ON THE ALGEBRAIC DEPENDENCE OF E-FUNCTIONS

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Abstract. Siegel introduced and studied the class of E-functions in 1929. They are power series, solutions of some linear differential equations, whose Taylor coefficients satisfy certain arithmetic and growth conditions. After the work of Siegel and Shidlovskii, and its refinement by Beukers, on the algebraic relations between values of E-functions over $\overline{\mathbb{Q}}$, it is important to know when the E-functions solutions of a given differential system of order 1 are algebraically dependent or not over $\overline{\mathbb{Q}}(z)$. In this paper, we give the complete classification of the vector solutions of two dimensional differential systems of order 1 whose components are algebraically dependent E-functions over $\overline{\mathbb{Q}}(z)$.

1. Introduction

An E-function is a power series

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n\in\overline{\mathbb{Q}}[[z]]$$

with coefficients in the field of algebraic numbers $\overline{\mathbb{Q}}$ such that

(1) $f(z)$ satisfies a nonzero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$;
(2) there exists $C>0$ such that
   (a) the maximum of the moduli of the galoisian conjuguates of a_n is bounded by C^{n+1};
   (b) there exists a sequence of positive integers d_n such that $d_n\leq C^{n+1}$ and d_na_m is an algebraic integers for all $m\leq n$.

The prototypical example is the exponential function. The E-functions were first introduced by Siegel ([1]) to generalize the diophantine properties of e^z, in particular the Lindemann-Weierstrass Theorem. The work of Siegel [13] and Shidlovskii [14] culminated with the following theorem, which can be seen as vast generalization of the Lindemann-Weierstrass Theorem.

Theorem 1 (Siegel-Shidlovskii). *Let $f_1(z),\dots,f_n(z)$ be E-functions such that*

$$(f_1'(z),\dots,f_n'(z))^t=A(z)(f_1(z),\dots,f_n(z))^t$$

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[1]His definition was slightly less restrictive, but it is now believed that both definitions define the same class of functions.

for some $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. *Denote the common denominator of the entries of* $A(z)$ *by* $T(z)$. *Then, for any* $\xi \in \overline{\mathbb{Q}}$ *such that* $\xi T(\xi) \neq 0$, *we have*

$$\deg \operatorname{tr}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\big(f_1(\xi), \ldots, f_n(\xi)\big) = \deg \operatorname{tr}_{\overline{\mathbb{Q}}(z)} \overline{\mathbb{Q}}(z)\big(f_1(z), \ldots, f_n(z)\big).$$

An alternative proof of the Siegel-Shidlovskii Theorem was given by Bertrand in [3] using Laurent's determinants.

In the seminal paper [1], André elucidated the structure of "E-operators" by means of their relations with G-operators. Any E-function is in the kernel of an E-operator. Using these results, André obtained in [2, Théorème 2.3.1] a completely new proof of the Siegel-Shidlovskii Theorem. Beukers [6] was even able to deduce from the work of André the following important refinement of a theorem of Nesterenko and Shidlovskii [9], which is itself a refinement of the above-mentioned Siegel-Shidlovskii Theorem.

Theorem 2 (Beukers)**.** *With the notations and hypotheses of Theorem 1, let us consider* $\xi \in \overline{\mathbb{Q}}$ *such that* $\xi T(\xi) \neq 0$. *For any polynomial relation* $P\big(f_1(\xi), \ldots, f_n(\xi)\big) = 0$ *with* $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$, *there exists* $Q \in \overline{\mathbb{Q}}[z][X_1, \ldots, X_n]$ *such that* $Q\big(f_1(z), \ldots, f_n(z)\big) = 0$ *and* $P(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n)_{|z=\xi}$.

In order to apply the above transcendence results, the first naive question is: *when are* $f_1(z), \ldots, f_n(z)$ *algebraically dependent over* $\overline{\mathbb{Q}}(z)$*?* The main result of this paper gives a complete answer to this question when $n = 2$. In what follows, for any $\gamma \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we denote by ${}_1F_1(1;\gamma;z)$ the hypergeometric function (which is an E-function) defined by:

$${}_1F_1(1;\gamma;z) = \sum_{k=0}^{\infty} \frac{z^k}{\gamma(\gamma+1)\cdots(\gamma+k-1)}.$$

Note that ${}_1F_1(1;1;z) = e^z$.

Theorem 3. *Let* $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ *be* E*-functions such that*

$$(f'(z), g'(z))^t = E(z)(f(z), g(z))^t$$

for some $E(z) \in M_2(\overline{\mathbb{Q}}(z))$. *If* $f(z)$ *and* $g(z)$ *are algebraically dependent over* $\overline{\mathbb{Q}}(z)$, *then one of the following cases occurs:*

(*i*) *There exist* $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ *and* $\alpha, \beta \in \overline{\mathbb{Q}}$ *such that*

$$f(z) = a(z)e^{\alpha z} + b(z)e^{\beta z} \quad \textit{and} \quad g(z) = c(z)e^{\alpha z} + d(z)e^{\beta z}.$$

(*ii*) *There exist* $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$ *and* $\alpha \in \overline{\mathbb{Q}}$ *such that*

$$f(z) = a(z)\,{}_1F_1(1;\gamma;\alpha z) + b(z) \quad \textit{and} \quad g(z) = c(z)\,{}_1F_1(1;\gamma;\alpha z) + d(z).$$

Remark. Let $f(z)$ be an E-function such that $f'(z) = u(z)f(z) + v(z)$ for some $u(z) \in \overline{\mathbb{Q}}(z)^{\times}$ and $v(z) \in \overline{\mathbb{Q}}(z)$. In particular, $f(z)$ and $f'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Using Theorem 3, it is easily seen that $f(z) = a(z)\,{}_1F_1(1;\gamma;\alpha z) + b(z)$ for some $a(z), b(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \{1\} \cup \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha \in \overline{\mathbb{Q}}$. This provides a complete proof of a result suggested by André in [1, Section 4.5], which answers a question asked by Shidlovskii.

Actually, we will first prove, in section 3, the following result.

Theorem 4. *Let $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ be E-functions such that*

$$(f'(z), g'(z))^t = E(z)(f(z), g(z))^t$$

for some $E(z) \in M_2(\overline{\mathbb{Q}}(z))$. Assume that the differential system $Y'(z) = E(z)Y(z)$ is reducible over $\overline{\mathbb{Q}}(z)$. Then, one of the following cases occurs:

(i) There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ and $\alpha, \beta \in \overline{\mathbb{Q}}$ such that

$$f(z) = a(z)e^{\alpha z} + b(z)e^{\beta z} \quad \text{and} \quad g(z) = c(z)e^{\alpha z} + d(z)e^{\beta z}.$$

(ii) There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$, $\alpha \in \overline{\mathbb{Q}}$ and $\delta \in \overline{\mathbb{Q}}$ such that

$$f(z) = (a(z)_1F_1(1; \gamma; \alpha z) + b(z))e^{\delta z} \quad \text{and} \quad g(z) = (c(z)_1F_1(1; \gamma; \alpha z) + d(z))e^{\delta z}.$$

The proof of Theorem 3, given in section 4, relies on Theorem 4. With the notations and hypotheses of Theorem 3, we actually prove that the differential system $Y'(z) = E(z)Y(z)$ is automatically reducible over $\overline{\mathbb{Q}}(z)$ and that the algebraic number δ given by Theorem 4 can be choosen equal to 0.

The present work was mainly motivated by the following question: is the group of units of the ring $\mathbf{E}$ of values of E-functions at algebraic points equal to $\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}})$? We refer to [10] for an application of Theorem 3 to this problem. The ring $\mathbf{E}$ was introduced in the paper [7], to which we refer the interested reader.

It would be very interesting to extend Theorem 3 to higher order differential systems. Note that, in [11, 12], Salikhov studied *hypergeometric* E-functions solutions of a linear differential equation of order $n \geq 1$ with coefficients in $\overline{\mathbb{Q}}(z)$ and algebraically dependent of their first $n-1$ derivatives. See [5] for further results in this direction.

2. Preliminary step for the proofs of Theorem 3 and Theorem 4

Let $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ be E-functions such that

$$(f'(z), g'(z))^t = E(z)(f(z), g(z))^t$$

for some $E(z) \in M_2(\overline{\mathbb{Q}}(z))$. According to the cyclic vector lemma, there exist a linear differential operator $\mathscr{L}$ of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$, a series $h(z) \in \overline{\mathbb{Q}}[[z]]$ such that $\mathscr{L}h(z) = 0$, and a matrix

$$\begin{pmatrix} p_1(z) & p_2(z) \\ p_3(z) & p_4(z) \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{Q}}(z))$$

such that

$$f(z) = p_1(z)h(z) + p_2(z)h'(z) \quad \text{and} \quad g(z) = p_3(z)h(z) + p_4(z)h'(z).$$

Hence,

$$h(z) = q_1(z)f(z) + q_2(z)g(z) \quad \text{and} \quad h'(z) = q_3(z)f(z) + q_4(z)g(z)$$

where

$$\begin{pmatrix} q_1(z) & q_2(z) \\ q_3(z) & q_4(z) \end{pmatrix} = \begin{pmatrix} p_1(z) & p_2(z) \\ p_3(z) & p_4(z) \end{pmatrix}^{-1} \in \mathrm{GL}_2(\overline{\mathbb{Q}}(z)).$$

Let $\Delta(z) \in \overline{\mathbb{Q}}[z]$ be a common denominator of the $q_i(z)$. Then,

$$k(z) := \Delta(z)h(z) = \Delta(z)q_1(z)f(z) + \Delta(z)q_2(z)g(z)$$

is an E-function (we recall that the set of E-functions is a sub-$\overline{\mathbb{Q}}[z]$-algebra of $\mathbb{Q}[[z]]$), and is a solution of a linear differential operator with coefficients in $\overline{\mathbb{Q}}(z)$ of order 2, namely $\mathscr{L}\Delta^{-1}$. By André's [1, Theorem 4.3], $k(z)$ is solution of a monic linear differential operator $\mathscr{M}$ with coefficients in $\overline{\mathbb{Q}}(z)$ of order $\nu = 1$ or 2 which is a right factor of an E-operator.

Let us first assume that $\nu = 1$. Then, it is well-known that $k(z) = q(z)e^{\alpha z}$ for some $q(z) \in \overline{\mathbb{Q}}[z]$ and $\alpha \in \overline{\mathbb{Q}}$. Therefore, there exist $a(z), c(z) \in \overline{\mathbb{Q}}(z)$ such that $f(z) = a(z)e^{\alpha z}$, and $g(z) = c(z)e^{\alpha z}$. Since $f(z)$ and $g(z)$ are entire functions, we have $a(z), c(z) \in \overline{\mathbb{Q}}[z]$, whence the desired result.

We shall now assume that $\nu = 2$. By André's [1, Theorem 4.3], the differential operator $\mathscr{M}$ has the following properties, that will be freely used in the rest of this paper:

Proposition 1. *We have:*

(1) *$\mathscr{M}$ has only apparent singularities on $\mathbb{C}^\times$;*
(2) *$\mathscr{M}$ is regular singular at 0, and its exponents at 0 are rational;*
(3) *$\mathscr{M}$ admits a basis of formal solutions at ∞ of the form*

$$\big(\widehat{a}_1(z)e^{\alpha_1 z}, \widehat{a}_2(z)e^{\alpha_2 z}\big) = \big(\mathfrak{f}_1(z), \mathfrak{f}_2(z)\big)z^{\Gamma_\infty}e^{\Delta z}$$

where the $\mathfrak{f}_i(z) \in \overline{\mathbb{Q}}[[\frac{1}{z}]]$ are Gevrey-1 series of arithmetic type, $\Gamma_\infty \in M_2(\mathbb{Q})$ is upper-triangular and $\Delta = \mathrm{diag}(\alpha_1, \alpha_2) \in M_2(\overline{\mathbb{Q}})$.

3. Proof of Theorem 4

So, we assume that the differential system $Y'(z) = E(z)Y(z)$ is reducible over $\overline{\mathbb{Q}}(z)$. This implies that the differential operator $\mathscr{M}$ is reducible over $\overline{\mathbb{Q}}(z)$, *i.e.*, that there exist $\eta(z), \omega(z) \in \overline{\mathbb{Q}}(z)$ such that

$$\mathscr{M} = \left(\frac{d}{dz} - \eta(z)\right)\left(\frac{d}{dz} - \omega(z)\right).$$

Since $\mathscr{M}(k(z)) = 0$, the function $u(z) := k'(z) - \omega(z)k(z)$ satisfies $u'(z) = \eta(z)u(z)$.

We claim that $u(z) = r(z)e^{\delta z}$ for some $r(z) \in \overline{\mathbb{Q}}(z)^\times$ and $\delta \in \overline{\mathbb{Q}}$. Indeed, let $\varpi(z) \in \overline{\mathbb{Q}}[z]$ be a denominator of $\omega(z)$. Then, $\varpi(z)u(z) \in \overline{\mathbb{Q}}[[z]]$ satisfies an homogeneous differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$ and is a E-function (because the set of E-functions is a sub-$\overline{\mathbb{Q}}[z]$-differential algebra of $\overline{\mathbb{Q}}[[z]]$). So, $\varpi(z)u(z) = r_1(z)e^{\delta z}$ for some $r_1(z) \in \overline{\mathbb{Q}}[z]$ and $\delta \in \overline{\mathbb{Q}}$. So $u(z) = r(z)e^{\delta z}$ with $r(z) = r_1(z)/\varpi(z)$. Note that $r(z) \neq 0$ because $k(z)$ is not a solution of a linear differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$ by hypothesis.

Then, it is easily seen that $l(z) := e^{-\delta z}k(z)$ is an E-function solution of an inhomogeneous linear differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$, say $y' + a_1(z)y = v(z)$, and that $e^{-\delta z}\zeta(z)$ is a solution of the corresponding homogeneous equation $y' + a_1(z)y = 0$.

Note that $e^{-\delta z}\zeta(z) = z^\gamma p(z)e^{\alpha z}$ for some $\gamma \in \mathbb{Q}$, $p(z) \in \mathbb{C}[z]$ and $\alpha \in \{\alpha_1 - \delta, \alpha_2 - \delta\} \subset \overline{\mathbb{Q}}$. Indeed, this follows from the fact that $\zeta'(z) = \omega(z)\zeta(z)$ together with Proposition 1.

Using the variation of constants method, we get that there exists $C \in \mathbb{C}^{\times}$ such that

$$l(z) = z^{\gamma} p(z) e^{\alpha z} \Bigg(\int_{z_0}^{z} x^{-\gamma} p(x)^{-1} v(x) e^{-\alpha x} dx + C \Bigg), \tag{3.1}$$

where $z_0 \in \mathbb{C}^{\times}$ is not a pole of $p(z)^{-1}v(z)$. We shall now express (3.1) by means of hypergeometric series. For this, we will use the following lemmas.

Lemma 1. *For all $\gamma \notin \mathbb{Z}$, for all $Q(z) \in \mathbb{C}(z)$, for all $z_0 \in \mathbb{C}^{\times}$ which is not a pole of $Q(z)$, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}^{\times}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $\nu \in \mathbb{C}$ such that*

$$\int_{z_0}^{z} x^{-\gamma} Q(x) e^{-\alpha x} dx = \lambda(z) z^{-\gamma} e^{-\alpha z} + \mu(z) \int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha x} dx + \nu.$$

Proof. Using the decomposition in partial fractions of $Q(z)$, we see that it is sufficient to prove the lemma for $Q(z) = Q_n(z) := \frac{1}{(z-\xi)^n}$ with $\xi \in \mathbb{C}^{\times}$ and $n \in \mathbb{N}^*$. We proceed by induction on n. The result is obvious for $n = 1$. Assume that the result is true for some $n \in \mathbb{N}^*$. An integration by parts shows that

$$\begin{aligned}\int_{z_0}^{z} x^{-\gamma} Q_{n+1}(x) e^{-\alpha x} dx &= z^{-\gamma} e^{-\alpha z} \frac{-1}{n} Q_n(z) - z_0^{-\gamma} e^{-\alpha z_0} \frac{-1}{n} Q_n(z_0) \\ &\quad - \frac{\alpha}{n} \int_{z_0}^{z} Q_n(x) x^{-\gamma} e^{-\alpha x} dx - \frac{\gamma}{n} \int_{z_0}^{z} Q_n(x) x^{-\gamma-1} e^{-\alpha x} dx.\end{aligned}$$

The induction hypothesis leads to the desired result. □

Lemma 2. *For all $Q(z) \in \mathbb{C}(z)$, for all $z_0 \in \mathbb{C}^{\times}$ which is not a pole of $Q(z)$, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $\nu \in \mathbb{C}$ such that*

$$\int_{z_0}^{z} Q(x) e^{-\alpha x} dx = \lambda(z) e^{-\alpha z} + \mu(z) \int_{z_0}^{z} R(x) e^{-\alpha x} dx + \nu.$$

Proof. Similar to the proof of Lemma 1. □

In what follows, for any $\gamma \in \mathbb{C} \setminus \mathbb{Z}$ and $\alpha \in \mathbb{C}$, we set:

$$\mathcal{E}_{\gamma,\alpha}(z) = z^{\gamma} \int_{0}^{z} x^{-\gamma} e^{-\alpha x} dx = \sum_{n=0}^{\infty} \frac{(-\alpha)^n z^{n+1}}{(n-\gamma+1)n!}$$

and, for $\gamma \in \mathbb{Z}$, we set $\mathcal{E}_{\gamma,\alpha}(z) = e^{-\alpha z}$. If $\gamma \in \mathbb{Q}$ and $\alpha \in \overline{\mathbb{Q}}$, then $\mathcal{E}_{\gamma,\alpha}(z)$ is an E-function.

Lemma 3. *Consider $\gamma \in \mathbb{C}$, $R(z) \in \mathbb{C}(z)$, $z_0 \in \mathbb{C}^{\times}$ which is not a pole of $R(z)$, $C \in \mathbb{C}$, and $\varphi(z) := z^{\gamma}(\int_{z_0}^{z} x^{-\gamma} R(x) e^{-\alpha x} dx + C)$. Assume that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Then, there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that*

$$\varphi(z) = \lambda(z) e^{-\alpha z} + \mu(z) \mathcal{E}_{\gamma,\alpha}(z) \ \textit{if}\ \gamma \notin \mathbb{Z}$$

and

$$\varphi(z) = \lambda(z)e^{-\alpha z} + \mu(z) \text{ if } \gamma \in \mathbb{Z}.$$

Proof. Let us first assume that $\gamma \notin \mathbb{Z}$. By Lemma 1, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}^{\times}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $C' \in \mathbb{C}$ such that

$$\varphi(z) = \lambda(z)e^{-\alpha z} + \mu(z)z^{\gamma}\int_{z_0}^{z} x^{-\gamma}R(x)e^{-\alpha x}dx + C'z^{\gamma}.$$

If $\mu(z) = 0$, then we must have $C' = 0$ because $\varphi(z) - \lambda(z)e^{-\alpha z}$ is meromorphic over $\mathbb{C}$, and, hence, the result is proved. We now assume that $\mu(z) \neq 0$. If $\xi \in \mathbb{C}^{\times}$ is a (simple) pole of $R(z)$, then ξ is a logarithmic singularity of $\int_{z_0}^{z} x^{-\gamma}R(x)e^{-\alpha x}dx$ and this contradicts the fact that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Therefore, $R(z) = z^{-n}S(z)$ for some integer $n \geq 0$ and some $S(z) \in \mathbb{C}[z]$. Hence,

$$\varphi(z) = \lambda(z)e^{-\alpha z} + \mu(z)z^{\gamma}\int_{z_0}^{z} x^{-\gamma-n}S(x)e^{-\alpha x}dx + C'z^{\gamma}.$$

But, $\int_{z_0}^{z} x^{-\gamma-n}S(x)e^{-\alpha x}dx$ is a linear combination with coefficients in $\mathbb{C}$ of functions of the form $\int_{z_0}^{z} x^{-\gamma-n+k}e^{-\alpha x}dx$ for $k \in \mathbb{N}$. Using integrations by parts, we conclude that $\int_{z_0}^{z} x^{-\gamma-n}S(x)e^{-\alpha x}dx$ is, up to an additive constant in $\mathbb{C}$, a linear combination with coefficients in $\mathbb{C}(z)$ of $z^{-\gamma}e^{-\alpha z}$ and $\int_{z_0}^{z} x^{-\gamma}e^{-\alpha x}dx$. Hence, there exist $\widetilde{\lambda}(z), \widetilde{\mu}(z), \widetilde{\nu}(z) \in \mathbb{C}(z)$ such that

$$\varphi(z) = \widetilde{\lambda}(z)e^{-\alpha z} + \widetilde{\mu}(z)\mathcal{E}_{\gamma,\alpha}(z) + \widetilde{\nu}(z)z^{\gamma}.$$

We must have $\widetilde{\nu}(z) = 0$ because $\varphi(z) - \widetilde{\lambda}(z)e^{-\alpha z} - \widetilde{\mu}(z)\mathcal{E}_{\gamma,\alpha}(z)$ is meromorphic over $\mathbb{C}$. This yields the desired result.

Let us assume that $\gamma \in \mathbb{Z}$. By Lemma 2, there exists $R(z) \in \mathbb{C}(z)$ with at most simple poles on $\mathbb{C}$ and whose set of poles in $\mathbb{C}^{\times}$ is included in the set of poles in $\mathbb{C}^{\times}$ of $Q(z)$, and there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ and $C' \in \mathbb{C}$ such that

$$\varphi(z) = \lambda(z)e^{-\alpha z} + \mu(z)\int_{z_0}^{z} R(x)e^{-\alpha x}dx + C'z^{\gamma}.$$

If $\mu(z) = 0$, the result is proved. We now assume that $\mu(z) \neq 0$. If $\xi \in \mathbb{C}$ is a (simple) pole of $R(z)$, then ξ is a logarithmic singularity of $\int_{z_0}^{z} R(x)e^{-\alpha x}dx$ and this contradicts the fact that $\varphi(z)$ is meromorphic over $\mathbb{C}$. Hence, $R(z) \in \mathbb{C}[z]$. Using integration by parts, we see that $\int_{z_0}^{z} R(x)e^{-\alpha x}dx$ is, up to an additive constant in $\mathbb{C}$, of the form $\eta(z)e^{-\alpha z}$ for some $\eta(z) \in \mathbb{C}[z]$, and this gives the desired result. □

We are now able to express (3.1) in terms of hypergeometric functions.

Let us first assume that $\gamma \notin \mathbb{Z}$. Using (3.1) and Lemma 3, we see that there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that

$$k(z) = l(z)e^{\delta z} = \lambda(z)e^{\delta z} + \mu(z)\mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z}. \tag{3.2}$$

A simple linear algebra argument shows that we can choose $\lambda(z)$ and $\mu(z)$ in $\overline{\mathbb{Q}}(z)$. Furthermore, we have the following relation

$$\mathcal{E}_{\gamma,\alpha}(z)e^{\alpha z} = \frac{\gamma\alpha^2}{z}\left({}_1F_1(1;\gamma;-\alpha z) - 1 + \frac{\alpha z}{\gamma}\right);$$

this is an easy consequence of the fact that both members of this equality satisfy the same nonhomogeneous differential equation of order one, namely

$$zy'(z) - (\gamma + \alpha z)y(z) = z. \tag{3.3}$$

Therefore, there exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z){}_1F_1(1;\gamma;-\alpha z)e^{\delta z} + b(z)e^{\delta z} \quad \text{and} \quad g(z) = c(z){}_1F_1(1;\gamma;-\alpha z)e^{\delta z} + d(z)e^{\delta z}. \tag{3.4}$$

It remains to prove that $a(z), b(z), c(z), d(z)$ belong to $\overline{\mathbb{Q}}[z, z^{-1}]$. Assume that $\xi (\in \overline{\mathbb{Q}})$ is a non-zero pole of $a(z)$ or $b(z)$. Let us denote by n the order of ξ as a pole of $a(z)$. Let us denote by m the order of ξ as a pole of $b(z)$. Let us first assume that $m > n$. Then, multiplying (the first) equation (3.4) by $(z-\xi)^m$ and letting $z = \xi$, we get $0 = \big((z-\xi)^m b(z)e^{\delta z}\big)_{|z=\xi}$ and this is a contradiction. So, we have $n \leq m$. Then, multiplying (the first) equation (3.4) by $(z-\xi)^n$ and letting $z = \xi$, we obtain that ${}_1F_1(1;\gamma;-\alpha\xi)$ belongs to $\overline{\mathbb{Q}}$, and this is a contradiction by [14, p. 192, Theorem 3]. Hence, $a(z)$ and $b(z)$ do not have poles on $\mathbb{C}^\times$ and, hence, belong to $\overline{\mathbb{Q}}[z, z^{-1}]$. A similar argument shows that $c(z)$ and $d(z)$ belong to $\overline{\mathbb{Q}}[z, z^{-1}]$.

We shall now assume that $\gamma \in \mathbb{Z}$. Using (3.1) and Lemma 3, we see that there exist $\lambda(z), \mu(z) \in \mathbb{C}(z)$ such that

$$k(z) = \lambda(z)e^{\delta z} + \mu(z)e^{(\alpha+\delta)z}. \tag{3.5}$$

It is easily seen that one can choose $\lambda(z), \mu(z) \in \overline{\mathbb{Q}}(z)$. It follows that there exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}(z)$ such that

$$f(z) = a(z)e^{(\alpha+\delta)z} + b(z)e^{\delta z} \quad \text{and} \quad g(z) = c(z)e^{(\alpha+\delta)z} + d(z)e^{\delta z}.$$

The proof of the fact that one can choose $a(z), b(z), c(z)$ and $d(z)$ in $\overline{\mathbb{Q}}[z, z^{-1}]$ is analogous to the proof of the similar result in the case $\gamma \notin \mathbb{Z}$, using Lindemann's Theorem.

4. Proof of Theorem 3

So, we assume that $f(z)$ and $g(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. It follows that the functions $k(z)$ and $k'(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Therefore, the differential Galois group of $\mathscr{M}$ over $\overline{\mathbb{Q}}(z)$ does not contain $\mathrm{SL}_2(\overline{\mathbb{Q}})$ (for an introduction to differential Galois theory, we refer to [4, 15] and the references therein). It follows from Kovacic's [8, §1.2, Theorem] that one of the following cases hold:

Case 1: $\mathscr{M}$ has a nonzero solution $\zeta(z)$ such that $\zeta'(z) = \omega(z)\zeta(z)$ for some $\omega(z) \in \overline{\mathbb{Q}}(z)$.

Case 2: $\mathscr{M}$ has a basis of solutions $(\zeta(z), \overline{\zeta}(z))$ such that $\zeta'(z) = \omega(z)\zeta(z)$ and $\overline{\zeta}'(z) = \overline{\omega}(z)\overline{\zeta}(z)$ where $\omega(z)$ is an algebraic function of degree 2 on $\overline{\mathbb{Q}}(z)$, and $\overline{\omega}(z)$ is its Galois conjugate.

Case 3: $\mathscr{M}$ has a basis of solutions of the form $(\zeta(z)\zeta_1(z), \zeta(z)\zeta_2(z))$ where $\zeta(z)$ satisfies $\zeta'(z) = \omega(z)\zeta(z)$ for some $\omega(z) \in \overline{\mathbb{Q}}(z)$ and where $\zeta_1(z)$ and $\zeta_2(z)$ are algebraic functions over $\overline{\mathbb{Q}}(z)$.

We shall now consider each of these cases (we will prove that the last two cases are impossible: the first case is automatically satisfied and, hence, $\mathscr{M}$ is automatically reducible over $\overline{\mathbb{Q}}(z)$).

4.1. **Case 1.** In this case, the operator $\mathscr{M}$ is reducible over $\overline{\mathbb{Q}}(z)$ (indeed, by euclidean division, there exists $\eta(z) \in \overline{\mathbb{Q}}(z)$ such that $\mathscr{M} = \left(\frac{d}{dz} - \eta(z)\right)\left(\frac{d}{dz} - \omega(z)\right)$). We have seen in section 3 that one of the following properties holds true:

(i) There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$ and $\alpha, \beta \in \overline{\mathbb{Q}}$ such that

$$f(z) = a(z)e^{\alpha z} + b(z)e^{\beta z} \quad \text{and} \quad g(z) = c(z)e^{\alpha z} + d(z)e^{\beta z}.$$

(ii) There exist $a(z), b(z), c(z), d(z) \in \overline{\mathbb{Q}}[z, z^{-1}]$, $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$, $\alpha \in \overline{\mathbb{Q}}$ and $\delta \in \overline{\mathbb{Q}}$ such that

$$f(z) = (a(z)\,{}_1F_1(1;\gamma;\alpha z) + b(z))e^{\delta z} \quad \text{and} \quad g(z) = (c(z)\,{}_1F_1(1;\gamma;\alpha z) + d(z))e^{\delta z}.$$

If (i) is satisfied, then Theorem 3 is proved. We shall now assume that (ii) is satisfied. The difference between (ii) and the desired conclusion is the multiplicative factor $e^{\delta z}$. Actually, if $\alpha = 0$ or $\mu(z) = 0$, then we can easily conclude: in this case, we can take $a(z) = c(z) = 0$ and, hence, $f(z)$ and $g(z)$ have the expected form. We now assume that $\alpha \neq 0$ and $\mu(z) \neq 0$. In order to conclude the proof, it is sufficient to show that $\delta = 0$. We will prove that this is indeed the case with the help of the following lemma.

Lemma 4. *For all $\gamma \in \mathbb{C} \setminus \mathbb{Z}$, $\alpha \in \mathbb{C}^\times$ and $\delta \in \mathbb{C}^\times$, the functions $\mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z}$ and $e^{\delta z}$ are algebraically independent over $\mathbb{C}(z)$.*

Proof. We consider the differential system associated with the differential equation (3.3):

$$Y' = A(z)Y \text{ with } A(z) = \begin{pmatrix} \gamma/z + \alpha & 1 \\ 0 & 0 \end{pmatrix}.$$

A fundamental matrix of solutions of this system is given by

$$\mathfrak{Y} = \begin{pmatrix} z^\gamma e^{\alpha z} & \mathcal{E}_{\gamma,\alpha}(z)e^{\alpha z} \\ 0 & 1 \end{pmatrix}.$$

We let G be the differential Galois group over $\mathbb{C}(z)$ of the above differential system, which is seen as an algebraic subgroup of $\mathrm{GL}_2(\mathbb{C})$ via $\mathfrak{Y}$. It is easily seen that

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}.$$

We now consider the differential Galois group H over $\mathbb{C}(z)$ of the differential system

$$Z' = (A(z) + \delta I_2)Z,$$

which is seen as an algebraic subgroup of $\mathrm{GL}_2(\mathbb{C})$ via the fundamental matrix of solutions

$$\mathfrak{Z} = e^{\delta z}\mathfrak{Y} = \begin{pmatrix} z^\gamma e^{(\alpha+\delta)z} & \mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z} \\ 0 & e^{\delta z} \end{pmatrix}.$$

Since the derived subgroup (G,G) of G is equal to the group of unipotent upper triangular matrices $\mathrm{U}_2 \subset \mathrm{GL}_2(\mathbb{C})$, we get that $(H,H)=\mathrm{U}_2$ and, hence, $\mathrm{U}_2 \subset H$. Therefore, the (differential) field extension $\mathbb{C}(z, z^{\gamma}e^{(\alpha+\delta)z}, \mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z}, e^{\delta z})/\mathbb{C}(z, z^{\gamma}e^{(\alpha+\delta)z}, e^{\delta z})$ is transcendental. Whence the desired result. $\square$

Recall that we want to prove that $\delta = 0$ under the assumptions that $\alpha \neq 0$ and $\mu(z) \neq 0$. We know that $k(z)$ and $k'(z) = u(z) + \omega(z)k(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Therefore, $k(z)$ and $u(z)$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Using the equality $k(z) = \lambda(z)e^{\delta z} + \mu(z)\mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z}$ and the facts that $u(z) = r(z)e^{\delta z}$ with $r(z) \neq 0$ and that $\mu(z) \neq 0$, we deduce that $\mathcal{E}_{\gamma,\alpha}(z)e^{(\alpha+\delta)z}$ and $e^{\delta z}$ are algebraically dependent over $\overline{\mathbb{Q}}(z)$. Lemma 4 implies that $\delta = 0$, as expected.

4.2. **Case 2.** We are going to prove that this case does not occur.

Since $\omega(z)$ has degree 2 over $\overline{\mathbb{Q}}(z)$, there exist $r(z), s(z) \in \overline{\mathbb{Q}}(z)$ such that $\omega(z) = r(z) + \sqrt{s(z)}$ and $\sqrt{s(z)} \notin \overline{\mathbb{Q}}(z)$. Hence, $\overline{\omega}(z) = r(z) - \sqrt{s(z)}$.

In what follows, we will denote by $\int r(z)$ and $\int \sqrt{s(z)}$ some primitive integrals of $r(z)$ and $\sqrt{s(z)}$ respectively such that $\zeta(z) = e^{\int r(z) + \int \sqrt{s(z)}}$. We can and will assume that $\overline{\zeta}(z) = e^{\int r(z) - \int \sqrt{s(z)}}$.

We claim that $s(z) = z^n q(z)^2$ for some $n \in \mathbb{Z} \setminus 2\mathbb{Z}$ and $q(z) \in \overline{\mathbb{Q}}(z)^{\times}$. Indeed, write $s(z) = c\prod_{i=1}^{m}(z-s_i)^{n_i}$ with $c \in \overline{\mathbb{Q}}^{\times}$, $s_1, \ldots, s_m \in \overline{\mathbb{Q}}$ pairwise distinct and $n_1, \ldots, n_m \in \mathbb{Z}$. If $s_i \neq 0$, then $n_i \in 2\mathbb{Z}$ because, otherwise, $e^{2\int \sqrt{s(z)}} = \zeta(z)/\overline{\zeta}(z)$ would have a non trivial monodromy around s_i, and this would be contradiction since both $\zeta(z)$ and $\overline{\zeta}(z)$ have trivial monodromy around s_i in virtue of Proposition 1. Therefore, $s(z) = z^n q(z)^2$ for some $n \in \mathbb{Z}$ and $q(z) \in \overline{\mathbb{Q}}(z)^{\times}$. We have $n \in \mathbb{Z} \setminus 2\mathbb{Z}$ because $\sqrt{s(z)} \notin \overline{\mathbb{Q}}(z)$.

If follows that $\int \sqrt{s(z)} = z^{1/2}h_{\infty}(z)$ (up to an additive constant, that we take equal to 0) for some germ of meromorphic function $h_{\infty}(z)$ at ∞. Moreover, near ∞, we have $e^{\int \omega} = z^{\gamma}t_{\infty}(z)e^{p(z)}$ for some $\gamma \in \overline{\mathbb{Q}}$, some germ of analytic function $t_{\infty}(z)$ at ∞ and some $p(z) \in \overline{\mathbb{Q}}[z]$. So, $\zeta(z) = z^{\gamma}t_{\infty}(z)e^{p(z)+z^{1/2}h_{\infty}(z)}$.

Let $\lambda, \mu \in \mathbb{C}$ be such that $k(z) = \lambda\zeta(z) + \mu\overline{\zeta}(z)$. We distinguish two cases:

- If $h_{\infty}(z)$ is analytic and vanishes at ∞, then $e^{-p(z)}k(z)$ has moderate growth at ∞. Since $e^{-p(z)}k(z)$ is an entire function, it is a polynomial. So, $k(z)$ satisfies a first order linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$, which is excluded by assumption.
- Otherwise, we see that $\zeta(z) = z^{\gamma}g_{\infty}(z^{1/2})e^{q(z^{1/2})}$ where g_{∞} is a germ of meromorphic function at ∞ and $q(z^{1/2}) \in \overline{\mathbb{Q}}[z^{1/2}] \setminus \overline{\mathbb{Q}}[z]$. But, Proposition 1 implies that $q(z^{1/2}) = \alpha_i z + \beta$ for some $i \in \{1,2\}$ and $\beta \in \mathbb{C}$, whence a contradiction.

4.3. **Case 3.** We are going to prove that this case does not occur.

Since $k(z)$ is a $\mathbb{C}$-linear combination of $\zeta(z)\zeta_1(2)$ and $\zeta(z)\zeta_2(z)$, we see that $k(z) = \zeta(z)\widetilde{\zeta}(z)$ where $\widetilde{\zeta}(z)$ is an algebraic function over $\mathbb{C}(z)$. Since $\zeta'(z) = \zeta(z)\omega(z)$ with $\omega(z) \in \overline{\mathbb{Q}}(z)$, we get that $\zeta(z) = m(z)e^{r(z)}$ for some $m(z)$ algebraic over $\mathbb{C}(z)$ and some $r(z) \in \overline{\mathbb{Q}}(z)$.

So, $k(z) = \widetilde{m}(z)e^{r(z)}$ where $\widetilde{m}(z)$ is algebraic over $\mathbb{C}(z)$. Since $k(z)$ is an entire function, we must have $r(z) \in \overline{\mathbb{Q}}[z]$. Therefore, the function $e^{-r(z)}k(z)$ is entire and algebraic over $\mathbb{C}(z)$ and, hence, $e^{-r(z)}k(z) \in \overline{\mathbb{Q}}[z]$. If follows that $k(z)$ satisfies a linear differential equation of order 1 with coefficients in $\overline{\mathbb{Q}}(z)$, and this is excluded by hypothesis.

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