-like feature at infinity,

$$v^{F_1}(\xi) \overset{|\xi|\to\infty}{=} \frac{1}{2\pi|\xi|^4} \begin{pmatrix} 2\,\xi_1\xi_2 \\ \xi_2^2-\xi_1^2 \end{pmatrix} + \mathcal{O}\,(e^{-|\xi|^2/4})\ ,$$

$$v^{F_2}(\xi) \overset{|\xi|\to\infty}{=} \frac{1}{2\pi|\xi|^4} \begin{pmatrix} \xi_2^2-\xi_1^2 \\ -2\,\xi_1\xi_2 \end{pmatrix} + \mathcal{O}\,(e^{-|\xi|^2/4})\ .$$

Therefore equality (11) does show that whenever $\alpha = 0$ and $(\beta_1, \beta_2) \neq (0, 0)$ the flow behaves asymptotically in time as would do one or two pairs of vortices. Nevertheless observe from (5) that, when the vorticity $\omega$ is such that $(1+|\cdot|)\,\omega$ is integrable (hence $\alpha$, $\beta_1$, $\beta_2$ defined) and the velocity $u$ is integrable, parameters $\alpha$, $\beta_1$ and $\beta_2$ must vanish and therefore the flow should return to equilibrium again faster. From now on our attention will be limited to these vortex-like finite-energy profiles and thus we must eschew assuming the velocity integrable.

For compressible flows mass conservation and a force balance for Newtonian fluids with constant Lamé coefficients give the following equations for the time evolution of the pair $(\rho, m)$, $\rho$ being the mass density field and $m = \rho\,u \in \mathbf{R}^2$ the momentum density field,

$$\left.\begin{array}{rcl} \partial_t\,\rho \ + \ \mathrm{div}\ m & = & 0 \\ \partial_t\,m \ + \ \mathrm{div}\,(m\otimes\frac{m}{\rho}) & = & \mu\,\triangle\,(\frac{m}{\rho}) + (\mu+\lambda)\,\nabla\,\mathrm{div}\,(\frac{m}{\rho}) - \nabla\,p \end{array}\right\} \qquad (18)$$

where $\mu$ and $\lambda$ are the shear and bulk Lamé viscosity coefficients, completed by a constitutive law for the pressure field $p$

$$p \ = \ P\,(\rho) \qquad (19)$$

obtained neglecting entropy variations. We require the pressure law $P$ to be a smooth increasing function, thus the pressure increases with density, and the Lamé coefficients to be such that the viscosity tensor is elliptic namely such that $\mu > 0$ and $\lambda + 2\mu > 0$, which is physically relevant. Besides we choose to formulate equations in terms of the momentum $m$ instead of the velocity $u$ in order to keep the conservation law structure of system (18). Of course since the density $\rho$ is expected to become asymptotically homogeneous we will also obtain profiles for the velocity $u$.

As equations (2) and (19) are seldom simultaneously satisfied, there is hardly any constant-density solution of system (18). Thereby we are not investigating stability of constant-density flows as compressible flows, but compatibility of asymptotic behaviours for initial data near equilibrium. To be somewhat more precise let us say that while considering viscosity coefficients $\mu$ and $\lambda$, pressure law $P$ and density of reference $\rho_\star$ as fixed we will ask the density oscillations around $\rho_\star$ and the velocity to be initially so small that both Reynolds number and Mach number shall be small. As for compressible flows in a more general context the reader may be referred to [16], [14] or [4].

Obviously the present work is not the first one tackling the asymptotic behaviour of near-equilibrium compressible flows. Since 1983 and the pioneer work of Kawashima about a vast class of hyperbolic-parabolic systems of degenerate type [10] equilibrium is known to be asymptotically stable for perturbations in Sobolev spaces $H^s(\mathbf{R}^2)$, for any integer $s$ bigger than or equal to three. We shall make use of this stability result. Besides working with Kawashima's solutions Hoff and Zumbrun established a precise analysis of asymptotic behaviour of perturbations of equilibrium [8] when initial perturbations belong to $H^s(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$, for any integer $s$ bigger than five. The present paper is partially modelled on their proof.

However the choice of Hoff and Zumbrun for initial data precludes vortex-like asymptotic profiles as stated in (11). Actually we will obtain the same decay rates but with different profiles. These decay rates are those of $\omega^{\beta_1,\beta_2}$ in Lebesgue spaces $L^p(\mathbf{R}^2)$, for $2 \leq p \leq \infty$. Since these rates are not critical, decay rates of non-linear terms should be negligible in the analysis of long-time behaviour. An important point in the proof is that decay rates in $L^p(\mathbf{R}^2)$, $2 \leq p \leq \infty$, are sufficient to establish that non-linear terms can indeed be neglected in $L^q(\mathbf{R}^2)$, for any $1 \leq q \leq \infty$, since those non-linearities are at least quadratic. Such remark enables us to keep us away from integrability of initial data as far as non-linear terms are concerned.

Yet in linearising around equilibrium and treating non-linear terms as source terms we must keep in mind that system (18), (19) is quasi-linear and non parabolic. Following [8] we turn round this difficulty using Kawashima's stability estimates to bound high derivatives of the solutions. However when doing so some terms are bounded by constants regardless of their natural decay rates. Thus we shall require high-regularity of initial data in order to recover natural decay rates for low derivatives of the solutions.

Let us now focus on the linearisation around $\rho = \rho_\star$ and $m = 0$ of system (18), (19). Denote the (reference) sound speed

$$c \quad = \quad \sqrt{P'(\rho_\star)} \tag{20}$$

then consider the following system for $\widetilde{\rho} = \rho - \rho_\star$ and $m$

$$\left.\begin{array}{rcl} \partial_t \widetilde{\rho} \;+\; \operatorname{div} m & = & 0 \\ \partial_t m \;+\; c^2 \nabla \widetilde{\rho} & = & \mu \triangle m \;+\; (\mu+\lambda) \nabla \operatorname{div} m \end{array} \quad . \right\} \tag{21}$$

An important feature concerning system (21) is that it splits up into two systems, one for a curl-free part and the other one for a constant-density divergence-free part. Let us divide $m = m_{\shortparallel} + m_{\perp}$ into its divergence free part $m_{\perp} = \mathbf{P}\, m$ and its curl-free part $m_{\shortparallel} = \mathbf{Q}\, m$, where $\mathbf{P}$ is the Leray projection, that is the projection onto divergence-free vector fields along gradient fields, and $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ its complementary projection. Then system (21) results for $(\widetilde{\rho}, m_{\shortparallel})$ in

$$\left.\begin{array}{rcrcl} \partial_t\, \widetilde{\rho} & + & \operatorname{div} m_{\shortparallel} & = & 0 \\ \partial_t\, m_{\shortparallel} & + & c^2\, \nabla \widetilde{\rho} & = & (\lambda + 2\mu)\, \triangle\, m_{\shortparallel} \end{array}\right\} \tag{22}$$

and for $(0, m_{\perp})$ in

$$\partial_t\, m_{\perp} \;-\; \mu\, \triangle\, m_{\perp} \;=\; 0\;. \tag{23}$$

Incidentally remark that $\mu > 0$ and $\lambda + 2\,\mu > 0$ clearly appear to be the conditions for ellipticity of the viscosity tensor.

Equation (23) coincides with the linearisation around equilibrium of the homogeneous Navier-Stokes equation and is thus expected to give rise to profiles as stated in (11) for suitable initial data. It remains to prove that solutions of system (22) decay faster.

System (22) can be handled essentially as Hoff and Zumbrun treated the whole system (21), the main difference being the former system includes Leray projections in its Green kernel. High (and mean) frequencies of the Green kernel of system (22) should indeed decay exponentially in time, whereas low frequencies can be approximated by the Green kernel $\widetilde{S}_{\shortparallel}$ of an *artificial viscosity* system. This system is derived from system (22) looking for a system whose eigenvalues coïncide with a second order low-frequency expansion of eigenvalues of system (22) — which gives a non-trivial real part — and that is simultaneously diagonalised with the hyperbolic part of system (22). This leads to a system[1] of (non-degenerate) hyperbolic-parabolic type whose hyperbolic and parabolic parts commute. Roughly speaking, $\widetilde{S}_{\shortparallel}$'s components are convolutions of a wave kernel and a heat kernel and look like Gaussian functions spreading (at scale $\sqrt{(\lambda/2 + \mu)\, t}\,$) around circles scattering at scale $c\, t$ and centred at the origin. Actually in [9] the following point-wise bounds are proved for any point $x \in \mathbf{R}^2$ and any time $t$ bigger than one,

$$|D^\sigma \widetilde{S}_{\shortparallel}(t, x)| \quad\leq\quad K\, t^{-5/4 - |\sigma|/2} \left\{ \begin{array}{lll} t^{3/4}\, s^{-3/2} \,, & & |x| \leq c\,(t - \sqrt{t}\,)\,, \\ e^{-\frac{s^2}{K\,t}} & , & |x| \geq c\,(t - \sqrt{t}\,)\,, \end{array}\right. \tag{24}$$

where $s = ||x| - c\, t|$ is the distance from $x$ to the circle of radius $c\, t$ centred at the origin. Once integrated, these bounds leads to decay rates $t^{-(5/4 - 3/2p)}$ in Lebesgue space $L^p(\mathbf{R}^2)$. Thereby $\widetilde{S}_{\shortparallel}$ spreads faster than a heat kernel hence decays faster in spaces requiring little localisation such as $L^p(\mathbf{R}^2)$ for $2 < p \leq \infty$, but more slowly in $L^p(\mathbf{R}^2)$, for $1 \leq p < 2$.

Let us now denote $S$ the Green kernel of system (21) and state the main result of this paper, whose existence and uniqueness part is due to Kawashima [10]. Lebesgue spaces are equipped with norms $\|\cdot\|_p$ and Sobolev spaces $H^s(\mathbf{R}^2)$ (based on $L^2(\mathbf{R}^2)$) with norms $|\cdot|_s$. The Green kernel of the linearised system (21) is denoted by $S$.

[1]See (48) below.

**Theorem 1** *Let $s$ be an integer bigger than or equal to five, $\rho_\star$ be a positive number, $\mu$ be a positive number, $\lambda$ a real number such that $\lambda + 2\mu > 0$, and $P : \mathbf{R}_\star^+ \to \mathbf{R}$ a smooth increasing function.*
*There exist positive constants $\varepsilon_0$ and $K$ and a family $(K_p)_{1<p\leq\infty}$ of positive constants such that defining*

$$X_0 \ = \ (\rho_0 - \rho_\star, m_0) \ , \qquad X_{0,\shortparallel} \ = \ (\rho_0 - \rho_\star, m_{0,\shortparallel}) \ ,$$

*where*

$$m_0 \ = \ m_{0,\shortparallel} + m_{0,\perp} \ , \qquad m_{0,\shortparallel} \ = \ \mathbf{Q}\, m_0 \ , \qquad m_{0,\perp} \ = \ \mathbf{P}\, m_0 \ ,$$

*if*

$$E \ = \ |\, X_0 \,|_s + \|\, X_{0,\shortparallel} \,\|_1 + \|\, (1 + |\ \cdot\ |)\ \mathrm{rot}\, m_0 \,\|_1 \ \leq \ \varepsilon_0 \ ,$$

*then system (18), (19) has a unique global classical solution $(\rho, m)$ with initial datum $(\rho_0, m_0)$, and $X = (\rho - \rho_\star, m)$ satisfies for any time $t > 0$ and any multi-index $\sigma$*

1. *when $|\sigma| \leq \frac{s-5}{2}$,*

$$\begin{aligned} &\|\, D^\sigma\big(X(t) - S(t) \star X_0\big) \,\|_p \\ &\qquad \leq K\, E^2\, \ln(1+t) \begin{cases} (1+t)^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}+\frac{1}{2}\right)} \ , & 2 \leq p \leq \infty \\ (1+t)^{-\left(\frac{5}{4}-\frac{3}{2p}+\frac{|\sigma|}{2}+\frac{1}{2}\right)} \ , & 1 \leq p \leq 2 \end{cases} ; \end{aligned} \tag{25}$$

2. *when $|\sigma| \leq \frac{s-5}{2}$, defining $m_\perp = \mathbf{P}\ m$, $m_\shortparallel = \mathbf{Q}\ m$ and $X_\shortparallel = (\rho - \rho_\star, m_\shortparallel)$,*

$$\begin{aligned} \|\, D^\sigma X_\shortparallel(t) \,\|_p \ &= \ \|\, \big(D^\sigma\big(\rho(t) - \rho_\star\big), D^\sigma\big(m(t) - m_\perp(t)\big)\big) \,\|_p \\ &\leq \ K\, E\ (1+t)^{-\left(\frac{5}{4}-\frac{3}{2p}+\frac{|\sigma|}{2}\right)} \ , \qquad 2 \leq p \leq \infty \ ; \end{aligned} \tag{26}$$

3. *when $|\sigma| \leq \frac{s-5}{2}$, if moreover $D^\sigma X_{0,\shortparallel}$ is integrable then, when $t \geq 1$, with $E' = E + \|\, D^\sigma X_{0,\shortparallel} \,\|_1$,*

$$\|\, D^\sigma X_\shortparallel(t) \,\|_p \ \leq \ K\, E'\ t^{-\left(\frac{5}{4}-\frac{3}{2p}+\frac{|\sigma|}{2}\right)} \ , \qquad 1 \leq p < 2 \ ; \tag{27}$$

4. *when $|\sigma| \leq \frac{s-5}{2}$, denoting again $m_\perp = \mathbf{P}\ m$,*

$$\|\, D^\sigma m_\perp(t) \,\|_p \ \leq \ K\, E\ (1+t)^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}\right)} \ , \qquad 2 \leq p \leq \infty \ , \tag{28}$$

*and moreover, for $2 \leq p \leq \infty$,*

$$\lim_{t\to\infty}\ t^{1-\frac{1}{p}+\frac{|\sigma|}{2}}\ \|\, D^\sigma\big(m_\perp(t) - \rho_\star\, u^{\beta_1,\beta_2}(t)\big) \,\|_p \ = \ 0 \ , \tag{29}$$

*where $u^{\beta_1,\beta_2}$ is defined by (17) (together with (12), (13) and (14), remind also $\nu = \mu/\rho_\star$) and*

$$\nu\, \beta_i \ = \ - \int_{\mathbf{R}^2} x_i\ \mathrm{rot}\left(\frac{m_0}{\rho_\star}\right)(x)\, dx \ ; \tag{30}$$

5. *if moreover* $(1+|\cdot|^2)\,\mathrm{rot}\, m_0$ *is integrable, then for any* $1<p\le\infty$*, with* $E' = E + \|\,(1+|\cdot|^2)\,\mathrm{rot}\, m_0\,\|_1$*, when* $|\sigma|\le\frac{s-5}{2}$ *and* $t\ge 1$*,*

$$\|\, D^\sigma\big(\,(\,\rho(t),\ m(t)\,)-(\,\rho_\star,\ \rho_\star\, u^{\beta_1,\beta_2}(t)\,)\big)\,\|_p \ \le\ K_p\, E'\, t^{-\left(\frac{5}{4}-\frac{3}{2p}+\frac{|\sigma|}{2}\right)}\ . \quad (31)$$

**Remarks:**

**1.** The hypothesis on $\mathrm{rot}\, m$ sufficient to define (30) is enough to prove (29) yet to quantify this asymptotic more localisation is needed, as required for (31).

**2.** Estimate (25) does show that non-linear terms can be neglected, whereas estimates (26) and (28), (31) establish that constant-density incompressible profiles dominate in $L^p(\mathbf{R}^2)$ for $2<p\le\infty$ (whereas sonic waves dominate when $1<p<2$). Indeed

$$\lim_{t\to\infty}\ t^{1-\frac{1}{p}}\ \|\, m(t)-\rho_\star\, u^{\beta_1,\beta_2}(t)\,\|_p \quad=\quad 0\ , \qquad 2<p\le\infty\ , \quad (32)$$

while $t^{-\left(1-\frac{1}{p}\right)}$ is the decay rate of $\rho_\star\, u^{\beta_1,\beta_2}(t)$ in $L^p(\mathbf{R}^2)$ (when $(\beta_1,\beta_2)$ is non zero) ; whereas, at least when $(1+|\cdot|^2)\,\mathrm{rot}\, m_0$ is integrable,

$$\lim_{t\to\infty}\ t^{\frac{5}{4}-\frac{3}{2p}}\ \|\,(\,\rho(t),\ m(t)\,)-\widetilde{S}_{\shortparallel}(t)\star X_{0,\shortparallel}\,\|_p \quad=\quad 0\ , \qquad 1<p<2\ , \quad (33)$$

where $\widetilde{S}_{\shortparallel}$ is the Green kernel of system (48) below, which satisfies (24) and decays in $L^p(\mathbf{R}^2)$ as $t^{-\left(\frac{5}{4}-\frac{3}{2p}\right)}$.

The proof is developed in the two following sections. The next section gathers estimates for linear equations whereas the last one encompasses the actual proof of Theorem 1 and in particular estimates of non-linear terms. As in Theorem 1 from now on $\mu$, $\lambda$, $\rho_\star$ and $P$ are considered as fixed.

Let us also make explicit the convention used in the present paper for Fourier transforms: when a function $f$ is integrable, its Fourier transform is defined by

$$\widehat{f}\,(\eta)\quad=\quad\int_{\mathbf{R}^2} f(x)\ e^{\mathrm{i}\,\eta\cdot x}\ dx\ , \qquad \eta\in\mathbf{R}^2\ .$$

At last as usual $C$ stands for a harmless constant that may differ from line to line even in the same sequence of inequalities.

# 1 Linear equations

This section is devoted to the study of the system resulting from linearisation of (18), (19) around $\rho=\rho_\star$ and $m=0$:

$$\left.\begin{array}{rcrcl} \partial_t\,\widetilde{\rho} & + & \mathrm{div}\ m & = & 0\\ \partial_t\, m & + & c^2\,\nabla\widetilde{\rho} & = & \mu\,\triangle\, m+(\mu+\lambda)\,\nabla\,\mathrm{div}\, m \end{array}\right\}\quad (34)$$

where $c=\sqrt{P'(\rho_\star)}>0$ is the reference sound speed. As was already mentioned, splitting $m=m_{\shortparallel}+m_\perp$ into curl-free $m_{\shortparallel}=\mathbf{Q}\, m$ and divergence-free $m_\perp=\mathbf{P}\, m$ yields the system

$$\left.\begin{array}{rcrcl} \partial_t\,\widetilde{\rho} & + & \mathrm{div}\, m_{\shortparallel} & = & 0\\ \partial_t\, m_{\shortparallel} & + & c^2\,\nabla\widetilde{\rho} & = & (\lambda+2\mu)\,\triangle\, m_{\shortparallel} \end{array}\right\}\quad (35)$$

and the equation

$$\partial_t m_\perp - \mu \triangle m_\perp = 0 . \tag{36}$$

Therefore the Green kernel $S$ of system (34) may be written in terms of the Green kernel $S_{\shortparallel}$ of system (35) and the heat kernel $K_\mu$, *i.e.* the Green kernel of equation (36). To make it explicit introduce kernels of Leray projection $\mathbf{P}$ and of its complementary projection $\mathbf{Q}$:

$$\mathbf{P} f = R_\perp \star f , \qquad \mathbf{Q} f = R_{\shortparallel} \star f ,$$

for any vector-field $f$. In terms of Fourier transforms note that

$$\widehat{R_\perp}(\eta) = \frac{\eta^\perp \, {}^{\mathrm{t}}\eta^\perp}{|\eta|^2} , \qquad \widehat{R_{\shortparallel}}(\eta) = \frac{\eta \, {}^{\mathrm{t}}\eta}{|\eta|^2} .$$

Now observe

$$S = S_{\shortparallel} \star \begin{bmatrix} \delta_0 & 0 \\ 0 & R_{\shortparallel} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_\mu \star R_\perp \end{bmatrix} , \tag{37}$$

where $\delta_0$ is the Dirac mass centred at the origin and of weight one.

Keeping (37) in mind, we now study $S_{\shortparallel}$ and $K_\mu$ separately.

## 1.1 Curl-free part

Estimates of $S_{\shortparallel}$ used afterwards may be established as those of $S$ developed by David Hoff and Kevin Zumbrun. That is why the needed results shall be stated and their proofs sketched but no explicit calculation written down, since similar calculations can be found in [8, 9]. The reader may also consult [11] and references therein about some refinement for estimates of $S$ and related subjects.

If $(\widetilde{\rho}, m_{\shortparallel})$ is a solution of system (35), the density oscillation $\widetilde{\rho}$ obeys

$$\partial_t^2 \widetilde{\rho} - c^2 \triangle \widetilde{\rho} - (\lambda + 2\mu) \triangle \partial_t \widetilde{\rho} = 0 . \tag{38}$$

By the way note that in the inviscid case, namely when $\lambda = \mu = 0$, the density oscillation satisfies a wave equation, the density waves travelling at speed $c$, which is the reason why it is called sound speed of the flow. By taking now Fourier transforms this yields a differential equation, where $\eta$ can be thought of as a parameter, for the quantity $y(t, \eta) = \widehat{\widetilde{\rho}}(t, \eta)$:

$$y'' + (\lambda + 2\mu) |\eta|^2 y' + c^2 |\eta|^2 y = 0 . \tag{39}$$

Thereby system (35) can be solved and

$$\widehat{S_{\shortparallel}}(t, \eta) = \begin{bmatrix} \frac{\lambda^+(\eta) e^{\lambda^-(\eta) t} - \lambda^-(\eta) e^{\lambda^+(\eta) t}}{\lambda^+(\eta) - \lambda^-(\eta)} & -\mathrm{i} \left( \frac{e^{\lambda^+(\eta) t} - e^{\lambda^-(\eta) t}}{\lambda^+(\eta) - \lambda^-(\eta)} \right) {}^{\mathrm{t}}\eta \\ -\mathrm{i}\, c^2 \left( \frac{e^{\lambda^+(\eta) t} - e^{\lambda^-(\eta) t}}{\lambda^+(\eta) - \lambda^-(\eta)} \right) \eta & \frac{\lambda^+(\eta) e^{\lambda^+(\eta) t} - \lambda^-(\eta) e^{\lambda^-(\eta) t}}{\lambda^+(\eta) - \lambda^-(\eta)} \end{bmatrix} , \tag{40}$$

where eigenvalues $\lambda^\pm$ are

$$\lambda^\pm(\eta) = -\frac{1}{2} \mu_{\shortparallel} |\eta|^2 \pm \frac{1}{2} \sqrt{\mu_{\shortparallel}^2 |\eta|^4 - 4 c^2 |\eta|^2} \tag{41}$$

and a new viscosity parameter $\mu_{\shortparallel}$ is defined for concision's sake by

$$\mu_{\shortparallel} \;=\; \lambda + 2\,\mu\,. \tag{42}$$

The former formula yields also an explicit formula for $\widehat{S}$ and enables us to perform the whole study of $S_{\shortparallel}$.

In order to capture the quite different behaviour of high and low frequencies, let us split $S_{\shortparallel}$. Let $\chi$ be a smooth real-valued cut-off function taking values between zero and one that is equal to one on $\{\eta \in \mathbf{R}^2 \,|\, |\eta| \leq R_0\}$ and vanishes on $\{\eta \in \mathbf{R}^2 \,|\, |\eta| \geq R_0 + 1\}$, for some $R_0 > 0$ to be chosen large enough. Now divide $S_{\shortparallel} = S_{\shortparallel}^{LF} + S_{\shortparallel}^{HF}$ in such a way that

$$\widehat{S_{\shortparallel}^{LF}}(t,\eta) \;=\; \chi(\eta)\,\widehat{S}_{\shortparallel}(t,\eta)\ , \qquad \widehat{S_{\shortparallel}^{HF}}(t,\eta) \;=\; (1-\chi(\eta))\,\widehat{S}_{\shortparallel}(t,\eta) \tag{43}$$

and study separately $S_{\shortparallel}^{LF}$ and $S_{\shortparallel}^{HF}$.

### 1.1.1 High frequencies

Expending $\lambda^{\pm}(\eta)$ around $|\eta| = \infty$ gives

$$\lambda^{+}(\eta) \overset{|\eta|\to\infty}{=} -\frac{c^2}{\mu_{\shortparallel}} + \mathcal{O}(|\eta|^{-2})\ , \qquad \lambda^{-}(\eta) \overset{|\eta|\to\infty}{=} -\mu_{\shortparallel}\,|\eta|^2 + \frac{c^2}{\mu_{\shortparallel}} + \mathcal{O}(|\eta|^{-2})\ .$$

*A priori* high frequencies should decay exponentially. Moreover the former expansion confirms that one component — $m_{\shortparallel}$ — should be regularised whereas another — $\widetilde{\rho}$ — should not.

To be more precise, an integral representation of solutions of system (35) using differential equation (39) may be used. Define

$$\begin{aligned}
A(t,r) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{S}^+\cup\,\mathcal{S}^-} \frac{e^{tz}}{p\,(r,z)}\,dz\\
B(t,r) &= \partial_t\,A(t,r) + \mu_{\shortparallel}\,r^2\,A(t,r)\\
D(t,r) &= e^{-\mu_{\shortparallel}\,r^2 t}\int_0^t e^{\mu_{\shortparallel}\,r^2 s}A(s,r)\,ds
\end{aligned}$$

where $\mathcal{S}^+$ and $\mathcal{S}^-$ are circles of radius $c^2/2\mu_{\shortparallel}$ centered at $-c^2/\mu_{\shortparallel}$ and $-\mu_{\shortparallel}\,r^2 + c^2/\mu_{\shortparallel}$ respectively, and $p$ is the polynomial $p\,(r,z) = z^2 + \mu_{\shortparallel}\,r^2\,z + c^2\,r^2$. Then with obvious matricial conventions

$$\begin{aligned}
\widehat{S}_{\shortparallel}^{\,1,1}(t,\eta) &= B(t,|\eta|)\ , & \widehat{S}_{\shortparallel}^{\,1,2}(t,\eta) &= -\mathrm{i}\;A(t,|\eta|)\;{}^{\mathrm{t}}\eta\ ,\\
\widehat{S}_{\shortparallel}^{\,2,1}(t,\eta) &= -\mathrm{i}\,c^2\,A(t,|\eta|)\;\eta\ , & \widehat{S}_{\shortparallel}^{\,2,2}(t,\eta) &= e^{-\mu_{\shortparallel}\,|\eta|^2 t} - c^2\,|\eta|^2\,D(t,|\eta|)\,.
\end{aligned}$$

Yet expanding $1/p(z,r)$ into powers of $r^{-1}$ yields for $r$ large enough

$$\begin{aligned}
A(t,r) &= \sum_{k=0}^{\infty} A_k(t,r)\;r^{-2k-2}\\
B(t,r) &= e^{-\frac{c^2 t}{\mu_{\shortparallel}}} + \sum_{k=0}^{\infty} B_k(t,r)\;r^{-2k-2}\\
D(t,r) &= \sum_{k=0}^{\infty} D_k(t,r)\;r^{-2k-4}
\end{aligned}$$

with for any $k \in \mathbf{N}$

$$|A_k(t,r)|,\ |B_k(t,r)|,\ |D_k(t,r)|\ \le C\ (e^{-\frac{c^2 t}{\mu_{\shortparallel}}} + e^{-\frac{\mu_{\shortparallel}}{2} r^2 t})\ r_0^k$$

where $C$ and $r_0$ are positive constants independent of $k$, $r$ and $t$. In quite the same way, it can also be proved that for any $j \in \mathbf{N}^*$

$$\begin{aligned}
|\partial_r^j A(t,r)| &\le C_j\,(e^{-\frac{c^2 t}{\mu_{\shortparallel}}} + e^{-\frac{\mu_{\shortparallel}}{2} r^2 t})\ r^{-j-2}\\
|\partial_r^j B(t,r)| &\le C_j\,(e^{-\frac{c^2 t}{\mu_{\shortparallel}}} + e^{-\frac{\mu_{\shortparallel}}{2} r^2 t})\ r^{-j-2}\\
|\partial_r^j D(t,r)| &\le C_j\,(e^{-\frac{c^2 t}{\mu_{\shortparallel}}} + e^{-\frac{\mu_{\shortparallel}}{2} r^2 t})\ r^{-j-4}\ .
\end{aligned}$$

Now Marcinkiewic multiplier theorem (see [18] for instance) and an adaptation proved in [8, Proposition 4.2] transform these estimates around $|\eta| = \infty$ into the following proposition preceded by useful definitions.

**Definition 2**

*i. A bounded symbol* $\widehat{f}$ *is an* $L^p$-multiplier *if the associated operator* $f\star$ *can be extended for any* $1 < p < \infty$ *from* $L^2(\mathbf{R}^2) \cap L^p(\mathbf{R}^2)$ *to* $L^p(\mathbf{R}^2)$*:*

$$\|\, f \star g \,\|_p \ \le\ C_p \,\|\, g \,\|_p\ , \qquad g \in L^p(\mathbf{R}^2)\ , \qquad 1 < p < \infty\ .$$

*ii. An* $L^p$*-multiplier is a* strong $L^p$-multiplier *if the above property also holds for any* $1 \le p \le \infty$*:*

$$\|\, f \star g \,\|_p \ \le\ C \,\|\, g \,\|_p\ , \qquad g \in L^p(\mathbf{R}^2)\ , \qquad 1 \le p \le \infty\ .$$

*iii. A family of multipliers — either strong or not — is* bounded *if the above constants —* $C$ *or* $C_p$ *— can be chosen uniformly for the whole family..*

A typical example of $L^p$-multiplier that is not a strong $L^p$-multiplier is given by $\widehat{R_\perp}$, the symbol of the Leray projection $\mathbf{P}$.

**Proposition 3** *If* $R_0$ *is large enough then there exists a positive constant* $b$ *such that* $S_{\shortparallel}^{HF}$*, the high-frequency part of* $S_{\shortparallel}$*, defined by (43), satisfies*

$$\widehat{S_{\shortparallel}^{HF}}\,(t,\eta) \quad = \quad e^{-b\,t}\ M(t,\eta) \tag{44}$$

*where* $(M(t))_{t\ge 0}$ *is a bounded family of strong* $L^p$*-multipliers, and for any integers* $1 \le i,j,k \le 2$ *with* $(i,j) \ne (1,1)$

$$\widehat{\partial_k\, S_{\shortparallel}^{HF\ i,j}}\,(t,\eta) \quad = \quad e^{-b\,t}\ (1 + t^{-1/2})\ N_k^{i,j}(t,\eta) \tag{45}$$

*where* $(N_k^{i,j}(t))_{t\ge 0}$ *is a bounded family of* $L^p$*-multipliers.*

**Remarks:**

**1.** The only component of $S_{\shortparallel}^{HF}$, therefore of $S_{\shortparallel}$, that does not give rise to regularisation is $(S_{\shortparallel}^{HF})^{1,1}$. Indeed its high-frequency expansion includes a Dirac mass $e^{-c^2 t/\mu_{\shortparallel}}\delta_0$. Actually since the former proposition has been obtained bounding Fourier transforms it is still true for $S_{\shortparallel}^{HF} \star \begin{bmatrix} \delta_0 & 0 \\ 0 & R_{\shortparallel} \end{bmatrix}$ or even for the high-frequency part of $S$. So is it stated in [8, Lemma 5.3].

**2.** Indeed components $\widehat{\partial_k S_{\shortparallel}^{HF}}^{1,2}$ and $\widehat{\partial_k S_{\shortparallel}^{HF}}^{2,1}$ are not strong $L^p$-multipliers. Their expansions contain terms of type $e^{-c^2 t/\mu_{\shortparallel}}\ \eta_k\ \eta_{k'}/|\eta|^2$.

**3.** However $\widehat{\partial_k S_{\shortparallel}^{HF}}^{2,2}$ is a strong $L^p$-multiplier and $\partial_k \widehat{\partial_{k'} S_{\shortparallel}^{HF}}^{2,2}$ is a (weak) $L^p$-multiplier. Thereby a source term in (35) would undergo, if placed into the first equation, a regularisation of one derivative in the determination of $m_{\shortparallel}$ but of none for $\widetilde{\rho}$; whereas, if placed in the second equation, it would undergo a regularisation of two derivatives in the determination of $m_{\shortparallel}$ and of only one for $\widetilde{\rho}$. This is the reason why one can not establish an existence result based on a naïve linearisation around equilibrium either in momentum variables $(\widetilde{\rho}, m)$, since $\triangle(m_{\shortparallel}\,\widetilde{\rho})$ could not be treated as a source term for the second equation, or in velocity variables $(\widetilde{\rho}, u)$, since $u\cdot\nabla\widetilde{\rho}$ could not be handled as a source term in the first equation of (35). To turn round this difficulty in [3] Raphaël Danchin considers and studies a linear system including a convection term. Yet doing so it seems difficult if not impossible to capture a precise decay behaviour due to dispersion. We shall rather work with Kawashima's solutions and sacrifice some regularity.

### 1.1.2 Low frequencies

From the Hausdorff-Young inequalities and explicit formula (40) we may at once deduce the following proposition.

**Proposition 4** *For any multi-index $\sigma$ there exists a positive constant $C_\sigma$ such that the low-frequency part $S_{\shortparallel}^{LF}$ of $S_{\shortparallel}$ satisfies for any time $t \geq 0$ and any real number $p$*

$$\|\, D^\sigma S_{\shortparallel}^{LF}(t)\,\|_p \ \leq\ C_\sigma \left\{ \begin{array}{cl} t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}\right)} & \text{if} \quad t\geq 1 \quad \text{and } 2\leq p\leq\infty\ , \\ 1 & \text{if } 0\leq t\leq 1 \text{ and } 1\leq p\leq\infty\ . \end{array}\right. \tag{46}$$

**Remark:** Obviously the proposition still holds when $S$ is substituted for $S_{\shortparallel}$.

Note $S_{\shortparallel}$ may contain mean frequencies but since they should both be regularised and decay exponentially the point is really in low frequencies. Let us then perform some expansions around $\eta = (0,0)$ in order to derive a good approximation of $S_{\shortparallel}^{LF}$.

As for eigenvalues we have

$$\lambda^\pm(\eta) \ \overset{|\eta|\to 0}{=}\ -\frac{1}{2}\,\mu_{\shortparallel}\,|\eta|^2 \pm\ \mathrm{i}\ c\ |\eta| + \mathcal{O}(|\eta|^3)\ . \tag{47}$$

We have expanded $\lambda^\pm$ until getting a non-trivial real part which leads us to second order expansion. Concerning diagonalisation basis we shall be satisfied with a first-order expansion and therefore we look for a Green kernel diagonalised on a diagonalisation basis of the hyperbolic part of (35). Thereby in order to build a good low-frequencies approximation of (35) we keep the same hyperbolic part but the parabolic part is modified and we obtain

$$\left.\begin{array}{rcrcl} \partial_t \widetilde{\rho} & + & \operatorname{div} m_{\shortparallel} & = & \frac{1}{2}\,\mu_{\shortparallel}\,\triangle\,\widetilde{\rho} \\ \partial_t\, m_{\shortparallel} & + & c^2\,\nabla\widetilde{\rho} & = & \frac{1}{2}\,\mu_{\shortparallel}\,\triangle\, m_{\shortparallel}\ . \end{array}\right\} \tag{48}$$

As for asymptotic behaviour the Green kernel $\widetilde{S}_{\shortparallel}$ of system (48) should give a close approximation of $S_{\shortparallel}$. The point in the approximation is that system (48) is of *non-degenerate* hyperbolic-parabolic type, with hyperbolic and parabolic parts commuting since simultaneously diagonalised. Such a system is called *artificial viscosity system.* See [15] (where by the way are also exposed Kawashima's estimates) to learn more about approximations of degenerate hyperbolic-parabolic systems in the unidimensional context and [8, Section 6] for the general case. Before establishing that $\widetilde{S}_{\shortparallel}$ indeed asymptotically approaches $S_{\shortparallel}$, we should study the asymptotic behaviour of $\widetilde{S}_{\shortparallel}$.

Since hyperbolic and parabolic parts of system (48) commute, defining $W$ the Green kernel of hyperbolic system

$$\left.\begin{array}{rcrcl} \partial_t \widetilde{\rho} & + & \operatorname{div} m_{\shortparallel} & = & 0 \\ \partial_t\, m_{\shortparallel} & + & c^2\,\nabla\widetilde{\rho} & = & 0 \end{array}\right\} \tag{49}$$

and $K_{\mu_{\shortparallel}/2}$ the heat kernel associated to

$$\partial_t\, f - \frac{1}{2}\,\mu_{\shortparallel}\,\triangle\, f = 0 \tag{50}$$

leads to $\widetilde{S}_{\shortparallel} = W \star \begin{bmatrix} K_{\mu_{\shortparallel}/2} & 0 \\ 0 & K_{\mu_{\shortparallel}/2} \end{bmatrix}$. Actually since system (49) implies

$$\partial_t^2\, \widetilde{\rho} - c^2 \triangle\, \widetilde{\rho} \;\;=\;\; 0\ , \tag{51}$$

by introducing $w$ the solution of equation (51) with initial datum $w(0) = 0$, $\partial_t w(0) = \delta_0$, an explicit description is obtained:

$$\widetilde{S}_{\shortparallel} \;\;=\;\; \begin{bmatrix} \partial_t\, w \star K_{\mu_{\shortparallel}/2} & -\,\nabla^{\mathrm{t}} w \star K_{\mu_{\shortparallel}/2} \\ -c^2\,\nabla w \star K_{\mu_{\shortparallel}/2} & \partial_t\, w \star K_{\mu_{\shortparallel}/2} \end{bmatrix}\ . \tag{52}$$

The former formula is fully explicit since

$$w\,(t,x) \;\;=\;\; \begin{cases} \frac{1}{2\pi c}\ \frac{1}{\sqrt{c^2t^2-|x|^2}} & \text{if } \ |x| < c\,t\ , \\ 0 & \text{if } \ |x| \geq c\,t\ . \end{cases} \tag{53}$$

This enables us to obtain point-wise bounds for $\widetilde{S}_{\shortparallel}$: for any multi-index $\sigma$, there exists a positive constant $C$ such that for any time $t \geq 1$ and any point $x \in \mathbf{R}^2$

$$|D^\sigma \widetilde{S}_{\shortparallel}(t,x)| \;\leq\; C\,t^{-5/4-|\sigma|/2} \begin{cases} t^{3/4}\ s^{-3/2} & \text{if } \ |x| \leq c(t-\sqrt{t})\ , \\ e^{-\frac{s^2}{Ct}} & \text{if } \ |x| \geq c(t-\sqrt{t})\ , \end{cases} \tag{54}$$

where $s = ||x| - c\,t|$ is the distance from $x$ to the circle centred at the origin and of radius $c\,t$. The reader is referred to [9] for a proof of these estimates. Integrating in space lead then to the following proposition.

**Proposition 5** *The Green kernel $\widetilde{S}_{\shortparallel}$ of the artificial viscosity system (48) is such that for any multi-index $\sigma$ there exists a positive constant $C$ such that*

$$\| D^\sigma \widetilde{S}_{\shortparallel}(t) \|_p \;\leq\; C_\sigma \, t^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{|\sigma|}{2}\right)} \;, \qquad t \geq 1 \;, \quad 1 \leq p \leq \infty \;. \tag{55}$$

**Remarks:**

**1.** Note that combining decay rates of the heat kernel in $L^q(\mathbf{R}^2)$ for suitable $q$ with an estimate of the wave operator as operator from $L^q(\mathbf{R}^2)$ to $L^p(\mathbf{R}^2)$ (see [19]) does not yield Proposition 5.

**2.** By getting back to (37), since $\mathbf{P}$ is not a strong $L^p$ multiplier, it may be observed that Proposition 5 does not give estimates of

$$\widetilde{S}_{part} \;=\; \widetilde{S}_{\shortparallel} \star \left[ \begin{array}{cc} \delta_0 & 0 \\ 0 & R_{\shortparallel} \end{array} \right] \;. \tag{56}$$

in $L^1(\mathbf{R}^2)$ and $L^\infty(\mathbf{R}^2)$. However $R_{\shortparallel}$ is explicit and point-wise bounds for $\widetilde{S}_{part}$ may indeed be obtained, leading to

$$\| D^\sigma \widetilde{S}_{part}(t) \|_p \;\leq\; \frac{c_\sigma \, L_\sigma(t)}{t^{\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{|\sigma|}{2}}} \;, \qquad t \geq 1 \;, \quad 1 \leq p \leq \infty \;, \tag{57}$$

where $L_\sigma(t) = 1 + \ln t$ if $\sigma = (0,0)$ and $L_\sigma(t) = 1$ otherwise. Once again see [9] for a proof.

We should now compare decay rates of $S_{\shortparallel} - \widetilde{S}_{\shortparallel}$ to the former rates of $\widetilde{S}_{\shortparallel}$. The first part of the following proposition is straightforward thanks to the Hausdorff-Young inequalities since $\widetilde{S}_{\shortparallel}$ is a low-frequency approximation of $S_{\shortparallel}$. The second part comes from through space decomposition combining the decay rate of $S_{\shortparallel} - \widetilde{S}_{\shortparallel}$ in $L^2(\mathbf{R}^2)$ and the following point-wise bound: for any integer $N > 2$ and any multi-index $\sigma$ there exists a positive constant $C$ such that for $t \geq 1$ and $x \in \mathbf{R}^2$, $x \neq (0,0)$

$$|D^\sigma (\widetilde{S}_{\shortparallel} - S_{\shortparallel})\,(t,x)| \quad\leq\quad C\, t^{-1-|\sigma|/2} \, \left( \frac{|x|}{t} \right)^{-N} \;, \tag{58}$$

which is easily obtained *via* a Hausdorff-Young inequality. See [8, Lemma 8.1] for a detailed combination of these two bounds.

**Proposition 6** *The low-frequency part $S_{\shortparallel}^{LF}$ of $S_{\shortparallel}$ satisfies*

1. *for any multi-index $\sigma$ there exists a positive constant $C_\sigma$ such that for any time $t \geq 1$ and any $2 \leq p \leq \infty$,*

$$\| D^\sigma (S_{\shortparallel}^{LF}(t) - \widetilde{S}_{\shortparallel}^{LF}(t)) \|_p \;\leq\; C_\sigma \, t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}+\frac{1}{2}\right)} \;, \tag{59}$$

*where $\widetilde{S}_{\shortparallel}^{LF}$ is the low-frequency part of $\widetilde{S}_{\shortparallel}$;*

2. *for any multi-index $\sigma$ and any real number $\theta > 0$ there exists a positive constant $C_{\sigma,\theta}$ such that for $t \geq 1$ and $1 \leq p \leq 2$,*

$$\| D^\sigma (S_{\shortparallel}^{LF}(t) - \widetilde{S}_{\shortparallel}^{LF}(t)) \|_p \quad\leq\; C_{\sigma,\theta} \, t^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{|\sigma|}{2}+\frac{1}{2}-\theta\right)} \;, \tag{60}$$

*where $\widetilde{S}_{\shortparallel}^{LF}$ is the low-frequency part of $\widetilde{S}_{\shortparallel}$.*

**Remark:** To prove $\widetilde{S}_{\shortparallel}$ gives a good description of the asymptotic behaviour of $S_{\shortparallel}$, it only remains to note that the high-frequency part of $\widetilde{S}_{\shortparallel}$ also satisfies the estimates stated in Proposition 3 for $S_{\shortparallel}^{HF}$ and it therefore decays exponentially.

## 1.2 Constant-density divergence-free part

We now focus our attention on the linear equation for $m_\perp$. Though it is nothing but the heat equation the following estimates are not so standard since they concern divergence-free solutions.

First note estimates for the heat kernel does not yield in a straightforward way estimates for $K_\mu \star R_\perp$ in $L^1(\mathbf{R}^2)$. However once again point-wise bounds may be obtained thanks to the Hausdorff-Young inequalities: for any multi-index $\sigma$ there exists a positive constant $C_\sigma$ such that for any time $t > 0$ and any point $x \in \mathbf{R}^2$

$$| D^\sigma (K_\mu(t) \star R_\perp)(x) | \quad \leq \quad C_\sigma \; ( \max (t^{1/2}, |x|) )^{-(|\sigma|+2)} \; .$$

See [9, Lemma 2.2] for a proof of the former bound. Then integrating in space gives the following proposition. (See also [5] for a different proof.)

**Proposition 7** *For any multi-index $\sigma$ that is non zero there exists a positive constant $C_\sigma$ such that for any time $t > 0$*

$$\| D^\sigma K_\mu(t) \star R_\perp \|_p \;\leq\; C_\sigma \, t^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2}\right)} \; , \quad 1 \leq p \leq \infty \; , \quad |\sigma| \neq 0 \; . \tag{61}$$

The following proposition, which is the key-proposition of the present subsection, also follows from the Hausdorff-Young inequalities. Yet since it does not seem to be written elsewhere we shall write its proof in full details.

**Proposition 8**

1. *For any multi-index $\sigma$ there exists a positive constant $C_\sigma$ such that if $\omega_0$ is such that $(1 + |\cdot|)\, \omega_0$ is integrable and $\widehat{\omega_0}(0) = 0$ then the associated divergence-free vector-field $m_{0,\perp} = K_{BS} \star \omega_0$ satisfies for any time $t \geq 0$*

$$\| D^\sigma K_\mu(t) \star m_{0,\perp} \|_p \;\leq\; C_\sigma \, E \, t^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2}\right)} \; , \quad 2 \leq p \leq \infty \; , \tag{62}$$

   *where $E = \| (1 + |\cdot|)\, \omega_0 \|_1$.*

2. *For any multi-index $\sigma$ there exists a positive constant $C_\sigma$ such that if $\omega_0$ is such that $(1 + |\cdot|^2)\, \omega_0$ is integrable, $\widehat{\omega_0}(0) = 0$ and $\nabla_\eta \widehat{\omega_0}(0) = (0,0)$ then $m_{0,\perp} = K_{BS} \star \omega_0$ satisfies for any time $t \geq 0$*

$$\| D^\sigma K_\mu(t) \star m_{0,\perp} \|_p \;\leq\; C_\sigma \, E' \, t^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2} + \frac{1}{2}\right)} \; , \quad 2 \leq p \leq \infty \; , \tag{63}$$

   *where $E' = \| (1 + |\cdot|^2)\, \omega_0 \|_1$*

3. *For any multi-index $\sigma$ there exists a positive constant $C_\sigma > 0$ such that if $\omega_0$ is such that $(1 + |\cdot|^2)\, \omega_0$ is integrable, $\widehat{\omega_0}(0) = 0$ and $\nabla_\eta \widehat{\omega_0}(0) = (0,0)$ then $m_{0,\perp} = K_{BS} \star \omega_0$ satisfies for any $t \geq 0$ and any $2 \leq p \leq \infty$*

$$\| \, |\cdot| \, (D^\sigma K_\mu(t) \star m_{0,\perp}) \|_p \;\leq\; C_\sigma \, E' \; (1 + t^{-\frac{1}{2}}) \; t^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2}\right)} \; , \tag{64}$$

   *where $E' = \| (1 + |\cdot|^2)\, \omega_0 \|_1$.*

**Proof.** 1. Since $\widehat{\omega_0}$ is Lipschitzian and $\widehat{\omega_0}(0) = 0$, $\widehat{m_{0,\perp}}$ belongs to $L^\infty(\mathbf{R}^2)$ and for any non-zero $\eta \in \mathbf{R}^2$

$$|\widehat{m_{0,\perp}}(\eta)| \;=\; C\,|\eta|^{-1}\,|\widehat{\omega_0}(\eta)| \;\leq\; C\,\|\,\nabla_\eta\,\widehat{\omega_0}\,\|_\infty \;\leq\; C\,E\ .$$

Yet for any $2 \leq p \leq \infty$ Hausdorff-Young inequalities lead when defining $p'$ the conjugate exponent of $p$, that is $p'$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, to

$$\|\,D^\sigma K_\mu \star m_{0,\perp}\,\|_p \;\leq\; C\,\|\,|\,\cdot\,|^{|\sigma|}\,e^{-\mu\,|\,\cdot\,|^2 t}\,\widehat{m_{0,\perp}}\,\|_{p'} \;\leq\; C\,t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}\right)}\,\|\,\widehat{m_{0,\perp}}\,\|_\infty\ .$$

Thereby the first part of the proposition is proved.

2. In quite the same way from $|\widehat{\omega_0}(\eta)| \;\leq\; C\,E'\,|\eta|^2$ we derive

$$\|\,D^\sigma K_\mu \star m_{0,\perp}\,\|_p \;\leq\; C\;E'\;\|\,|\,\cdot\,|^{|\sigma|+1}\,e^{-\mu\,|\,\cdot\,|^2 t}\,\|_{p'} \;\leq\; C\,E'\,t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}+\frac{1}{2}\right)}\ .$$

3. Both $|\widehat{m_{0,\perp}}(\eta)| \;\leq\; C\,E'\,|\eta|$ and

$$|\nabla_\eta\,\widehat{m_{0,\perp}}(\eta)| \;\leq\; C\,(\,|\eta|^{-2}\,|\widehat{\omega_0}(\eta)| + |\eta|^{-1}\,|\nabla_\eta\,\widehat{\omega_0}(\eta)|\,) \;\leq\; C\,E'$$

stand, in such a way that

$$\begin{aligned}
\|\,|\,\cdot\,|\;(D^\sigma K_\mu \star m_{0,\perp})\,\|_p \;&\leq\; C\;\|\,\nabla_\eta\,(\widehat{D^\sigma K_\mu}\;\widehat{m_{0,\perp}})\,\|_{p'} \\
&\leq\; C_\sigma\,E'\,(\;\|\,|\,\cdot\,|^{|\sigma|}\;(1 + t\;|\,\cdot\,|^2)\;e^{-\mu\,|\,\cdot\,|^2 t}\,\|_{p'} \\
&\qquad\qquad + \|\,|\,\cdot\,|^{|\sigma|}\;e^{-\mu\,|\,\cdot\,|^2 t}\,\|_{p'}\;) \\
&\leq\; C_\sigma\,E'\,(1 + t^{-\frac{1}{2}})\,t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}\right)}\ ,
\end{aligned}$$

which achieves the proof of the proposition. ■

We now derive from the former proposition asymptotic profiles for $m_\perp$ both in $L^p(\mathbf{R}^2)$ for $p \geq 2$ without assuming further localisation but also without obtaining convergence rates and in $L^p(\mathbf{R}^2)$ for $p \leq 2$ when assuming more localisation for $\operatorname{curl} m_{0,\perp}$.

**Corollary 9** *Let $1 < p \leq 2$.*
*For any multi-index $\sigma$, there exists a positive constant $C_{\sigma,p} > 0$ such that if $\omega_0$ is such that $(1 + |\,\cdot\,|^2)\;\omega_0$ is integrable, $\widehat{\omega_0}(0) = 0$ and $\nabla_\eta\,\widehat{\omega_0}(0) = (0,0)$ then $m_{0,\perp} = K_{BS} \star \omega_0$ satisfies for any time $t \geq 0$*

$$\|\,D^\sigma K_\mu(t) \star\; m_{0,\perp}\,\|_p \;\leq\; C_{\sigma,p}\;E'\;(1 + t^{-\frac{1}{2}})^{2\,(\frac{1}{p}-\frac{1}{2})}\;t^{-\left(1-\frac{1}{p}+\frac{|\sigma|}{2}+\frac{1}{2}\right)}\ , \tag{65}$$

*where $E' = \|\,(1 + |\,\cdot\,|^2)\;\omega_0\,\|_1$.*

**Proof.** Given a non-zero function $f : \mathbf{R}^2 \to \mathbf{R}$, since $1 < p \leq 2$, Hölder's inequalities yield for any $R > 0$

$$\begin{aligned}
\|\,f\,\|_p \;&\leq\; (\int_{|x|\leq R} |f|^p(x)\,dx\;)^{1/p} + (\int_{|x|\geq R} |f|^p(x)\,dx\;)^{1/p} \\
&\leq\; C_p\,\big(\,R^{\frac{2}{p}-1}\;\|\,f\,\|_2 + R^{\frac{2}{p}-2}\;\|\,|\,\cdot\,|\,f\,\|_2\,\big)
\end{aligned}$$

hence, by choosing $R = \|\, |\cdot|\, f\,\|_2 \,/\, \|\, f\,\|_2$ in order to optimise the last term with respect to $R > 0$,

$$\|\, f\,\|_p \quad \leq \quad C_p\,\|\, f\,\|_2^{2\,(1-\frac{1}{p})}\,\|\, |\cdot|\, f\,\|_2^{2\,(\frac{1}{p}-\frac{1}{2})} \tag{66}$$

is obtained and the proof is achieved thanks to Proposition 8 applying (66) to the function $f = D^\sigma K_\mu \star m_{0,\perp}$. ∎

**Corollary 10** *If $\omega_0$ is a real-valued function such that $(1+|\cdot|)\,\omega_0$ is integrable, $\widehat{\omega_0}(0) = 0$ and $\nabla_\eta\,\widehat{\omega_0}(0) = (0,0)$ then the associated divergence-free vector-field $m_{0,\perp} = K_{BS} \star \omega_0$ satisfies for any multi-index $\sigma$*

$$\lim_{t\to\infty}\; t^{1-\frac{1}{p}+\frac{|\sigma|}{2}}\;\|\, D^\sigma K_\mu(t) \star m_{0,\perp}\,\|_p \;=\; 0\;, \qquad 2 \leq p \leq \infty\;. \tag{67}$$

**Proof.** The conclusion of the corollary has already been proved when moreover $(1+|\cdot|^2)\,\omega_0$ is integrable. We shall obtain Coollary 10 by a density argument. Let $\varepsilon > 0$. Choose $R_\varepsilon > 0$ such that truncating $\omega$ into function $\omega_\varepsilon$ vanishing on $\{x\,|\,|x| > R_\varepsilon\}$ and coinciding with $\omega_0$ on $\{x\,|\,|x| \leq R_\varepsilon\}$ yields

$$\|\,(1+|\cdot|)\,(\omega_0 - \omega_\varepsilon)\,\|_1 \quad \leq \quad \varepsilon\;,$$

hence $|\widehat{\omega_\varepsilon}(0)| \leq C\,\varepsilon$ and $|\nabla_\eta\,\widehat{\omega_\varepsilon}(0)| \leq C\,\varepsilon$. Now define

$$\omega_{app} \quad = \quad \omega_\varepsilon - [\,\widehat{\omega_\varepsilon}(0)]\;G - \mathrm{i}\,[\partial_{\eta_1}\,\widehat{\omega_\varepsilon}(0)]\;F_1 - \mathrm{i}\,[\partial_{\eta_2}\,\widehat{\omega_\varepsilon}(0)]\;F_2$$

to obtain a function $\omega_{app}$ localised as a Gaussian function satisfying $\widehat{\omega_{app}}(0) = 0$, $\nabla_\eta\,\widehat{\omega_{app}}(0) = (0,0)$ and

$$\|\,(1+|\cdot|)\,(\omega_0 - \omega_{app})\,\|_1 \quad \leq \quad C\,\varepsilon\;.$$

Let $2 \leq p \leq \infty$ and $\sigma$ be a multi-index. The second part of Proposition 8 yields a $t_\varepsilon > 0$ such that for $t \geq t_\varepsilon$

$$t^{1-\frac{1}{p}+\frac{|\sigma|}{2}}\;\|\, D^\sigma K_\mu(t) \star K_{BS} \star \omega_{app}\,\|_p \quad \leq \quad \varepsilon\;.$$

Then, with a constant independent of $\varepsilon$, the triangle inequality together with the first part of Proposition 8 give for any $t \geq t_\varepsilon$

$$t^{1-\frac{1}{p}+\frac{|\sigma|}{2}}\;\|\, D^\sigma K_\mu(t) \star m_{0,\perp}\,\|_p \quad \leq \quad C\,\varepsilon\;,$$

which achieves the proof and the present section. ∎

# 2 Non-linear terms

Now taking advantage of estimates for the Green kernel $S$ of the linearised system (21) we prove Theorem 1. It only remains to bound non-linear terms. This task shall be performed in two steps. First we establish estimates of $X(t) = (\rho(t)-\rho_\star, m(t))$ and non-linear terms in Lebesgue spaces $L^p(\mathbf{R}^2)$, for $2 \leq p \leq \infty$, by a continuity fix-point-like argument. Then we use these bounds to estimates non-linear terms in $L^p(\mathbf{R}^2)$, for $1 \leq p < 2$.

For the sake of conciseness, write

$$X(t) \;=\; S(t) \star X_0 \;+\; X^{NL}(t) \;.$$

Then

$$X^{NL}(t) \;=\; \sum_{k=1}^{2} \int_0^t S(t-t') \star \partial_k\, Q_k(t')\, dt' \tag{68}$$

where, for $k = 1, 2$,

$$Q_k \;=\; Q_k^1 + Q_k^2 \;, \qquad Q_k^1 \;=\; \begin{pmatrix} 0 \\ q_k^1 \end{pmatrix} \;, \qquad Q_k^2 \;=\; \sum_{k'=1}^{2} \begin{pmatrix} 0 \\ \partial_{k'}\, q_k^{2,k'} \end{pmatrix} \;,$$

in such a way that

$$\sum_{k=1}^{2} \partial_k\, q_k^1 \;=\; -\operatorname{div}\Big(\, m \otimes \frac{m}{1+\widetilde{\rho}} \,\Big) - \nabla\left(P\,(1+\widetilde{\rho}) - c^2\,\widetilde{\rho}\right)$$

$$\sum_{k,k'=1}^{2} \partial_k\, \partial_{k'}\, q_k^{2,k'} \;=\; -\,\mu \,\triangle \Big(\, \frac{m\,\widetilde{\rho}}{1+\widetilde{\rho}} \,\Big) \;-\; (\mu+\lambda)\ \nabla \operatorname{div} \Big(\, \frac{m\,\widetilde{\rho}}{1+\widetilde{\rho}} \,\Big) \;.$$

## 2.1 The case $p \geq 2$

As was already mentioned, the Green kernel $S$ does not regularise enough to enable us to deal ingenuously with all terms arising in $X^{NL}$ and we resort to Kawashima's estimates [10]. Again note that doing so we bound some quantities by constants regardless of their natural decay rates.

**Theorem 11 (Kawashima, 1983 [10])** *Let $s \geq 3$ be an integer. There exist $\varepsilon_0 > 0$ and $C > 0$ such that if $X_0 = (\widetilde{\rho}_0, m_0)$ belongs to $H^s(\mathbf{R}^2)$ with*

$$E \;=\; |\, X_0\,|_s \;\leq\; \varepsilon_0$$

*then system (68) has a unique global classical solution $X = (\widetilde{\rho}, m)$ of initial datum $X_0$, satisfying for any time $t \geq 0$*

$$|\, X(t)\,|_s^2 \;+\; \int_0^t |\, \nabla\, X(t')\,|_{s-1}^2\ dt' \quad\leq\quad C\,E^2 \;.$$

As in Theorem 1 we assume $s \geq 5$ and prove, when

$$E \;=\; |\, X_0\,|_s \;+\; \|\, X_{0,\shortparallel}\,\|_1 \;+\; \|\,(1+|\cdot|)\operatorname{rot} m_0\,\|_1$$

is small, that for any $2 \leq p \leq \infty$ and any multi-index $\sigma$ such that $|\sigma| \leq s-4$

$$\|\, D^\sigma X(t)\,\|_p \;\leq\; CE\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)\right)} \tag{69}$$

$$\|\, D^\sigma X^{NL}(t)\,\|_p \;\leq\; CE^2\ \ln(1+t)\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}\right)} \;. \tag{70}$$

For this purpose, following [8], we introduce

$$
\begin{aligned}
A(t) &= \sup_{\substack{0\le t'\le t\\ 2\le p\le\infty\\ |\sigma|\le s-4}} (1+t')^{1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)} \parallel D^\sigma X(t') \parallel_p \\
B(t) &= \sup_{\substack{0< t'\le t\\ 2\le p\le\infty\\ |\sigma|\le s-4}} \frac{(1+t')^{1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}}}{\ln(1+t')} \parallel D^\sigma X^{NL}(t) \parallel_p \,.
\end{aligned}
$$

The present subsection is essentially devoted to the proof of the following inequality

$$
B(t) \quad\le\quad C\ (\ E^2 + A(t)^2 + A(t)^{s-2}\ )\ . \tag{71}
$$

Together with linear estimates it shall yield

$$
A(t) \quad\le\quad C\ (\ E + A(t)^2 + A(t)^{s-2}\ )
$$

enabling us to propagate, whenever $2\,C\,E<1$ and $4\,C^2 E < 1/2$, both $A(t) + A(t)^{s-3} \le 1/2$ and

$$
A(t)\ \le\ \frac{C\,E}{1-A(t)-A(t)^{s-3}}\ \le\ 2\,C\,E\ ,
$$

which may be plugged in (71). Therefore as for the purpose of the present subsection it is enough to prove (71).

In order to establish (71) divide $S$ into a low-frequency part $S^{LF}$ and a high-frequency part $S^{HF}$ as was done for $S_{\shortparallel}$ in (43) and split $X^{NL}$ into

$$
\begin{aligned}
X^{NL}(t) &= \sum_{k=1}^{2}\int_0^{t/2} S^{LF}(t-t')\star\partial_k\, Q_k(t')\,dt' \\
&+ \sum_{k=1}^{2}\int_{t/2}^{t} S^{LF}(t-t')\star\partial_k\, Q_k^1(t')\,dt' \\
&+ \sum_{k,k'=1}^{2}\int_{t/2}^{t} S^{LF}(t-t')\star\partial_k\,\partial_{k'}\, Q_k^{2,k'}(t')\,dt' \\
&+ \sum_{k=1}^{2}\int_0^{t} S^{HF}(t-t')\star\partial_k\, Q_k(t')\,dt' \\
&= X_1^{NL}(t) + X_2^{NL}(t) + X_3^{NL}(t) + X_4^{NL}(t)\ .
\end{aligned} \tag{72}
$$

Let $2\le p\le\infty$ and $\sigma$ a multi-index such that $|\sigma|\le s-4$.

**1.** For some multi-indices $\sigma'$ of length $|\sigma'| = |\sigma| + 1$ Young's inequality yields

$$
\begin{aligned}
\| D^\sigma X_1^{NL}(t) \|_p &\leq \sum_{\sigma'} \int_0^{t/2} \| D^{\sigma'} S^{LF}(t-t') \star Q(t') \|_p \, dt' \\
&\leq C \sum_{\sigma'} \int_0^{t/2} \| D^{\sigma'} S^{LF}(t-t') \|_p \, \| Q(t') \|_1 \, dt' \\
&\leq C \, (1 + t/2)^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2} + \frac{1}{2}\right)} \int_0^{t/2} \| Q(t') \|_1 \, dt' \ .
\end{aligned}
$$

As $X(t)$ is bounded in $L^\infty(\mathbf{R}^2)$ thanks to Sobolev' embeddings and Theorem 11, from Theorem 11 may be derived

$$
\begin{aligned}
\int_0^t \| Q(t') \|_1 \, dt' &\leq C \int_0^t ( \| X(t') \|_2^2 + \| \nabla X(t') \|_2^2 ) \ dt' \\
&\leq C \int_0^t ( A(t')^2 (1+t')^{-1} + \| \nabla X(t') \|_2^2 ) \ dt' \\
&\leq C \, ( E^2 + A(t)^2 ) \ \ln(1+t) \qquad (73)
\end{aligned}
$$

by using, when $0 \leq t \leq 1$, $\int_0^t \| \nabla X \|_2^2 \leq C \, E^2 \, t$. Thereby

$$
\| D^\sigma X_1^{NL}(t) \|_p \ \leq \ C \, (E^2 + A(t)^2) \ \ln(1+t) \ (1+t)^{-\left(1 - \frac{1}{p} + \frac{|\sigma|}{2} + \frac{1}{2}\right)}. \qquad (74)
$$

**2.** When defining $1 \leq r \leq 2$ by $1 + 1/p = 1/2 + 1/r$, Young's inequality yields

$$
\begin{aligned}
\| D^\sigma X_2^{NL}(t) \|_p &\leq \sum_{k=1}^{2} \int_{t/2}^{t} \| \partial_k \, S^{LF}(t-t') \star D^\sigma Q_k^1(t') \|_p \, dt' \\
&\leq C \int_{t/2}^{t} \| \nabla \, S^{LF}(t-t') \|_2 \, \| D^\sigma Q^1(t') \|_r \, dt' \\
&\leq C \int_{t/2}^{t} (1 + t - t')^{-1} \ \| D^\sigma Q^1(t') \|_r \, dt' \ .
\end{aligned}
$$

Now for such an $r$, Hölder's inequalities combined with Leibniz' rule for differentiation give

$$
\begin{aligned}
\| D^\sigma Q^1(t) \|_r &\leq C \sum_{\sum |\sigma_i| = |\sigma|} \| D^{\sigma_1} X(t) \|_p \, \| D^{\sigma_2} X(t) \|_2 \prod_{i \geq 3} \| D^{\sigma_i} X(t) \|_\infty \\
&\leq C \, (A(t)^2 + A(t)^{\max(|\sigma|, 2)}) \\
&\quad \times \, (1+t)^{-\left(1 - \frac{1}{p} + \frac{1}{2} + \min\limits_{\sum |\sigma_i| = |\alpha|} \sum\limits_i \frac{1}{2} \min(|\sigma_i|, \, s - 4 - |\sigma_i|)\right)} \\
&\leq C \, (A(t)^2 + A(t)^{\max(|\sigma|, 2)}) \ (1+t)^{-\left(1 - \frac{1}{p} + \frac{1}{2} \min(|\sigma|, \, s - 4 - |\sigma|) + \frac{1}{2}\right)},
\end{aligned}
$$

the last inequality being proved in the following way: consider $\sigma_i$'s such that $\sum |\sigma_i| = |\alpha|$, then either all $\sigma_i$'s satisfies $|\sigma_i| \leq (s-4)/2$ and

$$
\sum_i \ \min(|\sigma_i|, \, s - 4 - |\sigma_i|) \ = \ \sum_i \ |\sigma_i| \ = \ |\sigma| \ ,
$$

or there is a $\sigma_{i_0}$ such that $|\sigma_{i_0}| > (s-4)/2$ hence

$$\sum_i \min(|\sigma_i|,\, s-4-|\sigma_i|) \geq \min(|\sigma_{i_0}|,\, s-4-|\sigma_{i_0}|) = s-4-|\sigma_{i_0}| \geq s-4-|\sigma|\,.$$

Therefore

$$\begin{aligned} \|\, D^\sigma X_2^{NL}(t)\,\|_p \quad &\leq \quad C\,(A(t)^2 + A(t)^{\max(|\sigma|,2)})\ \ln(1+t) \\ &\quad \times\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)+\frac{1}{2}\right)}. \end{aligned} \tag{75}$$

**3.** Assume first that $\sigma$ is non-zero. Again letting $1 \leq r \leq 2$ be such that $1+1/p = 1/2+1/r$ Young's inequality yields for some $\sigma'$ such that $|\sigma'| = |\sigma|-1$

$$\begin{aligned} \|\, D^\sigma X_3^{NL}(t)\,\|_p \quad &\leq \quad C \sum_{\sigma'} \int_{t/2}^t \|\, D^2\, S^{LF}(t-t')\,\|_2\ \|\, D^{\sigma'} Q^2(t')\,\|_r\, dt' \\ &\leq \quad C \sum_{\sigma'} \int_{t/2}^t (1+t-t')^{-3/2}\ \|\, D^{\sigma'} Q^2(t')\,\|_r\, dt'\ . \end{aligned}$$

Now again

$$\|\, D^{\sigma'} Q^2(t')\,\|_r \ \leq\ C\,(A(t)^2 + A(t)^{\max(|\sigma|,2)})\,(1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)+\frac{1}{2}\right)}.$$

Therefore when $|\sigma| \neq 0$

$$\begin{aligned} \|\, D^\sigma X_3^{NL}(t)\,\|_p \quad &\leq \quad C\,(A(t)^2 + A(t)^{\max(|\sigma|,2)})\ \min(1,t) \\ &\quad \times\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)+\frac{1}{2}\right)}. \end{aligned} \tag{76}$$

Putting one derivative less on $S^{LF}$ it may also be proved that

$$\|\, X_3^{NL}(t)\,\|_p \quad \leq \quad C\,A(t)^2\ \ln(1+t)\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(1,\ s-5)+\frac{1}{2}\right)}. \tag{77}$$

**4.** For some $\sigma'$ such that $|\sigma'| = |\sigma|+1$

$$\|\, D^\sigma X_4^{NL}(t)\,\|_p \quad \leq \quad C \sum_{\sigma'} \int_0^t e^{-b(t-t')}\,\|\, D^{\sigma'} Q(t')\,\|_p\ dt'\ .$$

Now on the one hand when $|\sigma'| \leq (s-4)$

$$\begin{aligned} \|\, D^{\sigma'} Q^1(t)\,\|_p \quad &\leq \quad C\,(A(t)^2 + A(t)^{\max(|\sigma|+1,2)}) \\ &\quad \times\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|+1,\, s-5-|\sigma|)+1\right)} \end{aligned}$$

and when $|\sigma'| \leq (s-5)$

$$\begin{aligned} \|\, D^{\sigma'} Q^2(t)\,\|_p \quad &\leq \quad C\,(A(t)^2 + A(t)^{|\sigma|+2}) \\ &\quad \times\ (1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|+2,\, s-6-|\sigma|)+1\right)}. \end{aligned}$$

And on the other hand when $|\sigma'| = (s-3)$ Sobolev embeddings and Theorem 11 yield

$$\|\, D^{\sigma'} Q(t)\,\|_p \quad \leq \quad C\ A(t)\ (E + A(t) + A(t)^{|\sigma|+1})\ (1+t)^{-\left(1-\frac{1}{p}\right)}.$$

At last, when $|\sigma'| = (l-2)$, $\| D^{\sigma'} Q^2(t) \|_p$ may be bounded by interpolation in such a way that in any case stands

$$\begin{aligned} \| D^{\sigma'} Q(t) \|_p \quad \leq \quad & C\,(\,E^2 + A(t)^2 + A(t)^{|\sigma|+2}) \\ & \times\,(1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\,s-5-|\sigma|)+\frac{1}{2}\right)}. \end{aligned} \tag{78}$$

It should be emphasised that the former estimates are the critical ones leading to some loss of decay in the estimates. Observe now that for $\gamma > 1$

$$\begin{aligned} \int_0^{t/2} e^{-b(t-t')}\,(1+t')^{-\gamma}\,dt' \quad & \leq \quad C\,e^{-bt/2} \\ \int_{t/2}^{t} e^{-b(t-t')}\,(1+t')^{-\gamma}\,dt' \quad & \leq \quad C\,(1+t)^{-\gamma}. \end{aligned}$$

Therefore gathering everything leads to

$$\begin{aligned} \| D^{\sigma} X_4^{NL}(t) \|_p \quad \leq \quad & C\,(\,E^2 + A(t)^2 + A(t)^{|\sigma|+2}) \\ & \times\,(1+t)^{-\left(1-\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\,s-5-|\sigma|)+\frac{1}{2}\right)}. \end{aligned} \tag{79}$$

This achieves the proof of (71). $\square$

## 2.2 The case $p < 2$

It only remains to bound $X^{NL}(t)$ and its derivatives in $L^p(\mathbf{R}^2)$, for $1 \leq p \leq 2$. An important point is that we shall only make use of bounds of $X(t)$ in $L^p(\mathbf{R}^2)$ for $2 \leq p \leq \infty$ thus we do not need to assume $X(t)$ to be integrable !

Our aim is to prove for any multi-index $\sigma$ such that $|\sigma| \leq (l-2)$ and any index $1 \leq p \leq 2$

$$\| D^{\sigma} X^{NL}(t) \|_p \,\leq\, C\,E^2\,\ln(1+t)\,\,(1+t)^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\,s-5-|\sigma|)+\frac{1}{2}\right)}\;. \tag{80}$$

Since $(\widehat{S}(t))_{0\leq t\leq 2}$ is a family of bounded strong $L^p$-multipliers and Kawashima's theorem gives through Sobolev' embeddings $\| D^{\sigma} Q(t) \|_p \;\leq\; C\,E^2$ whenever $|\sigma| \leq (s-4)$, then does stand

$$\| D^{\sigma} X^{NL}(t) \|_p \quad \leq \quad C\,E^2\,t\;, \qquad 0 \leq t \leq 2\,, \quad 1 \leq p \leq \infty\;.$$

The point is therefore in dealing with $X^{NL}(t)$ and its derivatives for $t \geq 2$.

For this purpose, having in mind (37), we introduce $\widetilde{S}$ defined by

$$\widetilde{S} \quad = \quad \widetilde{S}_{\shortparallel} \star \begin{bmatrix} \delta_0 & 0 \\ 0 & R_{\shortparallel} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{\mu} \star R_{\perp} \end{bmatrix}\,, \tag{81}$$

where $\widetilde{S}_{\shortparallel}$ is the Green kernel of the artificial viscosity system (48). Then $\widetilde{S}$ is the Green kernel of the artificial viscosity system

$$\left.\begin{aligned} \partial_t\,\widetilde{\rho} \;\;+\;\; \mathrm{div}\,m \;\;&=\;\; (\mu+\tfrac{\lambda}{2})\,\triangle\,\widetilde{\rho} \\ \partial_t\,m \;\;+\;\; c^2\,\nabla\widetilde{\rho} \;\;&=\;\; \mu\,\triangle\,m \;+\; \tfrac{\lambda}{2}\,\nabla\,\mathrm{div}\,m \end{aligned}\right\} \tag{82}$$

and should well approach $S$. Indeed Proposition 6 still holds when replacing $S_{\shortparallel}$ with $S$ and $\widetilde{S}_{\shortparallel}$ with $\widetilde{S}$. Now split $S$ and $\widetilde{S}$ into high-frequency and low-frequency parts and divide $X^{NL}$ into

$$\begin{aligned}
X^{NL}(t) &= \sum_{k=1}^{2} \int_{t-1}^{t} S(t-t') \star \partial_k Q_k(t')\, dt' \\
&+ \sum_{k=1}^{2} \int_{0}^{t/2} \widetilde{S}(t-t') \star \partial_k Q_k(t')\, dt' \\
&+ \sum_{k=1}^{2} \int_{t/2}^{t-1} \widetilde{S}(t-t') \star \partial_k Q_k^1(t')\, dt' \\
&+ \sum_{k,k'=1}^{2} \int_{t/2}^{t-1} \widetilde{S}(t-t') \star \partial_k \partial_{k'} Q_k^{2,k'}(t')\, dt' \\
&+ \sum_{k=1}^{2} \int_{0}^{t/2} (S^{LF} - \widetilde{S}^{LF})(t-t') \star \partial_k Q_k(t')\, dt' \\
&+ \sum_{k=1}^{2} \int_{t/2}^{t-1} (S^{LF} - \widetilde{S}^{LF})(t-t') \star \partial_k Q_k^1(t')\, dt' \\
&+ \sum_{k,k'=1}^{2} \int_{t/2}^{t-1} (S^{LF} - \widetilde{S}^{LF})(t-t') \star \partial_k \partial_{k'} Q_k^{2,k'}(t')\, dt' \\
&+ \sum_{k=1}^{2} \int_{0}^{t-1} (S^{HF} - \widetilde{S}^{HF})(t-t') \star \partial_k Q_k(t')\, dt' \\
&= X_1^{NL}(t) + \cdots + X_8^{NL}(t)\ . \qquad (83)
\end{aligned}$$

Let $1 \le p \le 2$ and $\sigma$ be such that $|\sigma| \le (s-4)$.

**1.** Since $(\widehat{S}(t))_{0 \le t \le 1}$ is a bounded family of strong $L^p$-multipliers, we may obtain

$$\begin{aligned}
\| D^\sigma X_1^{NL}(t) \|_p &\le C \int_{t-1}^{t} \| \nabla D^\sigma Q(t') \|_p \, dt' \\
&\le C\ E^2 \int_{t-1}^{t} (1+t')^{-\left(1-\frac{1}{p}+\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}\right)}\, dt' \\
&\le C\ E^2\ t^{-\left(1-\frac{1}{p}+\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}\right)}\ , \qquad (84)
\end{aligned}$$

where $\| D^\sigma Q(t') \|_p$ is bounded mainly as was established (78) in the former subsection.

**2.** When $t \geq 2$, for some $\sigma'$ such that $|\sigma'| = |\sigma| + 1$, a Young inequality yields

$$
\begin{aligned}
\| D^\sigma X_2^{NL}(t) \|_p &\leq C \sum_{\sigma'} \int_0^{t/2} \| D^{\sigma'} \widetilde{S}(t-t') \|_p \, \| Q(t') \|_1 \, dt' \\
&\leq C \, t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{|\sigma|}{2} + \frac{1}{2}\right)} \int_0^{t/2} \| Q(t') \|_1 \, dt' \\
&\leq C \, E^2 \, \ln(1+t) \; t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{|\sigma|}{2} + \frac{1}{2}\right)} \,, \qquad (85)
\end{aligned}
$$

when using (73) with $A(t) \leq C\,E$.

**3.** Since $5/4 - 3/2p + 1/2 \leq 1$, Hölder's and Young's inequalities yield through a change of variables

$$
\begin{aligned}
\| D^\sigma X_3^{NL}(t) \|_p &\leq C \int_{t/2}^{t-1} \| \nabla \widetilde{S}(t-t') \|_p \, \| D^\sigma Q^1(t') \|_1 \, dt' \\
&\leq CE^2 \int_{t/2}^{t-1} \Big[ (t-t')^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}\right)} \\
&\qquad\qquad \times (1+t')^{-\left(1 + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)\right)} \Big] \, dt' \\
&\leq CE^2 \, t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2} + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)\right)} \int_{1-\frac{1}{t}}^{1/2} \frac{dt''}{t''^{\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}}} \\
&\leq CE^2 \ln(1+t) \;\; t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|) + \frac{1}{2}\right)}, \qquad (86)
\end{aligned}
$$

where $\| D^\sigma Q^1(t') \|_1$ has been estimated mainly as was $\| D^\sigma Q^1(t') \|_p$ in the former subsection.

**4.** When $\sigma$ is non-zero acting in quite the same way leads for some multi-indices $\sigma'$ such that $|\sigma'| = |\sigma| - 1$ to

$$
\begin{aligned}
\| D^\sigma X_4^{NL}(t) \|_p &\leq C \int_{t/2}^{t-1} \| D^2 \widetilde{S}(t-t') \|_p \, \| D^{\sigma'} Q^2(t') \|_1 \, dt' \\
&\leq CE^2 \int_{t/2}^{t-1} \Big[ (t-t')^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + 1\right)} \\
&\qquad\qquad \times (1+t')^{-\left(1 + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)\right)} \Big] \, dt' \\
&\leq CE^2 \int_{t/2}^{t-1} \Big[ (t-t')^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}\right)} \\
&\qquad\qquad \times (1+t')^{-\left(1 + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)\right)} \Big] \, dt' \\
&\leq CE^2 \ln(1+t) \;\; t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|) + \frac{1}{2}\right)}, \qquad (87)
\end{aligned}
$$

where has been used $(t-t')^{-\frac{1}{2}} \leq 1$ in the integrand. In a similar way, since $5/4 - 3/2p + 1/2 \leq 1/2$,

$$
\begin{aligned}
\| X_4^{NL}(t) \|_p &\leq C \int_{t/2}^{t-1} \| \nabla \widetilde{S}(t-t') \|_p \, \| Q^2(t') \|_1 \, dt' \\
&\leq CE^2 \ln(1+t) \;\; t^{-\left(\frac{5}{4} - \frac{3}{2}\frac{1}{p} + \frac{1}{2}\min(1,\, s-5) + \frac{1}{2}\right)}. \qquad (88)
\end{aligned}
$$

**5.** By proceeding as for $X_2^{NL}$ may be obtained when $t \geq 2$

$$\| D^\sigma X_5^{NL}(t) \|_p \quad \leq \quad C\, E^2 \, \ln(1+t) \;\; t^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{|\sigma|}{2}+1-\theta\right)}, \tag{89}$$

for some $0 < \theta \leq 1/2$.

**6.** Proceeding as for $X_3^{NL}$ and taking into account $(t-t')^{-(1/2-\theta)} \leq 1$, when $0 < \theta \leq 1/2$, in the integrand lead to

$$\| D^\sigma X_6^{NL}(t) \|_p \;\leq\; C \; E^2 \;\; \ln(1+t) \;\; t^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, s-4-|\sigma|)+\frac{1}{2}\right)}. \tag{90}$$

**7.** Proceeding as for $X_4^{NL}$ and taking into account $(t-t')^{-(1-\theta)} \leq 1$, whenever $0 < \theta \leq 1/2$, in the integrand give

$$\| D^\sigma X_7^{NL}(t) \|_p \;\leq\; C\, E^2 \;\ln(1+t) \;\; t^{-\left(\frac{5}{4}-\frac{3}{2}\frac{1}{p}+\frac{1}{2}\min(|\sigma|,\, l-2-|\sigma|)+\frac{1}{2}\right)}. \tag{91}$$

**8.** The high-frequency study yields

$$\begin{aligned} \| D^\sigma X_8^{NL}(t) \|_p \quad &\leq \quad C \int_0^{t-1} e^{-b(t-t')} \, \| \nabla D^\sigma Q(t') \|_p \, dt' \\ &\leq \quad C\, E^2 \int_0^t e^{-b(t-t')} \, (1+t')^{-\left(1-\frac{1}{p}+\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}\right)} \, dt'. \end{aligned}$$

Now observe when $\gamma \geq 0$ for any time $t \geq 0$

$$\begin{aligned} \int_0^{t/2} e^{-b(t-t')} \, (1+t')^{-\gamma} \, dt' \quad &\leq \quad C \; t \; e^{-bt/2} \;, \\ \int_{t/2}^{t} e^{-b(t-t')} \, (1+t')^{-\gamma} \, dt' \quad &\leq \quad C \; (1+t)^{-\gamma}. \end{aligned}$$

Thereby

$$\| D^\sigma X_8^{NL}(t) \|_p \quad \leq \quad C \; E^2 \;\; (1+t)^{-\left(1-\frac{1}{p}+\min(|\sigma|,\, s-5-|\sigma|)+\frac{1}{2}\right)}. \tag{92}$$

This achieves the proof of Theorem 1. Estimate (25) comes from (69) and (80). Estimates (26) and (27) are consequences of estimate (25) and linear estimates of Propositions 3, 4, 5 & 6. Estimate (26) may be derived from estimate (25) and estimates of Propositions 7 & 8. Equality (29) is deduced from estimate (25) and Corollary 10. At last estimate (31) may be obtained from estimate (25) and Proposition 9. $\square$

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