ON THE ARCHIMEDEAN AND NONARCHIMEDEAN q-GEVREY ORDERS

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Abstract. q-Difference equations appear in various contexts in mathematics and physics. The "basis" q is sometimes a parameter, sometimes a fixed complex number. In both cases, one classically associates to any series solution of such equations its q-Gevrey order expressing the growth rate of its coefficients : a (nonarchimedean) q^{-1}-adic q-Gevrey order when q is a parameter, an archimedean q-Gevrey order when q is a fixed complex number. The objective of this paper is to relate these two q-Gevrey orders, which may seem unrelated at first glance as they express growth rates with respect to two very different norms. More precisely, let $f(q,z) \in \mathbb{C}(q)[[z]]$ be a series solution of a linear q-difference equation, where q is a parameter, and assume that $f(q,z)$ can be specialized at some $q = q_0 \in \mathbb{C}^\times$ of complex norm > 1. On the one hand, the series $f(q,z)$ has a certain q^{-1}-adic q-Gevrey order s_q. On the other hand, the series $f(q_0,z)$ has a certain archimedean q_0-Gevrey order s_{q_0}. We prove that $s_{q_0} \leq s_q$ "for most q_0". In particular, this shows that if $f(q,z)$ has a nonzero (nonarchimedean) q^{-1}-adic radius of convergence, then $f(q_0,z)$ has a nonzero archimedean radius converges "for most q_0".

Contents

1. Introduction

Let q be a nonzero element of a field K and consider a linear q-difference equation

$$a_n(z)f(q^n z) + a_{n-1}(z)f(q^{n-1}z) + \cdots + a_0(z)f(z) = 0 \tag{1}$$

with coefficients $a_0(z), \ldots, a_n(z) \in K(z)$ such that $a_0(z)a_n(z) \neq 0$.

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These equations, and more generally the q-calculs, appear in various mathematical and physical domains including Gromov-Witten theory [GL03, Roq19], knot theory [GL05], quantum affine algebras and elliptic quantum groups [TV97], *etc.*

1.1. **q-Gevrey estimates.** Assume that K is endowed with an absolute value

$$|\cdot| : K \to \mathbb{R}^+.$$

A solution $f(z) \in K[[z]]$ of (1) may be divergent. The important works of Bézivin in [Béz92] and of Bézivin and Boutabaa in [BB92] give precise (q-Gevrey) estimates on the growth of the coefficients of $f(z)$, that we shall now recall.

Definition 1. *A formal power series*

$$f(z) = \sum_{k \geq 0} f_k z^k \in K[[z]]$$

is q-Gevrey of order $s \in \mathbb{R}$ if there exist $A, B > 0$ such that, for all $k \geq 0$,

$$|f_k| \leq AB^k |q|^{\frac{k(k-1)}{2}s}.$$

This is equivalent to the fact that the series

$$\sum_{k \geq 0} \frac{|f_k|}{|q|^{\frac{k(k-1)}{2}s}} z^k$$

has a nonzero radius of convergence. If this radius of convergence is nonzero and finite, we say that $f(z)$ has exact q-Gevrey order s.

1.1.1. *Archimedean q-Gevrey estimates.* Bézivin proved the following fundamental result (which is for instance the starting point of the resumation theories for the solutions of q-difference equations; see Ramis and Zhang's papers [Zha00, RZ02, Zha02, RSZ06]). It is a q-analogue of a famous result due to Ramis [Ram78, Ram79, Ram84] for differential equations.

Theorem 2 ([Béz92])**.** *Assume that $K = \mathbb{C}$ is the field of complex numbers and that $|\cdot| : \mathbb{C} \to \mathbb{R}^+$ is the usual archimedean norm. If $|q| > 1$, then any solution $f(z) \in \mathbb{C}[[z]]$ of (1) is either convergent or has exact q-Gevrey order $s = 1/r$ for some positive slope r of L at 0.*

The slopes mentioned in Theorem 2 are the slopes of the Newton polygon of L at 0. Let us recall that the Newton polygon $\mathcal{N}_0(L)$ of L at 0 is the convex hull in $\mathbb{R}^2$ of

$$\{(i, j) \mid i \in \mathbb{Z} \text{ and } j \geq \nu_0(a_{n-i})\},$$

where ν_0 denotes the z-adic valuation. This polygon is delimited by two vertical half lines and by k vectors $(r_1, d_1), \ldots, (r_k, d_k) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ having pairwise distinct slopes $\lambda_1 = \frac{d_1}{r_1}, \ldots, \lambda_k = \frac{d_k}{r_k}$, called the slopes of L at 0. See Sauloy's [Sau04] for further informations.

1.1.2. *Nonarchimedean q-Gevrey estimates.* In a subsequent work, Bézivin and Boutabaa proved the following nonarchimedean variant of Theorem 2.

Theorem 3 ([BB92])**.** *Assume that $|\cdot| : K \to \mathbb{R}^+$ is nonarchimedean. If $|q| > 1$, then any solution $f(z) \in K[[z]]$ of (1) is either convergent or has exact q-Gevrey order $s = 1/r$ for some positive slope r of L at 0.*

Example 4. *One can illustrate the previous results with the Tchakaloff series*

$$f(z) = \sum_{k \geq 0} q^{\frac{k(k-1)}{2}} z^k$$

that satisfies

$$qzf(q^2z) - (1+z)f(qz) + f(z) = 0. \tag{2}$$

Of course, if the hypotheses of Theorem 2 or Theorem 3 are satisfied, then $f(z)$ has exact q-Gevrey order 1. This is in accordance with the fact that the slopes of (2) at 0 are 0 and 1.

Remark 5. *For an extension of Theorem 3 to nonlinear q-difference equations, we refer to Di Vizio's* [DV08].

Remark 6. *The situation is radically different when $|q| = 1$; see Di Vizio's* [DV09].

In the present paper, we will focus our attention on the following two cases :

(1) $K = \mathbb{C}(\mathbf{q})$ is the field of rational fractions in the indeterminate $\mathbf{q}$ with coefficients in $\mathbb{C}$, $q = \mathbf{q}$, and $|\cdot| = |\cdot|_{\mathbf{q}^{-1}}$ is the $\mathbf{q}^{-1}$-adic norm defined, for any $a(\mathbf{q}) \in \mathbb{C}(\mathbf{q})$, by

$$|a(\mathbf{q})|_{\mathbf{q}^{-1}} = e^{\deg_{\mathbf{q}} a(\mathbf{q})}$$

($|\cdot|_{\mathbf{q}^{-1}}$ is a nonarchimedean norm with $|q|_{\mathbf{q}^{-1}} = e > 1$, so we can apply Theorem 3);

(2) $K = \mathbb{C}$ and $|\cdot|$ is its usual archimedean norm (we can apply Theorem 2 provided that $|q| > 1$).

1.2. **Statement of the main result.** We are now in a position to describe the problem considered in the present paper. Let

$$f(\mathbf{q}, z) = \sum_{k \geq 0} f_k(\mathbf{q}) z^k \in \mathbb{C}(\mathbf{q})[[z]]$$

be such that

$$a_n(\mathbf{q}, z) f(\mathbf{q}, \mathbf{q}^n z) + a_{n-1}(\mathbf{q}, z) f(\mathbf{q}, \mathbf{q}^{n-1} z) + \cdots + a_0(\mathbf{q}, z) f(\mathbf{q}, z) = 0 \tag{3}$$

for some $a_0(\mathbf{q}, z), \ldots, a_n(\mathbf{q}, z) \in \mathbb{C}(\mathbf{q}, z)$ such that $a_0(\mathbf{q}, z) a_n(\mathbf{q}, z) \neq 0$.

On the one hand, we can apply Theorem 3 to $f(\mathbf{q}, z)$ with $K = \mathbb{C}(\mathbf{q})$ and with the $\mathbf{q}^{-1}$-adic norm $|\cdot| = |\cdot|_{\mathbf{q}^{-1}}$: the series $f(\mathbf{q}, z)$ is either convergent or has some exact $\mathbf{q}$-Gevrey order $s_{\mathbf{q}}$. If $f(\mathbf{q}, z)$ is convergent, then we set $s_{\mathbf{q}} = 0$.

On the other hand, assume that we can specialize the $a_i(\mathbf{q}, z)$ and the $f_k(\mathbf{q})$ at a given $q \in \mathbb{C}$ such that $|q| > 1$. Then, it is meaningful to consider the series

$$f(q, z) = \sum_{k\geq 0} f_k(q) z^k \in \mathbb{C}[[z]].$$

It is a solution of the q-difference equation

$$a_n(q, z) f(q, q^n z) + a_{n-1}(q, z) f(q, q^{n-1} z) + \cdots + a_0(q, z) f(q, z) = 0. \tag{4}$$

If the $a_i(q, z)$ are not all zero, then Theorem 2 ensures that $f(q, z)$ is either convergent or has some exact q-Gevrey order s_q (with respect to the usual archimedean norm $|\cdot|$ on $\mathbb{C}$). If $f(q, z)$ is convergent, then we set $s_q = 0$.

The following theorem, which is the main result of the present paper, gives a relation between s_q and $s_{\mathbf{q}}$.

Theorem 7. *There exist $v(\mathbf{q}, X) \in \mathbb{C}[\mathbf{q}][X] \setminus \{0\}$, $w(\mathbf{q}) \in \mathbb{C}[\mathbf{q}] \setminus \{0\}$ and $M > 0$ such that, for all $m \geq M$, $v(\mathbf{q}, \mathbf{q}^m) \neq 0$ and*

$$f_m(\mathbf{q}) v(\mathbf{q}, \mathbf{q}^m) v(\mathbf{q}, \mathbf{q}^{m-1}) \cdots v(\mathbf{q}, \mathbf{q}^M) w(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]$$

and such that

$$s_q \leq s_{\mathbf{q}}$$

for all but finitely many $q \in \mathbb{C}$ such that
- *$|q| > 1$,*
- *and $v(q, q^m) \neq 0$ for all $m \geq M$.*

We emphasize that it may happen that $s_q > s_{\mathbf{q}}$ for certain choices of q; see the example given in Section 4.

1.3. **Organization of the paper.** The proof of Theorem 7 is given in Section 3. Our proof relies on a preliminary result, namely Proposition 9, which is stated and proven in Section 9. In Section 4, we illustrate Theorem 7 on a $\mathbf{q}$-hypergeometric example.

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2. A preliminary result

In what follows, we let $\mathbf{N}(\cdot)$ be the norm on $\mathbb{C}[\mathbf{q}]$ defined, for any $u(\mathbf{q}) = \sum_{i=0}^{d} u_i \mathbf{q}^i \in \mathbb{C}[\mathbf{q}]$, by

$$\mathbf{N}(u(\mathbf{q})) = \max\{|u_i| \mid i \in \{0, \ldots, d\}\}.$$

(From now on, $|\cdot|$ will denote the usual archimedean norm on $\mathbb{C}$.)

Definition 8. *We will say that a sequence $(u_m(\mathbf{q}))_{m\geq 0} \in \mathbb{C}[\mathbf{q}]^{\mathbb{Z}_{\geq 0}}$ has moderate growth with respect to $\mathbf{N}(\cdot)$ if there exist $A, B > 0$ such that, for all $m \geq 0$,*

$$\mathbf{N}(u_m(\mathbf{q})) \leq AB^m.$$

The aim of this section is to prove the following result, which will be used in Section 3 for proving Theorem 7.

Proposition 9. *Let*

$$f(\mathbf{q},z)=\sum_{m\geq 0} f_m(\mathbf{q})z^m \in \mathbb{C}(\mathbf{q})[[z]]$$

be such that

$$a_n(\mathbf{q},z)f(\mathbf{q},\mathbf{q}^n z)+a_{n-1}(\mathbf{q},z)f(\mathbf{q},\mathbf{q}^{n-1}z)+\cdots+a_0(\mathbf{q},z)f(\mathbf{q},z)=0 \tag{5}$$

for some $a_0(\mathbf{q},z),\ldots,a_n(\mathbf{q},z) \in \mathbb{C}(\mathbf{q},z)$ such that $a_0(\mathbf{q},z)a_n(\mathbf{q},z)\neq 0$. There exist $v(\mathbf{q},X) \in \mathbb{C}[\mathbf{q}][X]\setminus\{0\}$, $w(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]\setminus\{0\}$, a sequence $(u_m(\mathbf{q}))_{m\geq 0} \in \mathbb{C}[\mathbf{q}]^{\mathbb{Z}_{\geq 0}}$ having moderate growth with respect to $\mathbf{N}(\cdot)$ and $M>0$ such that, for all $m\geq M$, $v(\mathbf{q},\mathbf{q}^m)\neq 0$ and

$$f_m(\mathbf{q})=\frac{u_m(\mathbf{q})}{v(\mathbf{q},\mathbf{q}^m)v(\mathbf{q},\mathbf{q}^{m-1})\cdots v(\mathbf{q},\mathbf{q}^M)w(\mathbf{q})}.$$

Before proving Proposition 9, we state and prove some preliminary lemmas. In what follows, we set, for any $u(\mathbf{q})=\sum_{i=0}^{d} u_i\mathbf{q}^i \in \mathbb{C}[\mathbf{q}]$,

$$\begin{aligned}\ell(u(\mathbf{q})) &= \text{the number of nonzero coefficients of the polynomial } u(\mathbf{q})\\ &= \mathrm{card}\{i\in\{0,\ldots,d\} \mid u_i\neq 0\}.\end{aligned}$$

Lemma 10. *For any $u(\mathbf{q},X)\in\mathbb{C}[\mathbf{q}][X]$, the sequence $(\ell(u(\mathbf{q},\mathbf{q}^k)))_{k\geq 0}$ is ultimately constant.*

Proof. Set $u(\mathbf{q},X)=\sum_{0\leq i,j\leq d} u_{i,j}\mathbf{q}^iX^j$ with $u_{i,j}\in\mathbb{C}$. Then, $u(\mathbf{q},\mathbf{q}^k)=\sum_{0\leq i,j\leq d} u_{i,j}\mathbf{q}^{i+kj}$. If k is large enough, then the $i+kj$ are two by two distinct when i,j vary in $\{0,\ldots,d\}$ and, hence, $\ell(u(\mathbf{q},\mathbf{q}^k))=\mathrm{card}\{(i,j)\in\{0,\ldots,d\}^2 \mid u_{i,j}\neq 0\}$ is independent of k large enough. □

Lemma 11. *For any $u(\mathbf{q},X)\in\mathbb{C}[\mathbf{q}][X]$, the sequence $(\mathbf{N}(u(\mathbf{q},\mathbf{q}^k)))_{k\geq 0}$ is ultimately constant.*

Proof. Setting $u(\mathbf{q},X)=\sum_{0\leq i,j\leq d} u_{i,j}\mathbf{q}^iX^j$ with $u_{i,j}\in\mathbb{C}$ and arguing as in the proof of Lemma 10, we see that, for k large enough, $\mathbf{N}(u(\mathbf{q},\mathbf{q}^k))=\max\{|u_{i,j}| \mid 0\leq i,j\leq d\}$ is independent of k. □

We state the following lemma for latter reference; its proof is obvious and left to the reader.

Lemma 12. *The map $\ell:\mathbb{C}[\mathbf{q}]\to\mathbb{Z}_{\geq 0}$ is submultiplicative, i.e., for any $u(\mathbf{q}),v(\mathbf{q})\in\mathbb{C}[\mathbf{q}]$, we have*

$$\ell(u(\mathbf{q})v(\mathbf{q}))\leq \ell(u(\mathbf{q}))\ell(v(\mathbf{q})).$$

Lemma 13. *For any $u_1(\mathbf{q}),\ldots,u_n(\mathbf{q})\in\mathbb{C}[\mathbf{q}]$, we have*

$$\mathbf{N}(u_1(\mathbf{q})\cdots u_n(\mathbf{q}))\leq \ell(u_1(\mathbf{q}))\cdots\ell(u_{n-1}(\mathbf{q}))\mathbf{N}(u_1(\mathbf{q}))\cdots\mathbf{N}(u_n(\mathbf{q})).$$

Proof. Let us first consider the case $n=2$. We have to prove that

$$\mathbf{N}(u_1(\mathbf{q})u_2(\mathbf{q}))\leq \ell(u_1(\mathbf{q}))\mathbf{N}(u_1(\mathbf{q}))\mathbf{N}(u_2(\mathbf{q})).$$

For $k\in\{1,2\}$, we set

$$u_k(\mathbf{q})=\sum_i u_{k,i}\mathbf{q}^i.$$

We have $u_1(\mathbf{q})u_2(\mathbf{q}) = \sum_m a_m \mathbf{q}^m$ with

$$a_m = \sum_{i+j=m} u_{1,i}u_{2,j} = \sum_{i \text{ s.t.} u_{1,i}\neq 0} u_{1,i}u_{2,m-i}$$

and

$$\begin{aligned}|a_m| &\leq \sum_{i \text{ s.t.} u_{1,i}\neq 0} |u_{1,i}||u_{2,m-i}| \\ &\leq \sum_{i \text{ s.t.} u_{1,i}\neq 0} \mathbf{N}(u_1(\mathbf{q}))\mathbf{N}(u_2(\mathbf{q})) = \ell(u_1)\mathbf{N}(u_1(\mathbf{q}))\mathbf{N}(u_2(\mathbf{q})).\end{aligned}$$

Whence the desired result when $n = 2$. The general case follows from the case $n = 2$ by an obvious induction. □

Proof of Proposition 9. We can assume that :

— for all $i \in \{1, \ldots, n\}$, $a_i(\mathbf{q}, z) \in \mathbb{C}[\mathbf{q}][z]$;

— and $\inf\{\nu_z(a_i(\mathbf{q}, z)) \mid i \in \{0, \ldots, n\}\} = 0$ where $\nu_z : \mathbb{C}[\mathbf{q}][z] \to \mathbb{Z}_{\geq 0}$ denotes the z-adic valuation.

Indeed, we can always reduce the problem to this case by multiplying the $\mathbf{q}$-difference equation (5) (on the left) by a suitable nonzero element of $\mathbb{C}[\mathbf{q}][z]$.

We set

$$f(\mathbf{q}, z) = \sum_{k\geq 0} f_k(\mathbf{q})z^k \text{ and } a_i(\mathbf{q}, z) = \sum_{j=0}^{d} a_{i,j}(\mathbf{q})z^j.$$

We have

$$\begin{aligned}&a_n(\mathbf{q}, z)f(\mathbf{q}, \mathbf{q}^n z) + a_{n-1}(\mathbf{q}, z)f(\mathbf{q}, \mathbf{q}^{n-1}z) + \cdots + a_0(\mathbf{q}, z)f(\mathbf{q}, z) \\ &\qquad = \sum_{m\geq 0} \left(\sum_{i=0}^{n} \sum_{j+k=m} a_{i,j}(\mathbf{q})f_k(\mathbf{q})\mathbf{q}^{ki} \right) z^m.\end{aligned}$$

Therefore, the series $f(\mathbf{q}, z) = \sum_{k\geq 0} f_k(\mathbf{q})z^k$ satisfies

$$a_n(\mathbf{q}, z)f(\mathbf{q}, \mathbf{q}^n z) + a_{n-1}(\mathbf{q}, z)f(\mathbf{q}, \mathbf{q}^{n-1}z) + \cdots + a_0(\mathbf{q}, z)f(\mathbf{q}, z) = 0$$

if and only if, for all $m \geq 0$,

$$\sum_{i=0}^{n} \sum_{j+k=m} a_{i,j}(\mathbf{q})f_k(\mathbf{q})\mathbf{q}^{ki} = 0.$$

The latter equation can be rewritten as follows :

(6) $f_m(\mathbf{q})v_0(\mathbf{q}, \mathbf{q}^m) + f_{m-1}(\mathbf{q})v_1(\mathbf{q}, \mathbf{q}^{m-1}) + \cdots + f_{m-d}(\mathbf{q})v_d(\mathbf{q}, \mathbf{q}^{m-d}) = 0$

where

$$v_k(\mathbf{q}, X) = \sum_{i=0}^{n} a_{i,k}(\mathbf{q})X^i.$$

Since $\inf\{\nu_z(a_i(\mathbf{q}, z)) \mid i \in \{0, \ldots, n\}\} = 0$, the polynomial $v_0(\mathbf{q}, X)$ is nonzero, so there exists $M > 0$ such that, for all $m \geq M$,

$$v_0(\mathbf{q}, \mathbf{q}^m) \neq 0.$$

We consider $w(\mathbf{q}) \in \mathbb{C}[\mathbf{q}] \setminus \{0\}$ such that

$$w(\mathbf{q})f_{M-d}(\mathbf{q}), \ldots, w(\mathbf{q})f_{M-1}(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]$$

and we set, for all $m \geq M$,

$$u_m(\mathbf{q}) = v_0(\mathbf{q}, \mathbf{q}^m)v_0(\mathbf{q}, \mathbf{q}^{m-1}) \cdots v_0(\mathbf{q}, \mathbf{q}^M)w(\mathbf{q})f_m(\mathbf{q}).$$

In terms of the $u_m(\mathbf{q})$, the equation (6) can be rewritten as follows :

$$u_m(\mathbf{q}) + u_{m-1}(\mathbf{q})\widetilde{v_{m,1}}(\mathbf{q}) + u_{m-2}(\mathbf{q})\widetilde{v_{m,2}}(\mathbf{q}) + \cdots + u_{m-d}(\mathbf{q})\widetilde{v_{m,d}}(\mathbf{q}) = 0 \tag{7}$$

where

$$\widetilde{v_{m,i}}(\mathbf{q}) = v_0(\mathbf{q}, \mathbf{q}^{m-1})v_0(\mathbf{q}, \mathbf{q}^{m-2}) \cdots v_0(\mathbf{q}, \mathbf{q}^{m-i+1})v_i(\mathbf{q}, \mathbf{q}^{m-i})$$

(with the convention $\widetilde{v_{m,1}}(\mathbf{q}) = v_1(\mathbf{q}, \mathbf{q}^{m-1})$).

Since $u_{M-1}(\mathbf{q}), \ldots, u_{M-d}(\mathbf{q})$ and the $\widetilde{v_{m,i}}(\mathbf{q})$ belong to $\mathbb{C}[\mathbf{q}]$, the equation (7) shows that, for all $m \geq M$,

$$u_m(\mathbf{q}) \in \mathbb{C}[\mathbf{q}].$$

It remains to prove that the sequence $(u_m(\mathbf{q}))_{m \geq M}$ has moderate growth with respect to N. In order to do so, let us first note that it follows from (7) and from the triangular inequality for $\mathbf{N}(\cdot)$ that, for $m \geq M$,

$$\mathbf{N}(u_m(\mathbf{q})) \leq \sum_{i=1}^{d} \mathbf{N}(u_{m-i}(\mathbf{q})\widetilde{v_{m,i}}(\mathbf{q})).$$

Using Lemma 13, we get

$$\mathbf{N}(u_{m-i}(\mathbf{q})\widetilde{v_{m,i}}(\mathbf{q})) \leq \ell(\widetilde{v_{m,i}}(\mathbf{q}))\mathbf{N}(u_{m-i}(\mathbf{q}))\mathbf{N}(\widetilde{v_{m,i}}(\mathbf{q})).$$

But, Lemma 10 and Lemma 11 ensure that there exists $c_0 > 0$ such that, for all $i \in \{0, \ldots, d\}$, for all $k \geq 0$,

$$\ell(v_i(\mathbf{q}, \mathbf{q}^k)) \leq c_0$$

and

$$\mathbf{N}(v_i(\mathbf{q}, \mathbf{q}^k)) \leq c_0.$$

Moreover, using the submultiplicativity of ℓ, we get :

$$\begin{aligned}\ell(\widetilde{v_{m,i}}(\mathbf{q})) &= \ell(v_0(\mathbf{q}, \mathbf{q}^{m-1})v_0(\mathbf{q}, \mathbf{q}^{m-2}) \cdots v_0(\mathbf{q}, \mathbf{q}^{m-i+1})v_i(\mathbf{q}, \mathbf{q}^{m-i})) \\ &\leq \ell(v_0(\mathbf{q}, \mathbf{q}^{m-1}))\ell(v_0(\mathbf{q}, \mathbf{q}^{m-2})) \cdots \ell(v_0(\mathbf{q}, \mathbf{q}^{m-i+1}))\ell(v_i(\mathbf{q}, \mathbf{q}^{m-i})) \leq c_0^i\end{aligned}$$

and, using Lemma 13, we get :

$$\begin{aligned}\mathbf{N}(\widetilde{v_{m,i}}(\mathbf{q})) &= \mathbf{N}(v_0(\mathbf{q}, \mathbf{q}^{m-1})v_0(\mathbf{q}, \mathbf{q}^{m-2}) \cdots v_0(\mathbf{q}, \mathbf{q}^{m-i+1})v_i(\mathbf{q}, \mathbf{q}^{m-i})) \\ &\leq \ell(v_0(\mathbf{q}, \mathbf{q}^{m-1}))\ell(v_0(\mathbf{q}, \mathbf{q}^{m-2})) \cdots \ell(v_0(\mathbf{q}, \mathbf{q}^{m-i+1})) \\ &\times \mathbf{N}(v_0(\mathbf{q}, \mathbf{q}^{m-1}))\mathbf{N}(v_0(\mathbf{q}, \mathbf{q}^{m-2})) \cdots \mathbf{N}(v_0(\mathbf{q}, \mathbf{q}^{m-i+1}))\mathbf{N}(v_i(\mathbf{q}, \mathbf{q}^{m-i})) \\ &\leq c_0^{i-1}c_0^{i-1}c_0 = c_0^{2i-1}\end{aligned}$$

Therefore,

$$\mathbf{N}(u_{m-i}(\mathbf{q})\widetilde{v_{m,i}}(\mathbf{q})) \leq c_0^{3i-1}\mathbf{N}(u_{m-i}(\mathbf{q})).$$

Hence, setting

$$K = \max\{c_0^{3i-1} \mid i \in \{1, \ldots, d\}\},$$

we get

$$\mathbf{N}(u_m(\mathbf{q})) \leq K \sum_{i=1}^{d} \mathbf{N}(u_{m-i}(\mathbf{q})).$$

This implies that the sequence $(\mathbf{N}(u_m(\mathbf{q})))_{m\geq M}$ has at most geometric growth. This concludes the proof. $\square$

3. Proof of Theorem 7

Proposition 9 ensures that there exist $v(\mathbf{q},X)\in\mathbb{C}[\mathbf{q}][X]\setminus\{0\}$, $w(\mathbf{q})\in\mathbb{C}[\mathbf{q}]\setminus\{0\}$, a sequence $(u_m(\mathbf{q}))_{m\geq 0}\in\mathbb{C}[\mathbf{q}]^{\mathbb{Z}_{\geq 0}}$ having moderate growth with respect to $\mathbf{N}(\cdot)$ and $M>0$ such that, for all $m\geq M$, $v(\mathbf{q},\mathbf{q}^m)\neq 0$ and

$$f_m(\mathbf{q})=\frac{u_m(\mathbf{q})}{v(\mathbf{q},\mathbf{q}^m)v(\mathbf{q},\mathbf{q}^{m-1})\cdots v(\mathbf{q},\mathbf{q}^M)w(\mathbf{q})}.$$

By definition, the series

$$\sum_{m\geq 0}\frac{|f_m(\mathbf{q})|_{\mathbf{q}^{-1}}}{|\mathbf{q}|_{\mathbf{q}^{-1}}^{\frac{m(m-1)}{2}s_{\mathbf{q}}}}z^m$$

has a positive radius of convergence. Using the Cauchy-Hadamard formula, we get

$$\limsup_{m\to+\infty}\left|\frac{f_m(\mathbf{q})}{\mathbf{q}^{\frac{m(m-1)}{2}s_{\mathbf{q}}}}\right|_{\mathbf{q}^{-1}}^{\frac{1}{m}}<+\infty,$$

i.e.,

$$\limsup_{m\to+\infty}\frac{1}{m}\left(\deg f_m(\mathbf{q})-\frac{m(m-1)}{2}s_{\mathbf{q}}\right)<+\infty.$$

Therefore, there exists $\alpha>0$ such that, for all m large enough,

$$\deg f_m(\mathbf{q})\leq\frac{m(m-1)}{2}s_{\mathbf{q}}+\alpha m. \tag{8}$$

On the other hand, it is easily seen that there exist some constants $\alpha',\beta'>0$ such that, for all m large enough,

$$\begin{aligned}\deg(v(\mathbf{q},\mathbf{q}^m)v(\mathbf{q},\mathbf{q}^{m-1})\cdots v(\mathbf{q},\mathbf{q}^M)w(\mathbf{q}))\\ \leq\frac{m(m-1)}{2}\deg_X v(\mathbf{q},X)+\alpha' m+\beta'.\end{aligned} \tag{9}$$

Putting (8) and (9) together, we get that there exist some constants $\alpha'',\beta''>0$ such that, for all m large enough,

$$\begin{aligned}\deg u_m(\mathbf{q})&=\deg f_m(\mathbf{q})+\deg(v(\mathbf{q},\mathbf{q}^m)v(\mathbf{q},\mathbf{q}^{m-1})\cdots v(\mathbf{q},\mathbf{q})w(\mathbf{q}))\\ &\leq\frac{m(m-1)}{2}\left(s_{\mathbf{q}}+\deg_X v(\mathbf{q},X)\right)+\alpha'' m+\beta''.\end{aligned} \tag{10}$$

Consider $q\in\mathbb{C}$ with $|q|>1$. We have

$$|u_m(q)|\leq\mathbf{N}(u_m(\mathbf{q}))\sum_{k=0}^{\deg u_m(\mathbf{q})}|q|^k=\mathbf{N}(u_m(\mathbf{q}))\frac{|q|^{\deg u_m(\mathbf{q})+1}-1}{|q|-1}.$$

Using the moderate growth of $(u_m(\mathbf{q}))_{m\geq 0}$ with respect to $\mathbf{N}(\cdot)$ and the estimate (10), we see that there exists $\gamma,\delta>0$ such that, for all m large enough,

$$|u_m(q)|\leq\gamma\delta^m|q|^{\frac{m(m-1)}{2}(s_{\mathbf{q}}+\deg_X v(\mathbf{q},X))}. \tag{11}$$

On the other hand, if we assume that q is such that

— $\deg_X v(q,X) = \deg_X v(\mathbf{q},X)$,
— $w(q) \neq 0$,
— and $v(q,q^m) \neq 0$ for all $m \geq M$

(note that the first two conditions exclude at most finitely many q), then we have

$$v(q,X) = cX^{\deg_X v(\mathbf{q},X)}\widetilde{v}(X)$$

for some $c \in \mathbb{C}^\times$ and some $\widetilde{v}(X) \in 1 + X^{-1}\mathbb{C}[X^{-1}]$ and, hence,

$$v(q,q^m)v(q,q^{m-1})\cdots v(q,q^M)w(q) \sim_{m\to+\infty} d'c'^m q^{\frac{m(m-1)}{2}\deg_X v(\mathbf{q},X)} \tag{12}$$

for some $c', d' \in \mathbb{C}^\times$. Putting (11) and (12) together, we obtain that there exist $\gamma', \delta' > 0$ such that

$$|f_m(q)| \leq \gamma' \delta'^m |q|^{\frac{m(m-1)}{2}s_{\mathbf{q}}}$$

and, hence, $s_q \leq s_{\mathbf{q}}$. This concludes the proof.

4. An example

Let us illustrate Theorem 7 with the $\mathbf{q}$-hypergeometric series

$$f(\mathbf{q},z) = \sum_{k\geq 0} f_k(q)z^k = \sum_{k\geq 0} \frac{(\mathbf{q}-3;\mathbf{q})_k}{(\mathbf{q}-2;\mathbf{q})_k} z^k$$

that satisfies the $\mathbf{q}$-hypergeometric equation

$$f(\mathbf{q},\mathbf{q}^2z) - \frac{(2\mathbf{q}-3)z - (1+(\mathbf{q}-2)/\mathbf{q})}{(\mathbf{q}-3)\mathbf{q}z - (\mathbf{q}-2)/\mathbf{q}} f(\mathbf{q},\mathbf{q}z) + \frac{z-1}{(\mathbf{q}-3)\mathbf{q}z - (\mathbf{q}-2)/\mathbf{q}} f(\mathbf{q},z) = 0. \tag{13}$$

We have used the classical notation for the $\mathbf{q}$-Pochhammer symbols :

$$(a;\mathbf{q})_k = (1-a)(1-a\mathbf{q})\cdots(1-a\mathbf{q}^{k-1}) \text{ if } k \geq 1,$$

and

$$(a;\mathbf{q})_0 = 1.$$

The polynomials

$$v(\mathbf{q},X) = 1-(\mathbf{q}-2)X \in \mathbb{C}[\mathbf{q}][X]\setminus\{0\} \text{ and } w(\mathbf{q}) = 1 \in \mathbb{C}[\mathbf{q}]\setminus\{0\}$$

satisfy

$$f_m(\mathbf{q})v(\mathbf{q},\mathbf{q}^m)v(\mathbf{q},\mathbf{q}^{m-1})\cdots v(\mathbf{q},\mathbf{q})w(\mathbf{q}) \in \mathbb{C}[\mathbf{q}]. \tag{14}$$

We clearly have $s_{\mathbf{q}} = 0$ because $\deg(\mathbf{q}-3;\mathbf{q})_k = \deg(\mathbf{q}-2;\mathbf{q})_k$. Moreover, if $q \in \mathbb{C}$ is such that

— $|q| > 1$,
— $v(q,q^m) \neq 0$ for all $m \geq 0$,
— $q \neq 2, 3$,

then we have $s_q = 0$ because

$$f_k(q) = \frac{(q-3;q)_k}{(q-2;q)_k} \sim_{k\to+\infty} c_q \left(\frac{q-3}{q-2}\right)^k$$

for some $c_q \in \mathbb{C}^{\times}$. In particular, for these q, we have $s_q \leq s_{\mathbf{q}}$ has claimed in Theorem 7. However, note that if $q = 2$ then

$$f_k(q) = f_k(2) = (-1; q)_k \sim_{k \to +\infty} c_2 q^{\frac{k(k-1)}{2}}$$

for some $c_2 \in \mathbb{C}^{\times}$, so that $s_2 = 1 > s_{\mathbf{q}} = 0$. This shows that even if we have found $v(\mathbf{q}, X)$ and $w(\mathbf{q})$ satisfying (14), one cannot conclude that $s_q \leq s_{\mathbf{q}}$ for all $q \in \mathbb{C}$ such that

— $|q| > 1$,

— $v(q, q^m) \neq 0$ for all $m \geq 0$;

we have to discard finitely many such q in general (here, we have to exclude $q = 2$).

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