Summary of the thesis

The index theorem of Atiyah and Singer, discovered in 1963, is a striking result which relates many different fields in mathematics going from the analysis of partial differential equations to differential topology and geometry. To be more precise, this theorem relates the dimension of the space of some elliptic partial differential equations and topological invariants coming from (co)homology theories, and has important applications. Many major results from different fields (algebraic topology, differential topology, functional analysis) may be seen as corollaries of this result, or obtained from techniques developed in the framework of index theory. On another side, zeta functions associated to pseudodifferential operators on a closed Riemannian manifold contain in their analytic properties many interesting informations. For instance, the Weyl theorem on the asymptotic number of eigenvalues of a Laplacian may be recovered within the residues of the zeta function. This gives in particular the volume of the manifold, which is a geometric data. Using the framework of noncommutative geometry developed by Connes, this idea may be pushed further, yielding index theorems in the spirit of the one of Atiyah-Singer. The interest in this viewpoint is to be suitable for more delicate geometrical situations. The present thesis establishes results in this direction.

The first chapter of the thesis aims at obtaining a general local index formula for "abstract elliptic operators". These formulas are derived from a cyclic cocycle expressed in terms of zeta functions residues, constructed by combining zeta functions renormalization techniques together with the excision property in cyclic cohomology. The formula also applies when the zeta function has multiple poles. We then relate this cocycle to the Chern-Connes character.

Chapters 2 and 3 establish index theorems for hypoelliptic operators in the Heisenberg calculus on foliations. We first get a result for these operators on $\mathbb{R}^n$. The idea is to retract the cocycle obtained in Chapter 1 to a cocycle depending only on the principal symbol of the operators. We give two ways of constructing this : one uses once more excision, the other one relies on the algebra cochains theory of Quillen. In the following, we extend this result to the more general situation of a discrete group acting on a foliation by foliated diffeomorphisms, in a joint work with D. Perrot. As a corollary, we give a new solution to a problem given by Connes and Moscovici, raising the question of computing the Chern character of the transverse fundamental class on a foliation.

The last chapter discusses some results on manifolds with conic singularities, and illustrates the results of Chapter 1 in the case where the zeta function exhibits double poles. Nevertheless, it should be noted that the index formula is no more local due to the singular situation considered. This phenomena is analogous to the apparition of the eta invariant in the case of Dirac operators, but may be developed for more general pseudodifferential operators.

KEYWORDS. Noncommutative geometry, index theory, foliations, conical manifolds, K-theory, cyclic (co)homology, hypoelliptic operators.

Introduction

1. Some historical points on index theory

The point of departure of index theory is the Atiyah-Singer theorem, proved in 1963 in [**2**]. The strength of this result lies in its interdisciplinary nature, relating many different mathematical fields (analysis, geometry, topology), by yielding equalities of the type

analytic term = topological term.

Historically, one may find many other instances of "index theorems" of that kind, giving an equality between terms of different nature. We shall first review some important examples of such results, and their consequences.

Gauss-Bonnet theorem (1848). Given a closed oriented Riemannian surface (M, g), the Gauss-Bonnet theorem consists in the following formula :

$$\chi(M) = \frac{1}{2\pi}\int_M k_g(x)dx,$$

where $\chi(M)$ denotes the Euler characteristic of M, and k_g the Gauss curvature of (M, g). The number $\chi(M)$ is a invariant of the *global topology* of M, whereas k_g is a geometric data. Indeed, the Theorema Egregium of Gauss implies that k_g depends only on the metric g and is therefore invariant under *local isometries*. Surprisingly, the theorem says that $\chi(M)$ is recovered by summation of k_g over all points of M. In other words, we have an equality between terms of different nature through a *local formula*.

This formula puts interesting constraints on the geometry of the surface. For example, let M be the 2-dimensional torus. The genus of M is 1 and $\chi(M) = 2 - 2 \cdot \mathrm{genus}(M)$, whence

$$\int_M k_g(x)dx = 0$$

As a consequence, there is no metric of positive Gauss curvature on the 2-torus. Combinations of vanishing theorems with index theorems offer some tools allowing to generalize this kind of arguments for the *positive scalar curvature problem*. In the same spirit as the Gauss-Bonnet theorem, we may think about the *Lichnerowicz theorem* for closed spin manifolds, see for example [**3**] for more details. *Noncommutative geometry* allows to push this reasoning even further, for example through the Baum-Connes conjecture and coarse index theory. For more details, one may consult the surveys of Schick [**37**] and Yu [**40**].

Fritz Noether theorem (1931). Let us now look at an example from operator theory, where the word "index" appears explicitely. Let $S^1 \subset \mathbb{C}$ be the unit circle in the complex plane, $L^2(S^1)$ be the space of square summable functions on S^1 and

$$H^2(S^1) = \left\{ z \longmapsto \sum_{n\geqslant 0} a_n z^n \,;\, \sum_{n\geqslant 0} |a_n|^2 < \infty \right\}$$

be *the Hardy space* on S^1. Let P denote the orthogonal projection from $L^2(S^1)$ onto $H^2(S^1)$. The space $C^1(S^1)$ of differentiable functions on the circle acts by multiplication on $L^2(S^1)$. If $u \in$

$C^1(S^1)$ nowhere vanishes on S^1, then PuP is invertible modulo compact operators, and is therefore a Fredholm [1] operator on $H^2(S^1)$. Fritz Noether's theorem computes its index :

$$\operatorname{Ind}(PuP) = -\operatorname{Winding}(u, 0) = -\frac{1}{2\pi \mathbf{i}} \int_{S^1} u^{-1} du.$$

We observe the left-hand-side is purely analytic, whereas the right-hand-side is completely topological. This formula is the first step of Brown-Douglas-Fillmore theory and K-homology **[5]**, which is of considerable importance for (noncommutative) index theory. We shall come back to this later in the memoir. For the moment, let us just mention that as an application of Brown-Douglas-Fillmore theory, we get a complete classification of essentially normal operators obtained as a compact perturbation of a normal operator on Hilbert spaces (see for example **[20]**). One relevant point here is the application of algebraic topology techniques to solve a problem of functional analysis.

Hirzebruch signature theorem (1956). We come back to differential geometry with a last example. Let M be a closed oriented manifold of dimension $n = 4k$. The *intersection form* of M is the following bilinear form on the de Rham cohomology :

$$H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta$$

The *signature* of M is defined as the number $\operatorname{Sign}(M) = p - q$, where (p, q) is the signature of the bilinear form above, and is a homotopy invariant. Then, Hirzebruch discovered that

$$\operatorname{Sign}(M) = \langle [M], L(M) \rangle$$

where $L(M)$ is the L-genus of Hirzebruch, which may be expressed as a polynomial on the Pontryagin classes of M. An interesting insight on this story is **[22]**. One important application of this theorem concerns the exotic spheres discovered by Milnor **[26]**, i.e manifolds homeomorphic, but non diffeomorphic to the euclidean sphere S^7 of dimension 7. Let us give the idea of the construction allowing to distinguish these differentiable structures. One constructs a manifold with boundary N of given intersection form (and thus signature), with boundary $\partial N = \Sigma^7$ homeomorphic to S^7, but not diffeomorphic. If it were, a smooth manifold $M = N \sqcup_\Sigma D^8$ could be constructed by gluing N and an 8-disk along Σ^7. The signature theorem would then give

$$\operatorname{Sign}(M) = \int_M \frac{1}{45}(7p_2 - p_1^2)$$

where the $p_i \in H^{4i}(M, \mathbb{Z})$ denote the Pontryagin classes of M. A contradiction arises for a subitable choice of N such that the right-hand-side is not an integer, taking into account that Pontryagin classes are integral.

Partial differential equations. In the sixties, Gel'fand suggested in **[16]** that the Fredholm index of an elliptic operator D, should be expressible in purely topological terms, arguing on its homotopy invariance. This idea was strengthened by *Bott periodicity*, which exhibits a periodicity in the higher homotopy groups of the classical groups. Indeed, if we take for instance

$$D = \sum_{i=1}^{n} a_i \partial_{x_i}$$

1. An operator $T : H \to H'$, where H and H' are Hilbert spaces, is *Fredholm* if its kernel $\operatorname{Ker} T$ and cokernel $\operatorname{Coker} T = H'/\operatorname{Im} T$ are finite dimensional. A theorem of Atkinson asserts that this is equivalent to the invertibility of T modulo compact operators. The *index* of T is then defined as the integer

$$\operatorname{Ind}(T) = \dim \operatorname{Ker}(T) - \dim \operatorname{Coker}(T).$$

a differential operator of order 1 on the torus $\mathbb{T}^n$ with constant matrix coefficients $a_i \in M_N(\mathbb{C})$, its symbol

$$\sigma_D(\xi) = \sum_{i=1}^{n} a_i \xi_i$$

is invertible for every $\xi \in \mathbb{R}^n \smallsetminus \{0\}$, and defines an element of the homotopy group $\pi_{n-1}(GL_N(\mathbb{C}))$. Moreover,

$$\pi_{n-1}(GL_N(\mathbb{C})) = \begin{cases} \mathbb{Z} \text{ if } n \text{ is even,} \\ 0 \text{ otherwise.} \end{cases}$$

In particular, the topological K-theory of Atiyah and Hirzebruch makes its first appearance. The question is then to know if the two different sources of integrality given above are related. The confirmation comes from the Atiyah-Singer theorem. It states that if M is a closed manifold, and $D : C^\infty(M, E) \to C^\infty(M, F)$ is a (pseudo)differential operator on M, with E and F being complex vector bundles over M, then D is Fredholm[2], and its index is given by a topological formula which depends only on its principal symbol σ_D. More precisely, $\mathrm{Ind}(D)$ is the evaluation of a specific cohomology class on the fundamental class in homology $[T^*M] \in H_\bullet(T^*M, \mathbb{C})$ of M :

$$\mathrm{Ind}(D) = \langle [T^*M], \mathrm{ch}_0[\sigma_D] \cup \mathrm{Td}(TM \otimes \mathbb{C}) \rangle$$

Let us explain the different terms of the formula. The leading symbol σ_D defines an isomorphism $\pi^*E \to \pi^*F$ outside a neighbourhood of the zero section, hence an element $[\sigma_D] = [\pi^*E, \pi^*F, \sigma_D]$ of the K-theory group $K^0(T^*M)$. Moreover, $\mathrm{ch}_0 : K^0(T^*M) \to H^{\mathrm{even}}(M, \mathbb{C})$ denotes the even Chern character and $\mathrm{Td}(TM \otimes \mathbb{C})$ the Todd class of the complexified cotangent bundle.

We may also give an odd version of the theorem. This requires $E = F$. Let S^*M denote the cotangent sphere of M and $\pi : S^*M \to M$ denote the canonical projection. By ellipticity, σ_D is an isomorphism of π^*E on S^*M, and thus defines an element $[\sigma_D]$ of the K-theory group $K^1(S^*M) = \pi_0(GL(C^\infty(S^*M))$. Atiyah-Singer formula restates as follows

$$\mathrm{Ind}(D) = \langle [S^*M], \mathrm{ch}_1[\sigma_D] \cup \pi^*\mathrm{Td}_{\mathbb{C}}(T^*M) \rangle$$

where $\mathrm{ch}_1 : K^1(S^*M) \to H^{\mathrm{odd}}(M, \mathbb{C})$ is the odd Chern character. We may retrieve the even version from the latter with Bott periodicity.

To illustrate the breakthrough this result made, let us mention that all the results given earlier in the Introduction are special cases of the Atiyah-Singer theorem. Gauss-Bonnet and Hirzebruch's signature theorems are respectively recovered from the even case with the following operators :

$$D = d + d^* : \Omega^{\mathrm{even}}(M) \longrightarrow \Omega^{\mathrm{odd}}(M),$$

where d is the de Rham differential and d^* its adjoint with respect to the metric on $\Omega^\bullet(M)$, and

$$D = d - *d* : S^{\mathrm{even}} \longrightarrow S^{\mathrm{odd}}$$

where $*$ is the Hodge star, which induces a $\mathbb{Z}_2$-graduation $\Omega^\bullet(M) = S^{\mathrm{even}} \oplus S^{\mathrm{odd}}$. For more details, the reader may consult **[3]**, **[15]** or **[17]**.

Noether theorem is obtained from the odd case when $M = S^1$: the Toeplitz operator may be written as an elliptic pseudodifferential operator of order 0. This point will be reviewed as a direct application of Chapter 1.

Nevertheless, many geometric situations leads to the necessity of developing index theory in more general settings. For example, the study of transverse situations on foliations gives index

2. as a bounded map $H^{s+\mathrm{ord}(D)}(M) \to H^s(M)$ between Sobolev spaces, for every $s \in \mathbb{R}$. The index does not depend on s.

problems for operators which are no more elliptic. The work of Atiyah and Singer does not apply directly here, but *Noncommutative Geometry* provides a framework allowing to extend their methods.

2. Towards Noncommutative Geometry

Atiyah quickly remarked that the analytic index may be interpreted as the result of an accurate pairing between K-theory and its "dual theory", called K-*homology*. To realize concretely this theory, Atiyah uses a description involving "abstract pseudodifferential operators" [**1**]. These ideas will be extended later with extensions of C^*-algebras by Brown-Douglas-Fillmore [**5**], and culminates with Kasparov's theory [**24**].

2.1. Fredholm modules and K-homology.

DEFINITION 1. An *odd Fredholm module* on an associative $*$-algebra $\mathcal{A}$ is given by :

(i) a $*$-representation of $\mathcal{A}$ as bounded operators on a (complex) Hilbert space H,

(ii) a self-adjoint operator $F : H \to H$ such that for every $a \in \mathcal{A}$, $a(F^2-1)$ and the commutator $[F, a]$ are compact operators.

An *even Fredholm module* on $\mathcal{A}$ consists of similar datas, with H endowed with a $\mathbb{Z}_2$-graduation ε, i.e $\varepsilon^2 = 1$. Elements of $\mathcal{A}$ are represented as even operators (i.e they commute with ε), and F is odd (i.e it anticommutes with ε).

REMARK 2. We may actually reduce to the case with $F^2 = 1$, modulo the equivalence relation defined in Kasparov's theory. See for example [**7**], Appendix 2.

The operator F is precisely the thing called "abstract pseudodifferential operator of order 0" by Atiyah. In the concrete case of a closed manifold M, let F be a pseudodifferential operator of order 0 and $f \in C^\infty(M)$, then the commutator $[F, f]$ has pseudodifferential order $\leqslant -1$, hence it is a compact operator.

We now consider the projection $P = \frac{1+F}{2}$. In these conditions, if $u \in \mathcal{A}$ is invertible, one easily sees that

$$PuP : PH \to PH$$

is invertible modulo compacts, and is consequently a Fredholm operator. We then get a group morphism between algebraic K-theory and $\mathbb{Z}$:

$$\mathrm{Ind}_F : K_1^{\mathrm{alg}}(\mathcal{A}) \to \mathbb{Z}, \quad [u] \mapsto \mathrm{Ind}(PuP) \tag{1}$$

There are also pairings of even Fredholm modules with $K_0^{\mathrm{alg}}(\mathcal{A})$. Let $(\mathcal{A}, H, F)$ be an even Freholm module. Note $H = H_+ \oplus H_-$ its decomposition with respect to the $\mathbb{Z}_2$-graduation ε. Let e be an idempotent of $\mathcal{A}$, then the operator

$$eFe : eH_+ \longrightarrow eH_-$$

is also Fredholm, giving rise to another group morphism :

$$\mathrm{Ind}_F : K_0^{\mathrm{alg}}(\mathcal{A}) \longrightarrow \mathbb{Z}, \quad [e] \longmapsto \mathrm{Ind}(eFe : eH_+ \to eH_-) \tag{2}$$

EXAMPLE 3.

(i) The triple $(C^\infty(S^1), L^2(S^1), F = D|D|^{-1})$, where $D = \frac{1}{i}\frac{d}{dt}$ and F is extended by 1 on $\mathrm{Ker}(D)$, defines the Fredholm module corresponding to the Noether theorem : the projection $P = \frac{1+F}{2}$ is precisely the orthogonal projection of $L^2(S^1)$ onto the Hardy space $H^2(S^1)$.

(ii) More generally, one may replace S^1 by any closed manifold M and D by any pseudodifferential operator D.

(iii) An important example of even Fredholm module is $(C^\infty(M), L^2(M,S), F = D|D|^{-1})$, where M is a closed spin manifold, S the bundle of spinors and D a Dirac operator.

We shall need an additional assumption for Fredholm modules. For $p \geqslant 0$, recall that the p-*th Schatten class* $\ell^p(H)$ on H is the two-sided ideal of the algebra $\mathcal{B}(H)$ of bounded operators on H

$$\ell^p(H) = \{T \in \mathcal{B}(H)\,;\ \mathrm{Tr}\,|T|^p < \infty\}.$$

DEFINITION 4. A Fredholm module $(\mathcal{A}, H, F)$ is said *finitely summable* if there exists $p \geqslant 0$ such that for any $a \in \mathcal{A}$, $[F, a]$ belongs to $\ell^p(H)$.

REMARK 5. In that case, for every $a_0, \dots, a_k \in \mathcal{A}$ with $k \geqslant p$,

$$[F, a_0] \dots [F, a_k]$$

is a trace-class operator on H.

EXAMPLE 6. The Fredholm module $(C^\infty(S^1), L^2(S^1), F)$ of the example above is p-summable for all $p > 1$. It also holds for $p = 1$ but this is not generic : a Fredholm module associated to a closed manifold M is p-summable in general for $p > \dim M$.

In **[7]**, Connes observed that given an odd Fredholm module $(\mathcal{A}, H, F)$, where $\mathcal{A}$ *is not* a C^*-algebra, the formula

$$\phi_{2k+1}(a_0, \dots, a_{2k+1}) = \mathrm{Tr}(a_0[F, a_1] \dots [F, a_{2k+1}])$$

where k is an integer such that $2k+1 \geqslant p$, $a_0, \dots, a_{2k+1} \in \mathcal{A}$, defines a cyclic cocycle[3] on $\mathcal{A}$, giving rise to the following index formula :

$$\mathrm{Ind}(PuP : PH \to PH) = \frac{(-1)^{k+1}}{2^{2k+1}} \phi_{2k+1}(u^{-1}, u, \dots, u^{-1}, u) \tag{3}$$

where $P = \frac{1+F}{2}$, and u is an invertible in $\mathcal{A}$.

For an even Fredholm module $(\mathcal{A}, H, F)$, one considers the following cyclic cocycle

$$\phi_{2k}(a_0, \dots, a_{2k}) = \mathrm{Tr}(\varepsilon a_0[F, a_1] \dots [F, a_{2k}])$$

with k integer such that $2k \geqslant p$, $a_0, \dots, a_{2k+1} \in \mathcal{A}$, leading to :

$$\mathrm{Ind}(eFe : eH_+ \to eH_-) = (-1)^k \phi_{2k}(e, \dots, e) \tag{4}$$

where e is an idempotent in $\mathcal{A}$.

When $\mathcal{A}$ is a C^*-algebra, we adopt the terminology K-*cycle* rather than Fredholm module. Considering a certain equivalence relation that we do not describe here, the set of odd K-cycles on a C^*-algebra A form a group $K^1(A)$ called *odd* K-*homology group* of A. There is also an *even* K-*homology group* $K^0(A)$, formed by even K-cycles. For more details, the reader may consult for

3. A *cyclic* p-*cocycle* on $\mathcal{A}$ is a $(p+1)$-linear form ϕ such that

(i) $\phi(a_0, \dots, a_p) = (-1)^p \phi(a_p, a_0, \dots, a_{p-1}), \forall a_0, \dots, a_p \in \mathcal{A}$

(ii) $b\phi = 0$, where b is the Hochschild coboundary.

The space of cyclic cocycles on $\mathcal{A}$ endowed with the Hochschild coboundary forms a differential complex, its cohomology is the *cyclic cohomology* $HC^\bullet(\mathcal{A})$ of $\mathcal{A}$.

example [20]. Though, cyclic (co)homology is not suitable for C^*-algberas[4]. That's why we shall focus on relevant dense subalgebras in the thesis.

When $\mathcal{A}$ is a dense subalgebra in a C^*-algebra A, the pairings (1)-(2) and (3)-(4) may be extended, in certain cases, to the topological K-theory $K_1^{top}(A)$ of the C^*-algebra. The case where $\mathcal{A}$ is stable by holomorphic functional calculus is the most simple, as the injection of $\mathcal{A} \subset A$ induces an isomorphism in K-theory.

In general, the problem of extending the pairings is difficult, but important for applications. We have in mind the papers [9] or [10], where such considerations are refined with the Novikov conjecture as a target. Another application concerns the transverse fundamental class of a foliation [6] and its geometric corollaries.

2.2. Cyclic (co)homology and (B, b)-bicomplex. K-homology is in some sense the stage of noncommutative topology. Connes developed *noncommutative differential geometry* in [7]. Some of the main tools introduced are periodic cyclic homology and cohomology, which are respectively the noncommutative analogue of the de Rham cohomology and homology.

Let $\mathcal{A}$ be an associative algebra (possibly non unital) over $\mathbb{C}$. The *algebra of universal differential forms* $(\Omega\mathcal{A}, \mathbf{d})$ on $\mathcal{A}$, is the algebra generated by $a \in \mathcal{A}$, and symbols $\mathbf{d}a$, $a \in \mathcal{A}$ with $\mathbf{d}$ linear in a and saisfiying the Leibniz rule

$$\mathbf{d}(a_0 a_1) = a_0 \cdot \mathbf{d}a_1 + \mathbf{d}a_0 \cdot a_1$$

for every $a_0, a_1 \in A$. We have a filtration,

$$\Omega\mathcal{A} = \bigoplus_{k \geqslant 0} \Omega^k \mathcal{A},$$

where $\Omega^k\mathcal{A} = \{a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k\,;\, a_0, \dots, a_k \in A\}$. As vector spaces, there is a natural isomorphism

$$\begin{aligned} \Omega^k\mathcal{A} &\xrightarrow{\simeq} \mathcal{A}^+ \otimes \mathcal{A}^{\otimes k} \\ a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k &\longmapsto a_0 \otimes a_1 \otimes \dots \otimes a_k \\ \mathbf{d}a_1 \dots \mathbf{d}a_k &\longmapsto 1 \otimes a_1 \otimes \dots \otimes a_k \end{aligned}$$

where A^+ is the unitalization of $\mathcal{A}$.

The universal differential $\mathbf{d}$ is extended to $\Omega\mathcal{A}$ by setting

$$\mathbf{d}(a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k) = \mathbf{d}a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k$$
$$\mathbf{d}(\mathbf{d}a_1 \dots \mathbf{d}a_k) = 0$$

One then defines two differentials on $\Omega\mathcal{A}$: the *Hochschild boundary* $b : \Omega^{k+1}\mathcal{A} \to \Omega^k\mathcal{A}$ given by the formula

$$\begin{aligned} b(a_0 \mathbf{d}a_1 \dots \mathbf{d}a_{k+1}) = a_0 a_1 \mathbf{d}a_2 \dots \mathbf{d}a_{k+1} + \sum_{i=1}^{k} (-1)^i a_0 \mathbf{d}a_1 \dots \mathbf{d}(a_i a_{i+1}) \dots \mathbf{d}a_{k+1} \\ + (-1)^{k+1} a_{k+1} a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k \end{aligned}$$

and the differential $B : \Omega^k\mathcal{A} \to \Omega^{k+1}\mathcal{A}$ defined by

$$\begin{aligned} B(a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k) = \mathbf{d}a_0 \mathbf{d}a_1 \dots \mathbf{d}a_k + (-1)^k \mathbf{d}a_n \mathbf{d}a_0 \mathbf{d}a_1 \dots \mathbf{d}a_{k-1} \\ + \dots + (-1)^{kk} \mathbf{d}a_1 \dots \mathbf{d}a_k \mathbf{d}a_0 \end{aligned}$$

4. Cyclic theory is poor in that case. For example if A is nuclear, $HP^1(A) = 0$ et $HP^0(A)$ is the space of traces on A. This is the same for entire cyclic cohomology. An alternative is the *local cyclic cohomology* of Puschnigg, but this point of view is not suitable for concrete calculations.

As we have $b^2 = B^2 = Bb + bB = 0$, we may consider the (B, b)-*bicomplex in homology*

$$\begin{tikzcd}
 & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] \\
\dots & \Omega^2\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0\mathcal{A} \arrow[l, "B"'] \\
\dots & \Omega^1\mathcal{A} \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0\mathcal{A} \arrow[l, "B"'] & \\
\dots & \Omega^0\mathcal{A} \arrow[l, "B"'] & &
\end{tikzcd}$$

Let $\widehat{\Omega}\mathcal{A}$ denote the direct product $\prod_k \Omega^k\mathcal{A}$. The *periodic cyclic homology* $\mathrm{HP}_\bullet(\mathcal{A})$ of $\mathcal{A}$ is the homology of the 2-periodic complex

$$\widehat{\Omega}^{\mathrm{even}}\mathcal{A} \underset{B+b}{\overset{B+b}{\rightleftarrows}} \widehat{\Omega}^{\mathrm{even}}\mathcal{A}$$

where

$$\widehat{\Omega}^{\mathrm{even}}\mathcal{A} = \prod_{k\geqslant 0} \Omega^{2k}\mathcal{A}, \quad \Omega^{\mathrm{odd}}\mathcal{A} = \prod_{k\geqslant 0} \widehat{\Omega}^{2k+1}\mathcal{A}$$

The reason why we use the direct product instead of the direct sum is that the latter always gives 0 in homology. Therefore, there is only an even and an odd periodic cyclic homology group, denoted $\mathrm{HP}_0(\mathcal{A})$ and $\mathrm{HP}_1(\mathcal{A})$.

For $k \geqslant 0$, let $\mathrm{CC}^k(\mathcal{A})$ be the dual of $\Omega^k\mathcal{A}$, that is, the space of $(k+1)$-linear forms on $\mathcal{A}^+$ verifying $\phi(a_0, \dots, a_k) = 0$ if $a_i = 1$ for at least one $i \geqslant 1$. In an obvious way, we can construct a (B, b)-*bicomplex in cohomology* by taking the dual of the one given in homology above. The (continous) dual of $\widehat{\Omega}\mathcal{A}$ (for the filtration topology) is the direct sum :

$$\mathrm{CC}^\bullet(\mathcal{A}) = \bigoplus_{k\geqslant 0} \mathrm{CC}^k(\mathcal{A})$$

The *periodic cyclic cohomology* $\mathrm{HP}^\bullet(\mathcal{A})$ of $\mathcal{A}$ is the cohomology of the dual 2-periodic complex giving cyclic homology, or equivalently, the total complex of the (B, b)-bicomplex in cohomology

$$\mathrm{CC}^{\mathrm{even}}(\mathcal{A}) \underset{B+b}{\overset{B+b}{\rightleftarrows}} \mathrm{CC}^{\mathrm{odd}}(\mathcal{A})$$

where

$$\mathrm{CC}^{\mathrm{even}}(\mathcal{A}) = \bigoplus_{k\geqslant 0} \mathrm{CC}^{2k}(\mathcal{A}), \quad \mathrm{CC}^{\mathrm{odd}}(\mathcal{A}) = \bigoplus_{k\geqslant 0} \mathrm{CC}^{2k+1}(\mathcal{A})$$

REMARK 7. Sometimes, some authors use $B - b$ instead of $B + b$.

A cyclic p-cocycle ϕ_p naturally yields a (B, b)-cocycle $(0, \dots, 0, \phi_p, 0, \dots)$, where ϕ_p is in $(\frac{p}{2} + 1)$-th position for p even, and in $\frac{p+1}{2}$-th position if p is odd. Such (B, b)-cocycles are said *homogeneous*. On the contrary, we say it is *inhomogeneous*.

Let us now give a fundamental example of cyclic cohomology.

EXAMPLE 8. Let M be a closed manifold and $\mathcal{A} = C^\infty(M)$ endowed with its Fréchet topology. A theorem of Connes **[7]** states that if one restricts to continuous (B, b)-cochains for the Fréchet topology, one recovers the de Rham homology of M :

$$HP^0_{\text{cont}}(A) \simeq H_0(M) \oplus H_2(M) \oplus \dots$$
$$HP^1_{\text{cont}}(A) \simeq H_1(M) \oplus H_3(M) \oplus \dots$$

In the dual way, the periodic cyclic homology of $\mathcal{A}$ endowed with its Fréchet topology also yields de Rham cohomology. Note that we have to use a projective tensor product in the definition of $\Omega\mathcal{A}$.

REMARK 9. Considering only the non periodic cyclic cohomology of degree n does not yield the de Rham homology of same parity truncated at order n. Indeed,

$$HC^{2k}_{\text{cont}}(A) \simeq H_0(M) \oplus \dots \oplus H_{2k-2}(M) \oplus \text{Ker}(d : \Omega_{2k}(M) \to \Omega_{2k-1}(M))$$
$$HC^{2k+1}_{\text{cont}}(A) \simeq H_1(M) \oplus \dots \oplus H_{2k-1}(M) \oplus \text{Ker}(d : \Omega_{2k+1}(M) \to \Omega_{2k}(M))$$

where $\Omega_\bullet(M)$ is the space of de Rham currents, so that the cyclic cohomology functor $HC^\bullet$ does not verify homotopy invariance. One then understands the necessity to "stabilize" $HC^\bullet$ for killing the error term, which is the role of the (B, b)-bicomplex.

As the examples may suggest, cyclic homology is the receptacle of the *noncommutative Chern character*. The odd and even versions are respectively defined as follows :

$$\text{(5)} \qquad ch_1 : K_1^{\text{alg}}(\mathcal{A}) \longrightarrow HP_{\text{odd}}(\mathcal{A}), \qquad u \longmapsto \sum_{k\geqslant 0} (-1)^k\, k! \cdot \text{tr}\left(u^{-1}\mathbf{d}u(\mathbf{d}u^{-1}\mathbf{d}u)^k\right)$$

$$\text{(6)} \qquad ch_0 : K_0^{\text{alg}}(\mathcal{A}) \longrightarrow HP_{\text{even}}(\mathcal{A}), \qquad e \longmapsto \text{tr}(e) + \sum_{k\geqslant 1} (-1)^k \frac{(2k)!}{k!} \text{tr}\left((e - \frac{1}{2})(\mathbf{d}e)^{2k}\right)$$

where tr denotes the trace of matrices.

We may now come back to the cyclic cocycles associated to a Fredholm module constructed by Connes.

THEOREM 10. (Connes, **[7]**) *Let $(\mathcal{A}, H, F)$ be an odd Fredholm p-summable, k be an integer such that $2k + 1 \geqslant p$ and $a_0, \dots, a_{2k+1} \in \mathcal{A}$. Then, the periodic cyclic cohomology class of the cyclic cocycle*

$$ch_{2k+1}(H, F)(a_0, \dots, a_{2k+1}) = -\frac{1}{2^{2k+1} \cdot k!} \text{Tr}(a_0[F, a_1] \dots [F, a_{2k+1}])$$

does not depend on k. The (B, b)-cocycle $ch(H, F)$ *is called the* odd Chern-Connes character *of the Fredholm module $(\mathcal{A}, H, F)$.*

To prove this theorem, it suffices to introduce the cyclic cochain γ

$$\gamma_{2k}(a_0, \dots, a_{2k}) = \frac{1}{2^{2k+1} \cdot k!} \text{Tr}(a_0 F[F, a_1] \dots [F, a_{2k}]).$$

One easily checks that $\phi_{2k+1} - \phi_{2k-1} = (B + b)\gamma_{2k}$.

THEOREM 11. (Connes, **[7]**) *Let $(\mathcal{A}, H, F)$ be an odd Fredholm p-summable, k be an integer such that $2k \geqslant p$ and $a_0, \dots, a_{2k} \in \mathcal{A}$. Then, the periodic cyclic cohomology class of the cyclic cocycle*

$$ch_{2k}(H, F)(a_0, \dots, a_{2k}) = \frac{k!}{(2k)!} \text{Tr}(\varepsilon a_0[F, a_1] \dots [F, a_{2k}])$$

does not depend on k. The (B, b)-cocycle $ch(H, F)$ *is called the* even Chern-Connes character *of the Fredholm module $(\mathcal{A}, H, F)$.*

The proof is similar to the odd case, introducing the (B, b)-cochain γ

$$\gamma_{2k-1}(a_0, \ldots, a_{2k-1}) = \frac{k!}{(2k)!}\mathrm{Tr}(\varepsilon a_0 F[F, a_1] \ldots [F, a_{2k-1}]).$$

The index formulas (3) and (4) now reinterpret respectively as follows :

$$\mathrm{Ind}(PuP) = \langle \mathrm{ch}_{2k+1}(H, F), \mathrm{ch}_1(u)\rangle, \qquad \mathrm{Ind}(eFe) = \langle \mathrm{ch}_{2k}(H, F), \mathrm{ch}_0(e)\rangle,$$

the restrictions being understood. Conceptually, we manage to state a noncommutative index theorem as an equality between

(i) A pairing of K-theory with a dual theory (Fredholm modules or K-homology depending on the nature of the algebra) in the left-hand-side,

(ii) A pairing between cyclic cohomology and homology in the right-hand-side, similar to the pairing between de Rham homology and cohomology in the Atiyah-Singer theorem.

Unfortunately, a problem occurs because the formulas defining $\mathrm{ch}(H, F)$ are non-local : As they involve the operator trace, they are sensitive to compact perturbations of the operator F. Though, it can be shown that their periodic cyclic cohomology class does not depend on these perturbations. This non-local behaviour then introduces difficulties to derive concrete index formulas directly from the Chern-Connes character $\mathrm{ch}(H, F)$.

2.3. Spectral triples (unbounded Fredholm modules). The non-locality of the formula which defines the Chern-Connes character led Connes and Moscovici to introduce the notion of spectral triple in **[11]**. They were aiming at very general formula covering the Atiyah-Singer index formula, with applications to other and more intricate situations in mind.

Spectral triples are an unbounded version of Fredholm modules. We shall focus on the odd case. To adapt the theory to the even case, one proceeds to the same modifications as those done for bounded Fredholm modules.

DEFINITION 12. A spectral triple (or unbounded Fredholm module) is a triple $(\mathcal{A}, H, D)$ consisting of the following datas :

(i) an associative $*$-algebra $\mathcal{A}$ over $\mathbb{C}$, $*$-represented as bounded operators on a Hilbert space H,

(ii) an unbounded operator D on H, such that

  (a) For every $a \in \mathcal{A}$, $a(\lambda - D^2)^{-1}$ is compact, for every λ in the resolvent set of D^2 (in other words, D has a locally compact resolvent))

  (b) For every $a \in \mathcal{A}$, $[D, a]$ is defined on $\mathrm{dom}\, D$, and extends to a bounded operator on H.

A spectral triple $(\mathcal{A}, H, D)$ is *regular* if $\mathcal{A}$, $[D, \mathcal{A}]$ are contained in $\bigcap_{n\geqslant 1} \mathrm{dom}\, \delta^n$, where δ is the unbounded derivation $\mathrm{ad}\,|D| = [|D|, .]$ sur $\mathcal{B}(H)$.

EXAMPLE 13. Let M be an odd dimensional closed spin manifold, D the associated Dirac operator acting on the spinor bundle S. Then $(C^\infty(M), L^2(S), D)$ is a regular spectral triple. Connes gives a reconstruction theorem **[8]**, which says under additional assumptions, all "commutative" spectral triples are of that form. Then, one can think of spectral triples as noncommutative Riemannian manifolds.

If we assume $\mathcal{A}$ to be unital, the hypothesis (ii) allows to perform holomorphic functional calculus on $\Delta = D^2$, thanks to a Cauchy integral. In particular, complex powers Δ^{-z}, for $z \in \mathbb{C}$ may be defined as follows

$$\Delta^{-z} = \frac{1}{2\pi \mathbf{i}}\int \lambda^{-z}(\lambda - \Delta)^{-1}\, d\lambda \tag{7}$$

where the integration contour may for example be taken as a vertical line pointing downwards, which separates 0 and the spectrum of Δ. This converges for the operator norm when $\mathrm{Re}(z) > 0$. To make sense of this when $\mathrm{Re}(z) \leqslant 0$, one writes $\Delta^{-z} = \Delta^{-z-k}\Delta^k$, for $k \in \mathbb{N}$ large enough.

REMARK 14. This works well for the spectral triple of Exemple 13, because Δ has here a compact resolvent. However, we shall also be interested in non-unital cases, where this no more holds in general. For example for the spectral triple $(C_c^\infty(M), L^2(M), D)$, with M as a non compact complete Riemannian manifold, we shall construct the complex powers Δ^{-z} as properly supported pseudodifferential operators (modulo smoothing operators), reasonning directly at the level of symbolic calculus by replacing the $(\lambda - \Delta)^{-1}$ in the integral is replaced by a parametrix.

We need some more axioms to define zeta functions. For this, let us recall the case of zeta functions on a Riemannian manifold M of dimension n. Let Δ be a Laplace-type operator on M, and P be a (pseudo)differential operator of order d on M *with compact support*. A theorem of Minakshisundaram and Pleijel **[27]** asserts that the zeta function of P

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z})$$

is holomorphic on the half-plane $\mathrm{Re}(z) > \frac{n+d}{2}$, and extends to a meromorphic function on the complex plane $\mathbb{C}$, with at most simple poles at the integers $\{\frac{n+d}{2}, \frac{n+d-1}{2}, \ldots\}$. This leads to the following definition.

DEFINITION 15. A spectral triple $(\mathcal{A}, H, D)$ has the *analytic continuation property* if for every P belonging to the smallest subalgebra generated by $\delta^n\mathcal{A}, \delta^n[D, \mathcal{A}]$, $n \geqslant 0$, the zeta function

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z})$$

exists for $\mathrm{Re}(z) \gg 0$, and extends to a meromorphic function on $\mathbb{C}$. We say that $(\mathcal{A}, H, D)$ has *simple dimension spectrum* if it has the analytic continuation property, and if the poles of all zeta functions given above are at most simple.

The locality, in the sense defined before, comes from Wodzicki's results **[39]**, showing that the functional

$$\fint P = \mathrm{Res}_{z=0}\mathrm{Tr}(P\Delta^{-z}) \tag{8}$$

defines the *unique trace* on the algebra $\Psi_{\mathrm{cl,c}}(M)$ of classical pseudodifferential operators with compact support on M *vanishing on regularizing operators*. Wodzicki also shows that the residue of $P \in \Psi_{\mathrm{cl,c}}(M)$ is given by a formula involving only its symbol σ_{-n} of order $-n$,

$$\fint P = \frac{1}{(2\pi)^n}\int_{S^*M} \iota_L\left(\sigma_{-n}(x,\xi)\frac{\omega^n}{n!}\right) \tag{9}$$

where L is the generator of the dilations $(x, \lambda\xi)_{\lambda\in\mathbb{R}}$ on T^*M, which writes locally $L = \sum_i \xi_i \partial_{\xi_i}$, ω is the standard symplectic form on T^*M and ι the interior product. The definition of $\fint$ always makes sense in the abstract framework developed by Connes and Moscovici, which led them to the *local residue index formula*.

THEOREM 16. (Connes-Moscovici, **[11]**) *Let $(\mathcal{A}, H, D)$ be a regular spectral triple with simple dimension spectrum, and denote $\Delta = D^2$. For every non-zero $p \in \mathbb{N}$, one defines a functional on $\mathcal{A}^{p+1}$ by*

$$\phi_p(a_0, \ldots, a_p) = \sum_{k_1,\ldots,k_p \geqslant 0} c_{p,k}\mathrm{Res}_{z=0}\mathrm{Tr}\left(a_0[D, a_1]^{(k_1)} \ldots [D, a_1]^{(k_p)}\Delta^{-\frac{p}{2}-|k|-z}\right)$$

where $|k| = k_1 + \ldots + k_p$, $X^{(k_i)} = \mathrm{ad}(\Delta)^{k_i}(X) = \big[\Delta, \big[\ \ldots, [\Delta, X]\big]\big]$, *et*

$$c_{p,k} = \frac{(-1)^k}{k!} \frac{\Gamma(|k| + \frac{p}{2})}{(k_1+1)(k_1+k_2+2)\ldots(k_1+\ldots+k_p+p)}.$$

Then, $\phi = (\phi_p)_{p \in 2\mathbb{N}+1}$ *is a* (B, b)*-cocycle cohomologous in the* (B, b)*-bicomplex to the Chern-Connes character* $\mathrm{ch}(H, D|D|^{-1})$ *associated to the Fredholm module* $(\mathcal{A}, H, D|D|^{-1})$*. For that reason, we denote* $\phi = \mathrm{ch}(\mathcal{A}, H, D)$.

Connes and Moscovici first obtained the Residue cocycle as a limit of the JLO cocycle **[23]** :

$$\mathrm{JLO}_p(a_0, \ldots, a_p) = t^{p/2} \int_{\Delta_{n+1}} \mathrm{Tr}\left(a_0 e^{-t_0 D^2} [D, a_1] e^{-t_1 D^2} \ldots [D, a_p] e^{-t_p D^2}\right) dt$$

A more direct way is given by Higson **[19]** (Appendix B), using Quillen's theory of cochains.

EXAMPLE 17. Applying this result to the spectral triple $(C^\infty(M), L^2(S), D)$ of Example 13 yields (up to a multiplicative constant) the following :

$$\phi_n(a_0, \ldots, a_p) = \int_M \widehat{A}(M) \wedge a_0 da_1 \ldots da_p$$

where $\widehat{A}$ is the A-genus of M. The idea is to perform a Mellin transform to boil down to the heat kernel, and to use a rescaling technique found by Getzler **[3]**. For details, the reader may consult the paper **[33]** of Ponge.

2.4. Foliations, Heisenberg calculus and transverse geometry. Before giving the applications of these results to the transverse geometry of foliations, it is necessary to have some insight on the Heisenberg calculus on foliations.

Let M be a foliated manifold of dimension n, and let V be the integrable sub-bundle of the tangent bundle TM of M which defines the foliation. Denote ν the dimension of the leaves and $h = n - \nu$ their codimension.

The fundamental idea of the Heisenberg calculus is that *longitudinal* vector fields (with respect to to the foliation) have *order 1*, whereas *transverse* vector fields have *order* $\leqslant 2$. We shall now describe the symbolic calculus allowing to do so, following Connes and Moscovici **[11]**.

Let $(x_1, \ldots, x_n)$ a foliated local coordinate system of M, i.e, the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\nu}$ (locally) span V, so that $\frac{\partial}{\partial x_{\nu+1}}, \ldots, \frac{\partial}{\partial x_n}$ are transverse to the leaves of the foliation. Then, we set

$$|\xi|' = (\xi_1^4 + \ldots + \xi_\nu^4 + \xi_{\nu+1}^2 + \ldots + \xi_n^2)^{1/4}$$
$$\langle \alpha \rangle = \alpha_1 + \ldots + \alpha_\nu + 2\alpha_{\nu+1} + \ldots 2\alpha_n$$

for every $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$.

DEFINITION 18. A smooth function $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is a *Heisenberg symbol of order* $m \in \mathbb{R}$ if over any compact subset $K \subset \mathbb{R}^n_x$ and for every multi-index α, β, there exists $C = C_{K,\alpha,\beta} > 0$ satisfying following estimate

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi)| \leqslant C_{K,\alpha,\beta}(1 + |\xi|')^{m - \langle \alpha \rangle}$$

We shall focus on the smaller class of *classical Heisenberg symbols*. For this, we first define the *Heisenberg dilations*

$$\lambda \cdot (\xi_1, \ldots, \xi_\nu, \xi_{\nu+1}, \ldots, \xi_n) = (\lambda\xi_1, \ldots, \lambda\xi_\nu, \lambda^2\xi_{\nu+1}, \ldots, \lambda^2\xi_n)$$

for any non-zero $\lambda \in \mathbb{R}$ and non-zero $\xi \in \mathbb{R}^n$.

Then, a Heisenberg pseudodifferential σ of order m is said *classical* if its symbol σ has an asymptotic expansion when $|\xi|' \to \infty$

$$\sigma(x,\xi) \sim \sum_{j\geqslant 0} \sigma_{m-j}(x,\xi) \tag{10}$$

where $\sigma_{m-j}(x,\xi)$ are *Heisenberg homogeneous* functions, that is, for any non zero $\lambda \in \mathbb{R}$,

$$\sigma_{m-j}(x,\lambda\cdot\xi) = \lambda^{m-j}\sigma_{m-j}(x,\xi)$$

The *Heisenberg principal symbol* is the symbol of higher degree in the expansion (10).

To such a symbol σ of order m, one associates its left-quantization as the linear map :

$$P : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \qquad Pf(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{f}(\xi)d\xi$$

where $\widehat{f}$ denotes the Fourier transform of the function f. We shall say that P is a *classical Heisenberg pseudodifferential operator of order* m. If P is properly supported, then it maps $C_c^\infty(\mathbb{R}^n)$ into itself. We denote by $\Psi_H^m(\mathbb{R}^n)$ the vector space of such properly supported operators and by $\Psi_{H,c}^m(\mathbb{R}^n)$ its subspace of *compactly* supported operators. Since properly supported operators can be composed, the unions of all-orders operators

$$\Psi_H(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} \Psi_H^m(\mathbb{R}^n)\,, \qquad \Psi_{H,c}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} \Psi_{H,c}^m(\mathbb{R}^n)$$

are associative algebras over $\mathbb{C}$. The ideals of *regularizing operators*

$$\Psi^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi_H^m(\mathbb{R}^n)\,, \qquad \Psi_c^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi_{H,c}^m(\mathbb{R}^n)$$

correspond respectively to the algebras of operators with properly and compactly supported smooth Schwartz kernel.

If $P_1, P_2 \in \Psi_H(\mathbb{R}^n)$ are Heisenberg pseudodifferential operators of symbols σ_1 and σ_2, P_1P_2 is a Heisenberg pseudodifferential operator whose symbol σ is given by the *star-product* of symbols :

$$\sigma(x,\xi) = \sigma_1 \star \sigma_2(x,\xi) \sim \sum_{|\alpha|\geqslant 0} \frac{(-\mathbf{i})^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma_1(x,\xi)\partial_x^\alpha\sigma_2(x,\xi) \tag{11}$$

Note that the order of each symbol in the sum is decreasing while $|\alpha|$ is increasing.

We define the *algebra of Heisenberg formal classical symbols* $\mathcal{S}_H(\mathbb{R}^n)$ and its compactly supported subalgebra $\mathcal{S}_{H,c}(\mathbb{R}^n)$ as quotients

$$\mathcal{S}_H(\mathbb{R}^n) = \Psi_H(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n)\,, \qquad \mathcal{S}_{H,c}(\mathbb{R}^n) = \Psi_{H,c}(\mathbb{R}^n)/\Psi_c^{-\infty}(\mathbb{R}^n)$$

Their elements are formal sums given in (10), and the product is the star product (11). Note that the $\sim$ can be replaced by equalities when working at a formal level.

A Heisenberg formal symbol is said *Heisenberg elliptic* (or H-elliptic for short) if it is invertible in $\mathcal{S}_H(\mathbb{R}^n)$. This is equivalent to say that its *Heisenberg principal symbol* is invertible on $\mathbb{R}_x^n \times \mathbb{R}_p^n \setminus \{0\}$. These symbols are generally no more elliptic in the usual sense. Nonetheless, when M is compact, the corresponding pseudodifferential operators are *hypoelliptic*, and remain Fredholm operators when acting on Sobolev space suitably defined for this context (cf. for example [**18**]).

EXAMPLE 19. *(Sub-elliptic Laplacian)* This is the differential operator,

$$\Delta_H = \partial_{x_1}^4 + \ldots + \partial_{x_\nu}^4 - (\partial_{x_{\nu+1}}^2 + \ldots + \partial_{x_n}^2)$$

It has Heisenberg principal symbol $\sigma(x,\xi) = |\xi|'^4$, and is therefore Heisenberg elliptic. However, its usual principal symbol, as an ordinary differential operator, is $(x,\xi) \mapsto \sum_{i=1}^{\nu} \xi_i^4$, so Δ_H is clearly not elliptic.

To finish, Heisenberg pseudodifferential operators are compatible with foliated coordinate changes. Therefore, the Heisenberg calculus can be defined globally on foliations by using a partition of unity. Then, for a foliated manifold M, we denote by $\Psi_H(M)$ the algebra of properly-supported Heisenberg pseudodifferential operators on M, and by $\Psi_{H,c}(M)$ its subalgebra of compactly-supported operators.

For a ($\mathbb{Z}_2$-graded) complex vector bundle E over M, one defines similarily the algebra of Heisenberg pseudodifferential operators $\Psi_H(M,E)$ and its compact support version acting on the smooth sections $C^\infty(M,E)$ of E. Note that for $a \in \mathcal{S}_H(M,E)$, $(x,\xi) \in T^*_xM$, we have $a(x,\xi) \in \mathrm{End}(E_x)$.

Wodzicki residue on $\Psi_H(M)$. We have results analogous to those of Minakshisundaram-Pleijel and Wodzicki in this context. By means of a partition of unity, one can construct a sub-elliptic sub-laplacian Δ from the flat Example 19. Its complex powers Δ^{-z} are defined as properly-supported Heisenberg pseudodifferential operators, using the parametrix $(\lambda-\Delta)^{-1}$ and an appropriate Cauchy integral

$$\Delta^{-z} = \frac{1}{2\pi \mathbf{i}} \int \lambda^{-z}(\lambda-\Delta)^{-1}\, d\lambda$$

where the contour is a vertical line pointing downwards.

THEOREM 20. (Connes-Moscovici, [**11**]) *Let* (M,V) *be a foliated manifold of dimension* n, *where* $V \subset TM$ *is the integrable sub-bundle defining the foliation,* ν *be the dimensions of the leaves,* h *their codimension, and* $P \in \Psi^m_{H,c}(M)$ *be a compactly-supported Heisenberg pseudodifferential operator of order* $m \in \mathbb{R}$. *Then, for any sub-elliptic sub-laplacian* Δ, *the zeta function*

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z/4})$$

is holomorphic on the half-plane $\mathrm{Re}(z) > m+\nu+2h$, *and extends to a meromorphic function of the whole complex plane, with at most simple poles in the set*

$$\{m+\nu+2h, m+\nu+2h-1, \ldots\}$$

The meromorphic extension of the zeta function given by this theorem allows the construction of a Wodzicki-Guillemin trace on $\mathcal{S}_{H,c}(M) = \Psi_{H,c}(M)/\Psi_c^{-\infty}(M)$.

THEOREM 21. (Connes-Moscovici, [**11**]) *The Wodzicki residue functional*

$$⨍ : \mathcal{S}_{H,c}(M) \longrightarrow \mathbb{C}, \quad P \longmapsto \mathrm{Res}_{z=0}\mathrm{Tr}(P\Delta^{-z/4})$$

is a trace. If $\dim M > 1$, *this is the unique trace on* $\mathcal{S}_{H,c}(M)$, *up to a multiplicative constant. Moreover we have the following formula, only depending on the formal symbol* σ *of* P *up to a finite order :*

$$⨍ P = \frac{1}{(2\pi)^n} \int_{S^*_H M} \iota_L \left(\sigma_{-(\nu+2h)}(x,\xi)\, \frac{\omega^n}{n!} \right) \tag{12}$$

Here, $S^*_H M$ is the Heisenberg cosphere bundle, that is, the sub-bundle

$$S^*_H M = \{(x,\xi) \in T^*M\,;\, |\xi|' = 1\}$$

of the tangent bundle TM of M, L is the generator of the Heisenberg dilations given locally by the formula

$$L = \sum_{i=1}^{\nu} \xi_i \partial_{\xi_i} + 2 \sum_{i=\nu+1}^{n} \xi_i \partial_{\xi_i}$$

ι stands for the interior product and ω denotes the standard symplectic form on T^*M.

All these results still hold for Heisenberg pseudodifferential operators acting on sections of a vector bundle E over M : In this case, the symbol $\sigma_{-(\nu+2n)}(x,\xi)$ above is at each point (x,ξ) an endomorphism acting on the fibre E_x, and (12) becomes :

$$⨍ P = \frac{1}{(2\pi)^n}\int_{S_H^*M} \iota_L\left(\mathrm{tr}(\sigma_{-(\nu+2n)}(x,\xi))\,\frac{\omega^n}{n!}\right)$$

where tr denotes the trace of endomorphisms.

Transverse geometry of foliations. Using these results to construct a regular spectral triple associated to the transverse geometry of a foliation $(Z,\mathcal{F})$, *which is not the one we shall deal with* !

Choosing a transversal W of the foliation, Connes and Moscovici reduced the problem to the study of the crossed product $C_c^\infty(W)\rtimes G$, where G is a discrete (pseudo)group of diffeomorphism $G\subset \mathrm{Diff}(W)$, which translates the holonomy of the foliation. A first problem occuring is that we are in a "type III" situation. Indeed, G does not preserve any measure on M in general, and even less a Riemannian metric, so that G-invariant elliptic operators does not exist, even at the leading symbol level. To cope with this matter, one boils down to a "type II" situation by a Thom isomorphism, using the following trick from **[6]**. We pass to the bundle of Riemannian metrics $M = F/O_n(R)$ over W, where F is the frame bundle of W. This fibration is in particular a foliation, the leaves being the fibers. *This will be the foliation of interest for us*. The action of G on W lifts to M preserving the fibration, so that the sub-bundle $V\subset TM$ tangent to the fibers is G-equivariant. One may endow V with a G-invariant metric since the fibers of M are symmetric spaces $GL_n(\mathbb{R})/O_n(\mathbb{R})$. Besides, the quotient $N = TM/V$ is isomorphic to the pull-back p^*TW, then every point of M defines a G-invariant metric on N.

Connes and Moscovici then construct a signature operator D almost invariant under the G-action, in the sense that the leading symbol of D is G-invariant. Their operator acts on sections of the bundle

$$E = \Lambda^\bullet(V^*\otimes\mathbb{C})\otimes\Lambda^\bullet(N^*\otimes\mathbb{C})$$

which is isomorphic $\Lambda^\bullet(T^*M\otimes\mathbb{C})$, but in a non-canonical way since this requires the choice of a connection. Though, this isomorphism is canonical at the level of volume forms (that is the top degrees). This explains why we cannot have better than an almost G-invariant operator in general. Roughly, the construction of D is as follows : we first define

$$Q = (d_H + d_H^*) \pm (d_V d_V^* - d_V^* d_V)$$

which is morally the sum of a horizontal and a vertical signature operator [5]. The Heisenberg symbol of Q is $(x,\xi)\mapsto |\xi|'^4$, and D is finally defined by the formula $Q = D|D|$. We are at this stage arrived at a well-defined index problem.

That this operator is not elliptic could at first sight raise problems. Actually, this does not matter : While constructing the transverse fundamental class in K-homology, Hilsum and Skandalis show in **[21]** that K-cycles may be obtained using hypoelliptic operators. Connes and Moscovici refined this observation to their context.

THEOREM 22. (Connes-Moscovici, **[11]**) $(C_c^\infty(M)\rtimes G, L^2(M,E), D)$ *is a regular spectral triple of simple dimension spectrum.*

REMARK 23. As a matter of fact, the theorem is stated more generally for G-invariant *triangular structures* on a manifold M, which means we have an integrable sub-bundle $V\subset TM$ of the tangent bundle of M, such that V and TM/V carries G-invariant Riemannian metrics.

5. Connes and Moscovici indeed construct $d_V d_V^* - d_V^* d_V$ as a deformation of the vertical signature operator $d_V + d_V^*$

Thus, the residue formula directly applies to the hypoelliptic signature operator D. However, we cannot directly derive a characteristic class formula as in Example 17, calculations provides thousands of terms only in codimension 1 ... To overcome this matter in higher codimensions, Connes and Moscovici developed the cyclic cohomology of Hopf algebras [**12**]. They defined such an algebra $\mathcal{H}$, which acts like a symmetry group allowing to reorganize the calculations, and built a characteristic map

$$\chi : HP^\bullet_{\mathrm{Hopf}}(\mathcal{H}) \longmapsto HP^\bullet(C^\infty_c(M) \rtimes G)$$

THEOREM 24. (Connes-Moscovici, [**12**]) *The Chern-Connes character of the transverse spectral triple* $(C^\infty_c(M) \rtimes G, L^2(M,E), D)$ *is contained in the image of the characteristic map* χ.

In some sense, the group $HP^\bullet_{\mathrm{Hopf}}(\mathcal{H})$ contains the geometric cocycles : Connes and Moscovici found that it is isomorphic to the Gel'fand-Fuchs cohomology $H^\bullet(WSO_n)$, which contains e.g characteristic classes, so χ may be viewed as of the noncommutative analogue of the Chern-Weil map from Gel'fand-Fuchs cohomology to equivariant cohomology (cf. Bott [**4**]). Then, reaching an index theorem amounts to finding the geometric preimage of the Chern-Connes character by χ. Connes and Moscovici finally state the following result :

The preimage of the Chern-Connes character by χ *is contained in the polynomial ring generated Pontryagin classes*

Explicit calculations are made in [**12**] in codimension 1, giving (twice) the transverse fundamental class of [**6**]. In codimension 2, the authors show that the coefficient of the first Pontryagin class does not vanish.

The result of Chapter 3 in this thesis allows to make the calculations in any codimension. We shall take an alternative road summed up in the diagram below,

$$\underbrace{H^\bullet(EG \times_G S^*_H M)}_{\text{equivariant cohomology}} \xrightarrow[\text{Connes [6]}]{\Phi} HP^\bullet(C^\infty_c(S^*_H M) \rtimes G) \xrightarrow{\pi^*} HP^\bullet(C^\infty_c(M) \rtimes G)$$

$\pi : S^*_H M \to M$ being the canonical projection, which does not make use of Hopf algebras. This was given by Connes for the problem of the transverse fundamental class.

2.5. General approach with extensions. The approach we adopt in this thesis is strongly related to the work of Cuntz and Quillen, showing that the excision property holds in periodic cyclic (co)homology [**13**]. We consider "abstract index problems". Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, possibly without unit, and $\mathcal{I}$ an ideal in $\mathcal{A}$, and consider the following extension

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$$

Let Ind and ∂ denote respectively the excision map in algebraic K-theory and periodic cyclic homology. We then have the following diagram,

$$\begin{array}{ccc} K_1^{\mathrm{alg}}(\mathcal{A}/\mathcal{I}) & \xrightarrow{\mathrm{Ind}} & K_0^{\mathrm{alg}}(\mathcal{I}) \\ \downarrow{\scriptstyle \mathrm{ch}_1} & & \downarrow{\scriptstyle \mathrm{ch}_0} \\ HP_1(\mathcal{A}/\mathcal{I}) & \xrightarrow{\partial} & HP_0(\mathcal{I}) \end{array} \tag{13}$$

The vertical arrows are respectively the odd and even Chern character in K-theory, given by formulas (5) and (6).

Nistor shows in [**29**] that this diagram commutes. We still denote $\partial : HP^0(\mathcal{I}) \to HP^1(\mathcal{A}/\mathcal{I})$ the excision map in cohomology. If $[\tau] \in HP^0(\mathcal{I})$ and $[u] \in K^1(\mathcal{A}/\mathcal{I})$, then one has the equality :

$$\langle [\tau], \mathrm{ch}_0 \mathrm{Ind}[u] \rangle = \langle \partial[\tau], \mathrm{ch}_1[u] \rangle \tag{14}$$

One should have in mind that the left hand-side is an analytic index, and think about the right hand-side as a topological index. Nistor recovers with this approach the index theorem for covering spaces of Connes-Moscovici in **[29]**, and the Atiyah-Patodi-Singer theorem on manifolds with cusps in **[28]**. We will see in the sequel how to choose adapted pseudodifferential extensions giving rise to local index formulas. For the moment, let us end the paragraph with the two following examples.

EXAMPLE 25. Let us sketch the construction of $\partial[\tau]$ when τ is a hypertrace on $\mathcal{I}$, i.e a linear form on $\mathcal{I}$ satisfying $\tau([\mathcal{A},\mathcal{I}]) = 0$. One "renormalizes" τ to a linear map $\tau_R : \mathcal{A} \to \mathbb{C}$, which coincides with τ when restricted to $\mathcal{I}$. This is not a trace, but we may then consider the following cyclic cocycle :

$$\phi(a_0, a_1) = b\tau_R(a_0, a_1) = \tau_R([a_0, a_1])$$

This cocycle descends to the quotient $\mathcal{A}/\mathcal{I}$ and gives a representative of $\partial[\tau]$.

EXAMPLE 26. Let $(\mathcal{A}, H, F)$ be an odd p-summable Fredholm module. Let $\ell^p(H)$ denote the p-th Schatten class. As usual, let $P = \frac{1+F}{2}$. We have the following extension

$$0 \longrightarrow \ell^p(H) \longrightarrow \ell^p(H) + \rho(\mathcal{A}) \longrightarrow \mathcal{A}/\mathrm{Ker}(\rho) \longrightarrow 0$$

where $\rho(a) = PaP$. The cyclic cocycle $\ell^p(H)$

$$\tau(x_0 \dots x_{p-1}) = \mathrm{Tr}(x_0 \dots x_{p-1})$$

yields a generator $[\tau]$ of $HP^0(\ell^p(H)) \simeq \mathbb{C}$. One can show that its image $\partial[\tau]$ by excision is precisely the Chern-Connes character. A similar interpretation exists for the even case. For more details, the reader may consult Cuntz's paper in **[14]**, Sections 3.3 et 3.4.

3. Main results of the thesis

We shall now give a detailed plan of the thesis, and announce its main results. We decided not to gather the preliminairies (other that those given in Introduction) in a single paragraph, but to introduce them in the required chapters as we go along. We hope the reader will be guided more efficiently in this way.

Chapters 1, 2 and 4 are extracted from the paper **[35]**, except for the last section which comes from **[32]**. The rest, up to some minor changes and corrections, is similar. Chapter 4 comes from **[32]**.

3.1. Chapter 1. Local index formula for abstract elliptic operators. The goal of this chapter is to establish a *local index formula* for "abstract elliptic operators", combining a suitable zeta function renormalization and excision in periodic cyclic cohomology. Let us present briefly the framework developed by Higson **[19]**, which allows to make sense of this statement.

Let H be a (complex) Hilbert space and Δ a unbounded, positive and self-adjoint operator acting on it. We denote by H^∞ the intersection :

$$H^\infty = \bigcap_{k \geqslant 0} \mathrm{dom}(\Delta^k).$$

DEFINITION. An algebra $\mathcal{D}(\Delta)$ of *abstract differential operators* associated to Δ is an algebra of operators on H^∞ fulfilling the following conditions

(i) The algebra $\mathcal{D}(\Delta)$ is filtered,

$$\mathcal{D}(\Delta) = \bigcup_{q \geqslant 0} \mathcal{D}_q(\Delta)$$

that is $\mathcal{D}_p(\Delta) \cdot \mathcal{D}_q(\Delta) \subset \mathcal{D}_{p+q}(\Delta)$. We shall say that an element $X \in \mathcal{D}_q(\Delta)$ is an *abstract differential operator of order at most* q.

(ii) There is a $r > 0$ ("the order of Δ") such that for every $X \in \mathcal{D}_q(\Delta)$, $[\Delta, X] \in \mathcal{D}_{r+q-1}(\Delta)$.

To state the last point, we define, for $s \in \mathbb{R}$, the *s-Sobolev space* H^s as the subspace $\mathrm{dom}(\Delta^{s/r})$ of H, which is a Hilbert space when endowed with the norm

$$\|v\|_s = (\|v\|^2 + \|\Delta^{s/r} v\|^2)^{1/2}$$

(iii) *Elliptic estimate.* If $X \in \mathcal{D}_q(\Delta)$, then, there is a constant $\varepsilon > 0$ such that

$$\|v\|_q + \|v\| \geqslant \varepsilon \|Xv\|\,,\ \forall v \in H^\infty$$

This formalism is motivated by important properties fulfilled by differential operators on a manifold. One interest of this is to unify diverse situations in a framework which is flexible enough to develop a notion of pseudodifferential calculus. In the cases of interest for us, the classical pseudodifferential calculus on a manifold and the Heisenberg calculus on a foliation are covered in this formalism.

Let $\Psi(\Delta)$ be an algebra of pseudodifferential operators associated to an algebra of abstract differential operators $\mathcal{D}(\Delta)$. We note

$$\Psi^{-\infty}(\Delta) = \bigcap_{m \in \mathbb{R}} \Psi^m(\Delta)$$

the ideal of "regularizing operators". This yields a pseudodifferential extension :

$$0 \longrightarrow \Psi^{-\infty}(\Delta) \longrightarrow \Psi(\Delta) \longrightarrow \mathcal{S}(\Delta) = \Psi(\Delta)/\Psi^{-\infty}(\Delta) \longrightarrow 0 \tag{15}$$

where $\mathcal{S}(\Delta)$ may be seen as an *algebra of abstract formal symbols*. The operator trace Tr on H is defined on $\Psi^{-\infty}$, giving rise to a cyclic cohomology class $[\mathrm{Tr}] \in HP^0(\Psi^{-\infty}(\Delta))$. To obtain a local index formula, we therefore calculate its image by excision $\partial[\mathrm{Tr}] \in HP^1(\mathcal{S}(\Delta))$, following the recipe given in Introduction, Section 2.5. This is the moment zeta functions come into play.

Assume that there exists $d \geqslant 0$ such that for every $P \in \Psi^m(\Delta)$, the operator $P\Delta^{-z}$ extends to a trace-class operator on H, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \frac{m+d}{r}$, so that the zeta function of P

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z/r})$$

is defined. Suppose also that for every $P \in \Psi^m(\Delta)$, ζ_P is holomorphic in the half-plane $\mathrm{Re}(z) > m + d$, and that it extends to a meromorphic function on $\mathbb{C}$ with at most simple poles in the set

$$\{m + d, m + d - 1, \ldots\}$$

We may then lift $\Psi^{-\infty}(\Delta)$ to a linear form τ_R on $\Psi(\Delta)$ using a zeta function renormalization :

$$\tau_R(P) = \mathrm{Pf}_{z=0}\mathrm{Tr}(P\Delta^{-z/r})$$

where Pf is the constant term in the Laurent expansion of ζ_P around $z = 0$. This allows to compute a representative of $\partial[\mathrm{Tr}]$, given in the following result.

THEOREM I. (THEOREM 1.13 *Suppose that for every $P \in \Psi(\Delta)$, the pole at 0 of the zeta function is of order $p \geqslant 1$. Then, the image $\partial[\mathrm{Tr}] \in HP^1(\mathcal{S})$ of the operator trace $[\mathrm{Tr}] \in HP^0(\Psi^{-\infty})$ by excision in periodic cyclic cohomology is represented by the following cyclic 1-cocycle, that we call the* generalized Radul cocycle *:*

$$c(a_0, a_1) = \overset{1}{⨍} a_0 \delta(a_1) - \frac{1}{2!} \overset{2}{⨍} a_0 \delta^2(a_1) + \ldots + \frac{(-1)^{p-1}}{p!} \overset{p}{⨍} a_0 \delta^p(a_1)$$

where $\delta(a) = [\log \Delta^{1/r}, a]$ *and* $\delta^k(a) = \delta^{k-1}(\delta(a))$ *are defined by induction, for all* $a \in \Psi(\Delta)$,

$$\fint^{k} P = \mathrm{Res}_{z=0} z^{k-1} \mathrm{Tr}(P\Delta^{-z/r})$$

This formula generalizes the Radul cocycle **[34]**, obtained in the case where Ψ is the algebra of classical pseudodifferential operators on a closed manifold. It was introduced in the context of Lie algebra cohomology.

If $p = 1$, $\fint^1$ is the usual Wodzicki residue. In general, the $\fint^k$ are not traces on $\Psi(\Delta)$ except for $k = p$.

Therefore, excision transfers the non-local cocycle on $\Psi^{-\infty}(\Delta)$ represented by the trace to the Radul cocycle, which is local. To this effect, this may be compared with the formula of Connes and Moscovici through the use of residues. Let us mention that the techniques of computations performed to get our formula are mainly their doing.

There is also one evident difference at first : our cocycle is defined direcly on an algebra of pseudodifferential operators, whereas the Chern-Connes character is defined on the algebra of the spectral triple. Actually, it turns out the generalized Radul cocycle retracts on the latter when $\mathcal{D}(\Delta)$ is associated to a spectral triple (more precisely its Fredholm module).

Indeed, let $(\mathcal{A}, H, F)$ be a p-summable odd Fredholm module. In addition, let $\Psi = \Psi(\Delta)$ be an algebra of abstract pseudodifferential operators such that

(1) Ψ^0 is an algebra of bounded operators on H containing the representation of $\mathcal{A}$,

(2) Ψ^{-1} is a two-sided ideal composed of p-summable operators on H,

(3) F is a multiplier of Ψ^0 and $[F, \Psi^0] \subset \Psi^{-1}$.

We then consider the following abstract principal symbol exact sequence :

$$0 \longrightarrow \Psi^{-1} \longrightarrow \Psi^0 \longrightarrow \Psi^0/\Psi^{-1} \longrightarrow 0 \tag{16}$$

which is easily compared with the extension (15). Let $P = \frac{1}{2}(1 + F)$, and ρ_F be the algebra homomorphism

$$\rho_F \ : \ \mathcal{A} \longrightarrow \Psi^0/\Psi^{-1} \ , \qquad \rho_F(a) = PaP \bmod \Psi^{-1}.$$

THEOREM II. (THEOREM 1.17) *The Chern-Connes character of the Fredholm module* (H, F) *is given by the odd cyclic cohomology class over* $\mathcal{A}$

$$\mathrm{ch}(H, F) = \rho_F^* \circ \partial([\mathrm{Tr}])$$

where $[\mathrm{Tr}] \in HP^0(\Psi^{-1})$ *is the class of the operator trace,* $\partial : HP^0(\Psi^{-1}) \to HP^1(\Psi^0/\Psi^{-1})$ *is the excision map associated to extension (16), and* $\rho_F^* : HP^1(\Psi^0/\Psi^{-1}) \to HP^1(\mathcal{A})$ *is induced by the homomorphism* ρ_F.

The formula obtained gives directly a local index formula for abstract elliptic operators after pairing with K-theory. As a simple application which will be detailed in the corresponding chapter, the case where $\mathcal{D}(\Delta)$ is the algebra of differential operators on the circle S^1 yields directly the Noether index theorem. This is because the Wodzicki residue of a classical pseudodifferential operator depends only on its symbol of order -1, and this makes the computations easy. However, for an arbitrary closed manifold M of dimension n, one has to compute its symbol of order $-n$, which is impossible to achieve directly. The same problem raises in the case where M is foliation of codimension h, where we have to compute the symbol of order $-(\nu + 2h)$, with $\nu = n - h$.

Plan of the chapter. Section 1 of the chapter recalls the formalism developed by Higson **[19]** on abstract differential operators, and the corresponding pseudodifferential calculus introduced by Uuye in **[38]**. We shall also give some notions we need for our local index formula.

Section 2 establishes the index formula announced in Theorem I.

Section 3 relates the generalized Radul cocycle constructed and the Chern-Connes character by proving Theorem II.

3.2. Chapter 2. The flat case. This part aims at understanding on a toy model example how we may overcome the difficulties raised by the concrete computation of the Radul cocycle, in the case of foliations. We shall focus on $\mathbb{R}^n$ viewed as a trivial foliation $\mathbb{R}^\nu \times \mathbb{R}^h$ of leaves $\mathbb{R}^\nu$. The algebra $\mathcal{D}(\Delta)$ of abstract differential operators considered is the algebra of differential operators, endowed with the order of the Heisenberg calculus. The operator Δ is the sub-elliptic sub-laplacian from Example 19. Let $\Psi^0_{H,c}(\mathbb{R}^n)$ be the algebra of Heisenberg pseudodifferential operator of order 0 with *compact support* on $\mathbb{R}^n = \mathbb{R}^\nu \times \mathbb{R}^h$, and $S^0_{H,c}(\mathbb{R}^n)$ be the algebra of Heisenberg formal symbols of order 0 with *compact support*. By Theorem 20, zeta functions have simple poles, and the Radul cocycle on $S_{H,c}(\mathbb{R}^n)$ writes

$$\phi(a_0, a_1) = ⨍ a_0[\log \Delta^{1/4}, a_1],$$

where $⨍$ is the Wodzicki residue, and is given by integration of the symbol of order $-(\nu + 2h)$ of the operator on the Heisenberg cotangent sphere $S^*_H(\mathbb{R}^n)$ (cf. Theorem 21).

The general idea is to construct a (B, b)-cocycle of higher degree cohomologuous to the Radul cocycle in the (B, b)-bicomplex, which involves only residues of symbols of order $-(\nu+2h)$. This allows to reduce matters at the Heisenberg principal symbol level,

$$\sigma : \mathcal{S}^0_H(\mathbb{R}^n) \longrightarrow C^\infty(S^*_H\mathbb{R}^n),$$

and to pass from a noncommutative formula (star-product of symbols) to a commutative one (multiplication of principal symbols) for the Radul cocycle.

THEOREM III. (THEOREM 2.5) *The Radul cocycle* ϕ *is* (B, b)*-cohomologous to the cyclic cocycle* ψ *on* $\mathcal{S}^0_H(\mathbb{R}^n)$ *defined by*

$$\psi_{2n-1}(a_0, \dots, a_{2n-1}) = -\frac{1}{(2\pi \mathbf{i})^n} \int_{S^*_H\mathbb{R}^n} \sigma(a_0) d\sigma(a_1) \dots d\sigma(a_{2n-1})$$

We give two constructions leading to the cocycle. The first one invokes once more excision in periodic cyclic cohomology : one transfers cyclic cocycles on $\Psi^{-\infty}(\mathbb{R}^n)$, very similar to that of Connes (Theorem 11), to (B, b)-cocycles on $\mathcal{S}^0_H(\mathbb{R}^n)$. The second construction uses the cochain theory of Quillen (i.e X-complex for coalgebras), which allows to construct (a variant) of the cocycles by a "non commutative Chern-Weil" process. The interest of this construction is to be purely algebraic, and this allows to bypass the step which consists in working with regularizing operators at first. Refinements of this technique will also give the relevant cocycles leading to the general theorem of Chapter 3.

In both cases, an essential ingredient for the construction of the intermediate cochains between the Radul cocycle and the one of the theorem above, is a particular operator that we shall denote F, which acts on symbols so that its commutator $[F, a]$ with a symbol $a \in \mathcal{S}_H(\mathbb{R}^n)$ gives the differential da. The notation will become clear in the sequel.

The pairing with K-theory, together with formula (14), gives the following theorem.

THEOREM IV. (THEOREM 2.12) *Let* $P \in M_N(\Psi^0_{H,c}(\mathbb{R}^n)^+)$ *be a H-elliptic operator of formal symbol* $u \in GL_N(\mathcal{S}^0_{H,c}(\mathbb{R}^n)^+)$*, and* $[u] \in K_1(\mathcal{S}^0_{H,c}(\mathbb{R}^n)^+)$ *its* K*-theory class. Then, the Fredholm index of* P

is given by the formula :

$$\mathrm{Ind}(P) = \frac{(-1)^n(n-1)!}{(2\pi\mathbf{i})^n(2n-1)!}\int_{S_H^*\mathbb{R}^n} \mathrm{tr}(\sigma(u)^{-1}d\sigma(u)(d\sigma(u)^{-1}d\sigma(u))^{n-1})$$

Despite its simplicity, this example will be important to guide the understanding of the general case in Chapter 3. This gives some ideas on the objects we have to introduce, and a first insight on how the computations work.

Plan of the chapter. Section 1 gives the general context and introduces the main tools (especially the operator F mentioned above) we shall need for the constructions.

Section 2 proves Theorem III using excision.

Section 3 gives the construction involving the algebra cochains formalism of Quillen.

Section 4 ends the computations, leading to the index theorem.

The two last sections are appendices for Sections 2 and 3.

3.3. Chapter 3. Equivariant index theorem for H-elliptic operators. In this chapter, we prove a general index theorem on foliations, generalizing the one from the previous chapter. The result presented gives a new solution to the problem of Connes and Moscovici on the computation of the Chern-Connes character of the transverse fundamental class recalled in Introduction, Section 2.4.

Let (M,V) be a foliated manifold (possibly non compact), and $G \subset \mathrm{Diff}(M)$ be a discrete subgroup of diffeomorphisms of M preserving the leaves of M, acting from the *right*. No additional assumptions are required. We then consider the following pseudodifferential extension

$$0 \longrightarrow \Psi_c^{-\infty}(M) \rtimes G \longrightarrow \Psi_{H,c}^0(M) \rtimes G \longrightarrow S_{H,c}^0(M) \rtimes G \longrightarrow 0,$$

which is the equivariant version of the one in the Heisenberg calculus. Though, note that the representation of elements in $\Psi_{H,c}^0(M) \rtimes G$ as operators on $C_c^\infty(M)$ does not yield Heisenberg pseudodifferential operators. Indeed, an element $P \in \Psi_{H,c}^0(M) \rtimes G$ writes

$$P = \sum_{g\in G} P_g \otimes U_g$$

and is represented (not faithfully) by the operator

$$P = \sum_{g\in G} P_g \otimes U_g : C^\infty(M) \longrightarrow C^\infty(M)$$

where $U_g : C^\infty(M) \to C^\infty(M)$ is the shift operator $U_g(f)(x) = f(x \cdot g) = (f \circ g)(x)$, for every $x \in M$. Such operators belongs to the larger class of *Fourier integral* operators. We can always develop their index theory, at least in the sense of extensions in the spirit adopted in the thesis. It turns out that when M is compact, the work of Savin and Sternin **[36]** shows that such operators whose formal symbol in $S_H^0(M) \rtimes G$ are Fredholm. Adapting the general formula of Chapter **??**, we may determine the Radul cocycle associated to the above extension. Let $\mathrm{Tr}_{[1]}$ be the trace on $\Psi_c^{-\infty}(M) \rtimes G$ obtained from the usual trace on $\Psi_c^{-\infty}(M)$ by localization at the unit of G :

$$\mathrm{Tr}_{[1]}\left(\sum_{g\in G} K_g U_g\right) = \mathrm{Tr}(K_1)$$

THEOREM V. (THEOREM 3.33) *The boundary of the localized trace* $\partial[\mathrm{Tr}_{[1]}] \in HP^1(\mathcal{S}_{H,c}(M) \rtimes G)$ *is represented by the* equivariant Radul cocycle

$$\phi(a_0, a_1) = ⨍ \left(a_0[\log \Delta_H^{1/4}, a_1]\right)_{[1]}$$

where $\Delta_H^{1/4}$ is the sub-elliptic sub-laplacian (Example 19) associated to M. The subscript $_{[1]}$ denotes the term localized at the unit.

This cocycle may also be viewed at the level of the (Heisenberg) principal symbol $C^\infty(S_H^*M) \rtimes G$ through the extension

$$0 \longrightarrow \Psi_{H,c}^{-1}(M) \rtimes G \longrightarrow \Psi_{H,c}^{0}(M) \rtimes G \longrightarrow C_c^\infty(S_H^*M) \rtimes G \longrightarrow 0,$$

which may be easily compared to the extension above. For details, the reader is referred to the body of the thesis. The main result is the geometric realization of the equivariant Radul cocycle. To simplify matters for the Introduction, we identify $\partial[\mathrm{Tr}_{[1]}]$ to the corresponding element of $HP^1(C_c^\infty(S_H^*M) \rtimes G)$.

THEOREM VI. (THEOREM 3.34) *Let M be a foliated manifold and G be a discrete group of foliated diffeomorphisms. Let EG be the universal bundle over the classifying space BG de G. Let*

$$0 \longrightarrow \Psi_{H,c}^{-1}(M) \rtimes G \longrightarrow \Psi_{H,c}^{0}(M) \rtimes G \longrightarrow C_c^\infty(S_H^*M) \rtimes G \longrightarrow 0$$

be the equivariant Heisenberg pseudodifferential extension. Then, the image of the localized trace at the unit $\partial[\mathrm{Tr}_{[1]}] \in HP^1(C_c^\infty(S_H^*M) \rtimes G)$ by excision is given by

$$\partial([\tau]) = \Phi(\pi^*\mathrm{Td}(TM \otimes \mathbb{C}))$$

where $\Phi : H^{\mathrm{ev}}(EG \times_G S_H^*M) \to HP^1(C_c^\infty(S_H^*M) \rtimes G)$ is Connes' characteristic map from equivariant cohomology to cyclic cohomology, $\mathrm{Td}(TM \otimes \mathbb{C})$ is the equivariant Todd class of the complexified tangent bundle of M and $\pi : S_H^*M \times_G EG \to M \times_G EG$ is the canonical projection.

To prove this result, the idea is to give an explicit homotopy between the Radul cocycle and the one given above. In the framework of cyclic cohomology, a recipe to obtain automatically transgression cochains between two representatives of a given cohomology class is to use a *JLO formula*. A first observation (already mentioned previously) is that our cocycles are not defined on algebras of functions, as it the usual JLO setting, but on pseudodifferential operators. Thus, we have to find an appropriate context, consisting in the points below :

(1) Having operators acting on symbols (which in some sense, replace the Hilbert space in the original setting for the JLO formula, Fredholm modules ...),
(2) A notion of "heat kernel" (more precisely, a "Laplacian on symbols")
(3) A related trace,
(4) A notion of "Dirac operator", in the sense that for $a \in \mathcal{S}_{H,c}(M)$, $[D, a]$ gives the differential da of the symbol a (up to some details),
(5) A way of computing things (in classical index theory, there are Mehler's formula and Getzler rescaling),
(6) Carry all this to the equivariant case.

We find an answer to these points by adapting the formalism developed by Perrot in **[30]** and **[31]** to the Heisenberg calculus. This formalism is a global version of the flat case exposed in Chapter 2 : For example, the operator F is a "toy model" of Dirac operator. As we shall go along in the chapter, we shall try to make remarks emphasizing these analogies which might facilitate the reading. The constructions are not easy but really flexible because the use of residues has the consequence that these are purely algebraic. Besides, this applies to operators which are not of Dirac type, and more generally to cases where Getzler rescaling does not apply. Though, it should be noted that this Dirac operator is only an intermediary, contrary to the classical JLO situation where it is the object of study. Another crucial feature of Perrot's formalism is to be *invariant under diffeomorphisms*.

These steps accomplished, we compute the JLO cocycle for two different Dirac operators, the first one gives the Radul cocycle of the pseudodifferential extension, the second one gives the Poincaré dual of the equivariant Todd class. The usual homotopy arguments from the original JLO formula borrows verbatim there, thus giving the theorem.

As a corollary, we obtain a new solution to the index problem of the transverse signature operator. For this, we combine the relation between the Radul cocycle and the Chern-Connes character found in Chapter 1 (Theorem II), together with the above theorem. We keep the same notations as in Theorem VI.

THEOREM VII. (THEOREM 3.34) *Let* G *be a discrete group acting by orientation preserving diffeomorphism on a manifold* W*. Let* M *be the bundle of metrics of* W *and* $\mathcal{A} = C^\infty_c(M) \rtimes G$*. If the actions of* G *has no fixed points, then the Chern-Connes character of the Fredholm module* (H, F) *associated to the hypoelliptic signature operator of Connes and Moscovici is*

$$\mathrm{ch}(H, F) = \pi_* \circ \Phi(L'(M)) \in HP^1(\mathcal{A}),$$

where $L'(M)$ *is the modified* L*-genus of Hirzebruch.*

Plan of the chapter. Section 1 recalls the X-complex formalism developed by Cuntz and Quillen for cyclic homology.

Section 2 computes the excision map corresponding to the equivariant pseudodifferential extension (16) and proves Theorem VI.

Sections 3, 4, 5 achieves the steps (1), (2), (3) et (4) given above to construct an algebraic JLO formula on symbols.

Section 6 introduces the required objects to carry all this to the equivariant setting. In particular, we recall the point of view we need to construct Connes' characteristic map from the equivariant cohomology $H^\bullet(M \times_G EG)$ to the periodic cyclic cohomology of the crossed product $HP^\bullet(C^\infty(M) \rtimes G)$. The X-complex is used there.

Section 7 finally gives the JLO formula, leading to Theorem VI.

Section 8 shows how to obtain Theorem VII from Theorem VI.

3.4. Chapter 4. Discussion on conic manifolds. This chapter is a discussion on manifolds with (isolated) conic singularity, and spectral triples associated. The motivation of this work was to apply the results of Chapter 1 in cases where zeta functions exhibits multiple poles. The first work in this direction is due to Lescure in **[25]**, where such spectral triples are constructed. However, the algebra considered in the spectral triple is that of smooth functions vanishing to infinite order in a neighbourhood of the conic point, with a unit adjoined. Thus, many informations are lost in the differential calculus, e.g the abstract algebra of differential operators associated to the spectral triple cannot contain all the conic differential operators. Therefore, it is natural to ask if one can refine the choice of the algebra. Unfortunately, we shall see that the analytic properties of the zeta function in this context gives an obstruction to do better, and that obtaining a regular spectral triple on such spaces inevitably leads us to erase the singularity, as Lescure does. However, looking at this example gives a good picture of what happens when the regularity of the spectral triple is lost. The abstract Radul cocycle of Chapter 1, and thus the index formulas are no more local, because the terms killed by the residue in presence of regularity cannot be neglected in that case. We refer the reader to the concerned chapter for the different definitions and notations.

THEOREM VIII. *Let* M *be a conic manifold, i.e a manifold with boundary endowed with a conic metric, and let* r *be a boundary defining function. Let* Δ *be the "conic laplacian" of Example* **??***. Then, the*

Radul cocycle associated to the pseudodifferential extension of Melrose's b*-calculus :*

$$0 \to r^{\infty}\Psi_b^{-\infty}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M) \to 0$$

is given by the following non-local *formula :*

$$\begin{aligned} c(a_0, a_1) = (\mathrm{Tr}_{\partial,\sigma} + \mathrm{Tr}_{\sigma})(a_0[\log\Delta, a_1]) - \frac{1}{2}\mathrm{Tr}_{\partial,\sigma}(a_0[\log\Delta, [\log\Delta, a_1]]) + \\ + \mathrm{Tr}_{\partial}\left(a_0 \sum_{k=1}^{N} a_1^{(k)}\Delta^{-k}\right) + \frac{1}{2\pi\mathbf{i}}\mathrm{Tr}\left(\int \lambda^{-z} a_0(\lambda-\Delta)^{-1} a_1^{(N+1)}(\lambda-\Delta)^{-N-1}\right) d\lambda \end{aligned}$$

pour $a_0, a_1 \in \Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M)$ *and* N *large enough.*

Tr_{σ} is the extension to M of the Wodzicki residue in the interior of M. $\mathrm{Tr}_{\partial,\sigma}$ measures its default to be a trace on $r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)$. Both are given by a local formula. The second line of the formula is the non-local part : Tr_{∂} is the conic version of the b-trace, obtained as a regularization of the usual trace on regularizing operators, and the integral, which vanishes under the residue when dealing with regular spectral triples, has a non-zero contribution here.

This approach yields another point of view on the eta invariant, the notable fact is that it is suitable also for pseudodifferential operators, and not only for Dirac operators. It might be an interesting problem to compare the formulas obtained with the usual eta invariant.

Plan of the chapter. Sections 1, 2 and 3 recall some facts on Melrose's b-calculus, related notions of residues and heat kernel expansions in this context.

Section 4 explains why the conic zeta function gives an obstruction to get a more refined spectral triple as the one given by Lescure.

Section 5 explains Theorem VIII.

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