FUNCTIONAL RELATIONS OF SOLUTIONS OF q-DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the algebraic relations satisfied by the solutions of q-difference equations and their transforms with respect to an auxiliary operator. Our main tool is the parametrized Galois theories developed in [HS08] and [OW15]. The first part of this paper is concerned with the case where the auxiliary operator is a derivation, whereas the second part deals a $\mathbf{q}'$-difference operator. In both cases, we give criteria to guaranty the algebraic independence of a series, solution of a q-difference equation, with either its successive derivatives or its $\mathbf{q}'$-transforms. We apply our results to q-hypergeometric series.

Contents

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Introduction

The study of the differential transcendence of special functions is an old and difficult problem. Only very recently, systematic methods in order to tackle this kind of question were discovered. Indeed, after the seminal work of Cassidy and Singer in [CS07], several authors developed Galoisian approaches in order to study the differential or difference relations between solutions of linear differential or difference equations; see *e.g.* Hardouin and Singer [HS08], Di Vizio, Hardouin and Wibmer [DVHW14b, DVHW14a] and Ovchinikov and Wibmer [OW15]. For instance, this led to a short and comprehensive proof of Hölder's theorem asserting the differential transcendence of Euler's Gamma function; see [HS08]. Also, this enabled the authors of the present paper to study the differential transcendence of generating series issued from the theory of automatic sequences, such as the Baum-Sweet or the Rudin-Shapiro generating series, which turn out to satisfy linear Mahler equations; see [DHR15]. In the present paper, we take a close look at the differential algebraic relations satisfied by solutions of linear q-difference equations.Very little was known about the differential or difference algebraic relations between these solutions. The first results in this direction, due to Bézivin ([BB92]) and Ramis ([Ram92]), assert that a non rational solution of a linear q-difference equation do not satisfy a linear dependence relation with its successive transforms with respect to a derivation or a $\mathbf{q}'$-difference operator provided that $\mathbf{q}'$ is multiplicatively independent of q. Later, the parametrized Galois theories developed by Hardouin and Singer in [HS08] and Ovchinnikov and Wibmer in [OW15] allowed their authors to give complete criteria for the differential or difference transcendence for the solutions of q-difference equations of order one or of systems of such equations. For irreducible q-difference equations, the results of [HS08] allowed to characterize the dependencies of the solutions via the existence of a linear compatible equation in the auxiliary operator. Our paper is mainly concerned with q-difference equations of order greater than two and combines the results of Bézivin and Ramis with the parametrized Galois theories mentioned above. This paper is divided in two parts.

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In the first part, we study the algebraic relations between the successive derivatives of the solutions of linear q-difference equations. These relations are encoded by the parametrized difference Galois groups introduced by Hardouin and Singer in [HS08]. The basic (and, at first sight, quite optimistic) question is: if we know the algebraic relations between the solutions, what can be said about the differential algebraic relations? In Galoisian terms, an equivalent question is: if we know what the non parametrized difference Galois group is, what can be said about the parameterized difference Galois group? Our answer reads as follows. Consider a linear q-difference equation

$$a_n(z)y(q^n z) + a_{n-1}(z)y(q^{n-1}z) + \cdots + a_0(z)y(z) = 0 \tag{0.1}$$

where $a_0(z), \dots, a_{n-1}(z), a_n(z) \in \mathbb{C}(z)$, $a_0(z)a_n(z) \neq 0$, and where q is a non zero complex number with $|q| \neq 1$. Let G be the difference Galois group of this equation. This is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ which reflects the algebraic relations between the solutions of the equation. Let G^δ be its parametrized difference Galois group. This is a differential algebraic subgroup of $\mathrm{GL}_n(\widetilde{\mathbb{C}})$, where $\widetilde{\mathbb{C}}$ is a differential closure of $\mathbb{C}$. As mentioned above, this parametrized difference Galois group reflects the differential algebraic relations between the solutions of the equation. The main result of the first part of the present paper (see Theorem 3.1) can be stated as follows.

Theorem. *Assume that the derived subgroup $G^{\circ,der}$ of the neutral component G° of G is an irreducible almost simple algebraic subgroup of $\mathrm{SL}_n(\mathbb{C})$. Then, G^δ is a subgroup of $G(\widetilde{\mathbb{C}})$ containing $G^{\circ,der}(\widetilde{\mathbb{C}})$.*

For instance, we have the following consequence (see Proposition 2.4).

Proposition. *Let $h(z)$ be a non zero Laurent series solution of (0.1). Let G be the difference Galois group of (0.1).*

- *Assume that G contains $\mathrm{SL}_n(\mathbb{C})$ (with $n \geq 2$) or $\mathrm{Sp}_n(\mathbb{C})$ (with n even). Then $h(z), \dots, h(q^{n-1}z)$ are differentially algebraically independent over $\mathbb{C}(z)$.*
- *Assume that G contains $\mathrm{SO}_n(\mathbb{C})$ with $n \geq 3$. Then $h(z), \dots, h(q^{n-2}z)$ are differentially algebraically independent over $\mathbb{C}(z)$.*

An important family of q-difference equations is given by the generalized q-hypergeometric equations. Assume that $0 < |q| < 1$. Let us fix $n \geq s$, two integers, let $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{R}})^n$, $\underline{b} = (b_1, \dots, b_s) \in (q^{\mathbb{R}} \setminus q^{-\mathbb{N}})^s$, $\lambda \in \mathbb{C}^\times$, and defines $\sigma_q(f(z)) = f(qz)$. Let us consider the generalized q-hypergeometric operator:

$$z\lambda \prod_{i=1}^{n}(a_i\sigma_q - 1) - \prod_{j=1}^{s}\left(\frac{b_j}{q}\sigma_q - 1\right).$$

When $b_1 = q$, this operator admits as solution the q-hypergeometric series:

$$\begin{aligned} {}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z) &= \sum_{m=0}^{\infty} \frac{(\underline{a}; q)_m}{(\underline{b}; q)_m} \lambda^m z^m \\ &= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{n}(1-a_i)(1-a_iq)\dots(1-a_iq^{m-1})}{\prod_{j=1}^{s}(1-b_j)(1-b_jq)\dots(1-b_jq^{m-1})} \lambda^m z^m. \end{aligned}$$

Using [Roq08, Roq11, Roq12], we see that, in many cases, the algebraic group $G^{\circ,der}$ is either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ or the symplectic group $\mathrm{Sp}_n(\mathbb{C})$ (for n even). Therefore, the above results ensure that, in many cases, the q-hypergeometric series are differentially transcendental. To the best of our knowledge, the only previously known result in this direction was due to Hardouin and Singer [HS08] about q-hypergeometric equations of order 2.

The first part of the present paper is organized as follows. Section 1 contains reminders about difference Galois theory. Section 2 contains reminders and complements about the parametrized difference Galois theory developed in [HS08]. In particular, we study the notion of projective isomonodromy. Roughly speaking, we show that if the difference Galois group of (0.1) is large, then we have two possibilities: either the parametrized difference Galois is large, or any solution of (0.1) satisfies a linear differential equation. In Section 3, we prove the above Theorem by showing that the latter case in the previous alternative does not occur. In Section 4, we apply our results to the q-hypergeometric equations.

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In the second part of the paper, we study the algebraic $\mathbf{q}'$-difference equations satisfied by the solutions of the equation (0.1), where $\mathbf{q}'$ is a non zero complex number such that $|q|$ and $|\mathbf{q}'|$ are multiplicatively independent. These relations are reflected by the parametrized difference Galois group introduced by Ovchinikov and Wibmer in [OW15]. Our main results are formally similar to those mentioned above. However, the proofs are more involved in this case because the parametrized difference Galois group are difference affine algebraic group schemes. These are more subtle than the differential algebraic groups. We obtain the following result, see Corollary 9.1.

Theorem. *Let G be the difference Galois group of (0.1). Assume that the derived subgroup $G^{\circ,der}$ of the neutral component G° of G is either $\mathrm{SL}_n(\mathbb{C})$, with $n \geq 2$, $\mathrm{SO}_n(\mathbb{C})$, with $n \geq 3$, or $\mathrm{Sp}_n(\mathbb{C})$, with n even, and assume that there exists h, a non zero a convergent Laurent series solution of (0.1). Then, h is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}(z)$.*

Furthermore, assume that there exist $b(z) \in \mathbb{C}(z)^{\times}$ and $c \in \mathbb{C}^{\times}$, $m \in \mathbb{Z}$ such that $\det(A) = cz^m \frac{b(qz)}{b(z)}$. Then, if $G^{\circ,der} = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ (resp. $G = \mathrm{SO}_n(\mathbb{C})$) then $h(z), \dots, h({\mathbf{q}'}^{n-1}z)$ (resp. $h(z), \dots, h({\mathbf{q}'}^{n-2}z)$) are $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}(z)$.

The second part of the paper is organized as follows. Section 5 contains reminders and complements about the parametrized difference Galois theory developed by Ovchinnikov and Wibmer in [OW15]. Then, we split our study in two cases, depending on the $\sigma_{\mathbf{q}'}$-transcendence of the determinant of the fundamental solution. Since the latter is solution of an order one q-difference equation, we have to compute the parametrized difference Galois group of such equations. This is the goal of Section 6. Then, in Section 7, we deal with projective isomonodromy, and we find basically the same type of result as in the first part. If the difference Galois group of (0.1) is large, then we have two possibilities: either the parametrized difference Galois group is large, or any solution of (0.1) satifies a linear $\mathbf{q}'$-difference equation. In Section 8, we prove that the latter case does not occur when the determinant of the fundamental solution is $\sigma_{\mathbf{q}'}$-algebraic. Hopefully, in all cases, we are able to prove the $\sigma_{\mathbf{q}'}$-transcendence of the meromorphic solutions of (0.1). We apply our main results to the q-hypergeometric series in Section 9.

General conventions. All rings are commutative with identity and contain the field of rational numbers. In particular, all fields are of characteristic zero.

Part 1. Differential relations of solutions of q-difference equations

1. Difference Galois theory

For details on what follows, we refer to [vdPS97, Chapter 1].

A σ_q-ring (R, σ_q) is a ring R together with a ring automorphism $\sigma_q : R \to R$. An ideal of R stabilized by σ_q is called a σ_q-ideal of (R, σ_q). If R is a field, then (R, σ_q) is called a σ_q-field. To simplify notation, we shall, most of the time, write R instead of (R, σ_q).

The ring of constants of the σ_q-ring R is defined by

$$R^{\sigma_q} = \{f \in R \mid \sigma_q(f) = f\}.$$

If R^{σ_q} is a field, it is called the field of constants.

A σ_q-morphism (resp. σ_q-isomorphism) from the σ_q-ring (R, σ_q) to the $\widetilde{\sigma_q}$-ring $(\widetilde{R}, \widetilde{\sigma_q})$ is a ring morphism (resp. ring isomorphism) $\varphi : R \to \widetilde{R}$ such that $\varphi \circ \sigma_q = \widetilde{\sigma_q} \circ \varphi$.

Given a σ_q-ring (R, σ_q), a $\widetilde{\sigma_q}$-ring $(\widetilde{R}, \widetilde{\sigma_q})$ is a R-σ_q-algebra if $\widetilde{R}$ is a ring extension of R and $\widetilde{\sigma_q}_{|R} = \sigma_q$; in this case, we shall often denote $\widetilde{\sigma_q}$ by σ_q. Two R-σ_q-algebras $(\widetilde{R}_1, \widetilde{\sigma_q}_1)$ and $(\widetilde{R}_2, \widetilde{\sigma_q}_2)$ are isomorphic if there exists a σ_q-isomorphism φ from $(\widetilde{R}_1, \widetilde{\sigma_q}_1)$ to $(\widetilde{R}_2, \widetilde{\sigma_q}_2)$ such that $\varphi_{|R} = \mathrm{Id}_R$.

We fix a σ_q-field $\mathbf{K}$ such that $\mathbf{k} = \mathbf{K}^{\sigma_q}$ is algebraically closed. We consider the following linear difference system

$$\sigma_q(Y) = AY, \text{ with } A \in \mathrm{GL}_n(\mathbf{K}), n \in \mathbb{N}^*. \tag{1.1}$$

By [vdPS97, §1.1], there exists a $\mathbf{K}$-σ_q-algebra R such that

1) there exists $U \in \mathrm{GL}_n(R)$ such that $\sigma_q(U) = AU$ (such a U is called a fundamental matrix of solutions of (1.1));
2) R is generated, as a $\mathbf{K}$-algebra, by the entries of U and $\det(U)^{-1}$;
3) the only σ_q-ideals of R are $\{0\}$ and R.

Such a R is called a Picard-Vessiot ring, or PV ring for short, for (1.1) over $\mathbf{K}$. By [vdPS97, Lemma 1.8], we have $R^{\sigma_q} = \mathbf{k}$. Two PV rings are isomorphic as $\mathbf{K}$-σ_q-algebras. A PV ring R is not always an integral domain. However, there exist idempotents elements $e_1, \ldots, e_s$ of R such that $R = R_1 \oplus \cdots \oplus R_s$ where the $R_i = Re_i$ are integral domains which are transitively permuted by σ_q. In particular, R has no nilpotent element and one can consider its total ring of quotients $\mathcal{Q}_R$, *i.e.*, the localization of R with respect to the set of its nonzero divisors, which can be decomposed as the direct sum $\mathcal{Q}_R = K_1 \oplus \cdots \oplus K_s$ of the fields of fractions K_i of the R_i. The ring $\mathcal{Q}_R$ has a natural structure of R-σ_q-algebra and we have ${\mathcal{Q}_R}^{\sigma_q} = \mathbf{k}$. Moreover, the K_i are transitively permuted by σ_q. We call the σ_q-ring $\mathcal{Q}_R$ a total PV ring for (1.1) over $\mathbf{K}$.

The difference Galois group $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ of R over $\mathbf{K}$ is the group of $\mathbf{K}$-σ_q-automorphisms of $\mathcal{Q}_R$ commuting with σ_q:

$$\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) = \{\phi \in \mathrm{Aut}(\mathcal{Q}_R/\mathbf{K}) \mid \sigma_q \circ \phi = \phi \circ \sigma_q\}.$$

Abusing notation, we shall sometimes denote by $\mathrm{Gal}(\mathcal{Q}_R/F)$ the group $\{\phi \in \mathrm{Aut}(\mathcal{Q}_R/F) \mid \sigma_q \circ \phi = \phi \circ \sigma_q\}$ for F a $\mathbf{K}$-σ_q-subalgebra of $\mathcal{Q}_R$.

An easy computation shows that, for any $\phi \in \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$, there exists a unique $C(\phi) \in \mathrm{GL}_n(\mathbf{k})$ such that $\phi(U) = UC(\phi)$. By [vdPS97, Theorem 1.13], the faithful representation ρ_U :

$$\begin{array}{rcl} \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) & \rightarrow & \mathrm{GL}_n(\mathbf{k}) \\ \phi & \mapsto & C(\phi) \end{array}$$

identifies $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ with a linear algebraic subgroup of $\mathrm{GL}_n(\mathbf{k})$. If we choose another fundamental matrix of solutions U, we find a conjugate representation.

A fundamental theorem of difference Galois theory ([vdPS97, Theorem 1.13]) says that R is the coordinate ring of a G-torsor over $\mathbf{K}$. In particular, the dimension of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ as linear algebraic group over $\mathbf{k}$ coincides with the transcendence degree of the K_i over $\mathbf{K}$. Thereby, the difference Galois group controls the algebraic relations satisfied by the solutions.

2. Parametrized Difference Galois theory

We shall use standard notions and notations of difference and differential algebra which can be found in [Coh65] and [vdPS97].

2.1. Differential algebra. A δ-ring (R, δ) is a ring R endowed with a derivation $\delta : R \to R$ (this means that δ is additive and satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$). If R is a field, then (R, δ) is called a δ-field. To simplify notation, we shall, most of the time, write R instead of (R, δ).

We denote by R^δ the ring of δ-constants of the δ-ring R, *i.e.*,

$$R^\delta = \{c \in R \mid \delta(c) = 0\}.$$

If R^δ is a field, it is called the field of δ-constants.

Given a δ-ring (R,δ), a $\widetilde{\delta}$-ring $(\widetilde{R},\widetilde{\delta})$ is a R-δ-algebra if $\widetilde{R}$ is a ring extension of R and $\widetilde{\delta}_{|R}=\delta$; in this case, we often denote $\widetilde{\delta}$ by δ. Let $\mathbf{K}$ be a δ-field. If $\mathbf{L}$ is a $\mathbf{K}$-δ-algebra and a field, we say that $\mathbf{L}/\mathbf{K}$ is a δ-field extension. Let R be a $\mathbf{K}$-δ-algebra and let $a_1,\dots,a_n\in R$. We denote by $\mathbf{K}\{a_1,\dots,a_n\}_\delta$ the smallest $\mathbf{K}$-δ-subalgebra of R containing $a_1,\dots,a_n$. Let $\mathbf{L}/\mathbf{K}$ be a δ-field extension and let $a_1,\dots,a_n\in\mathbf{L}$. We denote by $\mathbf{K}\langle a_1,\dots,a_n\rangle_\delta$ the smallest $\mathbf{K}$-δ-subfield of $\mathbf{L}$ containing $a_1,\dots,a_n$.

The ring of δ-polynomials in the differential indeterminates $y_1,\dots,y_n$ and with coefficients in a differential field $(\mathbf{K},\delta)$, denoted by $\mathbf{K}\{y_1,\dots,y_n\}_\delta$, is the ring of polynomials in the indeterminates $\{\delta^j y_i \mid j\in\mathbb{N}, 1\leq i\leq n\}$ with coefficients in $\mathbf{K}$.

Let R be be a $\mathbf{K}$-δ-algebra and let $a_1,\dots,a_n\in R$. If there exists a nonzero δ-polynomial $P\in\mathbf{K}\{y_1,\dots,y_n\}_\delta$ such that $P(a_1,\dots,a_n)=0$, then we say that $a_1,\dots,a_n$ are δ-algebraically dependent over $\mathbf{K}$. Otherwise, we say that $a_1,\dots,a_n$ are δ-algebraically independent over $\mathbf{K}$.

A δ-field $\mathbf{k}$ is called δ-closed if, for every (finite) set of δ-polynomials $\mathcal{F}$, if the system of δ-equations $\mathcal{F}=0$ has a solution with entries in some δ-field extension $\mathbf{L}$, then it has a solution with entries in $\mathbf{k}$. Note that the field of δ-constants $\mathbf{k}^\delta$ of any δ-closed field $\mathbf{k}$ is algebraically closed. Any δ-field $\mathbf{k}$ has a δ-closure $\widetilde{\mathbf{k}}$, *i.e.*, a δ-closed field extension. Moreover if $\mathbf{k}^\delta$ is algebraically closed then $\widetilde{\mathbf{k}}^\delta=\mathbf{k}^\delta$.

From now on, we consider a δ-closed field $\mathbf{k}$.

A subset $W\subset\mathbf{k}^n$ is Kolchin-closed (or δ-closed, for short) if there exists $S\subset\mathbf{k}\{y_1,\dots,y_n\}_\delta$ such that

$$W=\mathbb{V}(S)=\{a\in\mathbf{k}^n \mid f(a)=0 \text{ for all } f\in S\}.$$

The Kochin-closed subsets of $\mathbf{k}^n$ are the closed sets of a topology on $\mathbf{k}^n$, called the Kolchin topology. The Kolchin-closure of $W\subset\mathbf{k}^n$ is the closure of W in $\mathbf{k}^n$ for the Kolchin topology.

Following Cassidy in [Cas72, Chapter II, Section 1, p. 905], we say that a subgroup $G\subset\mathrm{GL}_n(\mathbf{k})\subset\mathbf{k}^{n\times n}$ is a linear δ-algebraic group (*LDAG*) if G is the intersection of a Kolchin-closed subset of $\mathbf{k}^{n\times n}$ (identified with $\mathbf{k}^{n^2}$) with $\mathrm{GL}_n(\mathbf{k})$. For $F\subset\mathbf{k}$ a δ-subfield, we say that a linear δ-algebraic group $G\subset\mathrm{GL}_n(\mathbf{k})$ is defined over F if G is the zero set of δ-polynomials with coefficients in F. For G a linear δ-algebraic group defined over F and L a δ-field extension of F, we denote by $G(L)$ the set of L-points of G.

A δ-closed subgroup, or δ-subgroup for short, of an LDAG is a subgroup which is Kolchin-closed. The Zariski-closure of a linear differential linear algebraic group $G\subset\mathrm{GL}_n(\mathbf{k})$ is denoted by $\overline{G}$ and is a linear algebraic group defined over $\mathbf{k}$.

2.1.1. *Difference differential algebra.* A (σ_q,δ)-ring (R,σ_q,δ) is a ring R endowed with a ring automorphism σ_q and a derivation $\delta: R\rightarrow R$ (in other words, (R,σ_q) is a σ_q-ring and (R,δ) is a δ-ring) such that σ_q commutes with δ. If R is a field, then (R,σ_q,δ) is called a (σ_q,δ)-field. If there is no possible confusion, we write R instead of (R,σ_q,δ).

We have straightforward notions of (σ_q,δ)-ideals, (σ_q,δ)-morphisms, (σ_q,δ)-algebras, *etc*, similar to the notions recalled in Sections 1 and 2.1. We omit the details and refer for instance to [HS08, Section 6.2], and to the references therein, for details.

In order to use the (σ_q,δ)-Galois theory developed in [HS08], we need to work with a base (σ_q,δ)-field $\mathbf{K}$ such that $\mathbf{k}=\mathbf{K}^{\sigma_q}$ is δ-closed. Most of the common function fields do not satisfy this condition. The following result shows that we can embed any (σ_q,δ)-field with algebraically closed field of constants into a (σ_q,δ)-field with δ-closed field of constants (here constants are σ_q-constants).

Lemma 2.1 ([DHR15, Lemma 2.3]). *Let F be a (σ_q, δ)-field with $\mathbf{k} = F^{\sigma_q}$ algebraically closed. Let $\widetilde{\mathbf{k}}$ be a δ-closed field containing $\mathbf{k}$. Then, the ring $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ is an integral domain whose fraction field $\mathbf{K}$ is a (σ_q, δ)-field extension of F such that $\mathbf{K}^{\sigma_q} = \widetilde{\mathbf{k}}$.*

2.2. **Parametrized Difference Galois theory.** For details on what follows, we refer to [HS08].

Let $\mathbf{K}$ be a (σ_q, δ)-field with $\mathbf{k} = \mathbf{K}^{\sigma_q}$ a δ-closed field. We consider the following linear difference system

$$\sigma_q(Y) = AY \tag{2.1}$$

with $A \in \mathrm{GL}_n(\mathbf{K})$ for some integer $n \geq 1$.

By [HS08, § 6.2.1], there exists a $\mathbf{K}$-(σ_q, δ)-algebra S such that

1) there exists $U \in \mathrm{GL}_n(S)$ such that $\sigma_q(U) = AU$ (such a U is called a fundamental matrix of solutions of (2.1));
2) S is generated, as $\mathbf{K}$-δ-algebra, by the entries of U and $\det(U)^{-1}$;
3) the only (σ_q, δ)-ideals of S are $\{0\}$ and S.

Such a S is called a (σ_q, δ)-Picard-Vessiot ring, or (σ_q, δ)-PV ring for short, for (2.1) over $\mathbf{K}$. It is unique up to isomorphism of $\mathbf{K}$-(σ_q, δ)-algebras. A (σ_q, δ)-PV ring is not always an integral domain. However, there exist idempotent elements $e_1, \dots, e_s$ of S such that $S = S_1 \oplus \cdots \oplus S_s$ where the $S_i = Se_i$ are integral domains stable by δ and transitively permuted by σ_q. In particular, S has no nilpotent element and one can consider its total ring of quotients $\mathcal{Q}_S$. It can be decomposed as the direct sum $\mathcal{Q}_S = K_1 \oplus \cdots \oplus K_s$ of the fields of fractions K_i of the S_i. The ring $\mathcal{Q}_S$ has a natural structure of S-(σ_q, δ)-algebra and we have ${\mathcal{Q}_S}^{\sigma_q} = \mathbf{k}$. Moreover, the K_i are transitively permuted by σ_q. We call the (σ_q, δ)-ring $\mathcal{Q}_S$ a total (σ_q, δ)-PV ring for (2.1) over $\mathbf{K}$.

The (σ_q, δ)-Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ of S over $(\mathbf{K}, \sigma_q, \delta)$ is the group of $\mathbf{K}$-(σ_q, δ)-automorphisms of $\mathcal{Q}_S$:

$$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}) = \{\phi \in \mathrm{Aut}(\mathcal{Q}_S/\mathbf{K}) \mid \sigma_q \circ \phi = \phi \circ \sigma_q \text{ and } \delta \circ \phi = \phi \circ \delta\}.$$

Note that, if $\delta = 0$, then we recover the difference Galois groups considered in Section 1.

A straightforward computation shows that, for any $\phi \in \mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$, there exists a unique $C(\phi) \in \mathrm{GL}_n(\mathbf{k})$ such that $\phi(U) = UC(\phi)$. By [HS08, Proposition 6.18], the faithful representation ρ_U :

$$\begin{array}{ccc} \mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}) & \rightarrow & \mathrm{GL}_n(\mathbf{k}) \\ \phi & \mapsto & C(\phi) \end{array}$$

identifies $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ with a linear δ-algebraic subgroup of $\mathrm{GL}_n(\mathbf{k})$. If we choose another fundamental matrix of solutions U, we find a conjugate representation.

The (σ_q, δ)-Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ of (2.1) reflects the differential algebraic relations between the solutions of (2.1). In particular, the δ-dimension of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ coincides with the δ-transcendence degree of the K_i over $\mathbf{K}$ (see [HS08, Proposition 6.26] for definitions and details).

A (σ_q, δ)-Galois correspondence holds between the δ-closed subgroups of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ and the $\mathbf{K}$-(σ_q, δ)-subalgebras F of $\mathcal{Q}_S$ such that every nonzero divisor of F is a unit of F (see for instance [HS08, Theorem 6.20]). Abusing notation, we still denote by $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/F)$ the group of F-(σ_q, δ)-automorphisms of $\mathcal{Q}_S$. The following proposition is at the heart of the (σ_q, δ)-Galois correspondence.

Proposition 2.2 ([HS08, Theorem 6.20]). *Let S be a (σ_q,δ)-PV ring over $\mathbf{K}$. Let F be a $\mathbf{K}$-(σ_q,δ)-subalgebra of $\mathcal{Q}_S$ such that every nonzero divisor of F is a unit of F. Let H be a δ-closed subgroup of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$. Then, the following hold:*

- $\mathcal{Q}_S^{\mathrm{Gal}^{\delta}(\mathcal{Q}_S/F)}=\{f\in\mathcal{Q}_S \mid \forall\phi\in\mathrm{Gal}^{\delta}(\mathcal{Q}_S/F),\phi(f)=f\}=F$;
- $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathcal{Q}_S^{H})=H$.

Let S be a (σ_q,δ)-PV ring over $\mathbf{K}$ for (2.1) and let $U\in\mathrm{GL}_n(S)$ be a fundamental matrix of solutions. Then, the $\mathbf{K}$-σ_q-algebra R generated by the entries of U and $\det(U)^{-1}$ is a PV ring for (2.1) over $\mathbf{K}$ and we have $\mathcal{Q}_R\subset\mathcal{Q}_S$. One can identify $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ with a subgroup of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ by restricting the elements of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ to $\mathcal{Q}_R$.

Proposition 2.3 ([HS08], Proposition 2.8). *The group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ is a Zariski-dense subgroup of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$.*

2.3. **Transcendence results.** Let $\mathbf{K}$ be a (σ_q,δ)-field with $\mathbf{k}=\mathbf{K}^{\sigma_q}$ a δ-closed field. Let S be a (σ_q,δ)-PV extension for (2.1), with total field of fractions $\mathcal{Q}_S$, let $U\in\mathrm{GL}_n(S)$ be a fundamental matrix of solutions of the system (2.1), and let $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ be the representation of the (σ_q,δ)-Galois group associated to the fundamental matrix of solutions U. We denote by $\mathrm{SO}_n(\mathbf{k})$ the special orthogonal group $\mathrm{SO}_n(\mathbf{k})=\{C\in\mathrm{SL}_n(\mathbf{k})|C^tC=\mathrm{I}_n\}$ and, if n is even, by $\mathrm{Sp}_n(\mathbf{k})$ the symplectic group $\mathrm{Sp}_n(\mathbf{k})=\{C\in\mathrm{GL}_n(\mathbf{k})|C^tJC=J\}$, $J=\begin{pmatrix}0&\mathrm{I}_{n/2}\\-\mathrm{I}_{n/2}&0\end{pmatrix}$, where $\mathrm{I}_{n/2}$ is the identity matrix of size $n/2$.

Proposition 2.4. *Assume that $n\geq 2$. Let $U\in\mathrm{GL}_n(S)$ be a fundamental solution matrix and let $u=(u_1,\dots,u_n)^t$ be a line (resp. column) vector of U. If there exists $\tilde{C}\in\mathrm{GL}_n(\mathbf{k})$ such that the image of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ by the representation $\rho_{U\tilde{C}}$ contains*

- *$\mathrm{SL}_n(\mathbf{k})$ or $\mathrm{Sp}_n(\mathbf{k})$ then $u_1,\dots,u_n$ are δ-algebraically independent over $\mathbf{K}$;*
- *$\mathrm{SO}_n(\mathbf{k})$ then any $n-1$ distinct elements among the u_i's are δ-algebraically independent over $\mathbf{K}$;*

Proof. Let $H\subset\mathrm{GL}_n(\mathbf{k})$ be $\mathrm{SL}_n(\mathbf{k})$ (resp. $\mathrm{SO}_n(\mathbf{k})$ or $\mathrm{Sp}_n(\mathbf{k})$). If the image of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ by the representation $\rho_{U\tilde{C}}$ contains H then the image of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ by the representation ρ_U contains $\widetilde{H}=\tilde{C}H\tilde{C}^{-1}$. By the parametrized Galois correspondence [HS08, Theorem 6.20], we have

- the field $\mathbf{K}_0=\mathcal{Q}_S^{\widetilde{H}}$, made of the elements of $\mathcal{Q}_S$ fixed by $\widetilde{H}$, is (σ_q,δ)-field with $\mathbf{K}_0^{\sigma_q}=\mathbf{k}$,
- $\mathcal{Q}_S$ is a (σ_q,δ)-PV field extension for (2.1) over $\mathbf{K}_0$,
- and in the representation attached to U, the (σ_q,δ)-Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}_0)$ coincides with $\widetilde{H}$.

Moreover, by [HS08, Prop.6.24], the $\mathbf{K}$-δ-algebra $\widetilde{S}=\mathbf{K}_0\{U,\frac{1}{\det(U)}\}_{\delta}$ is a $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}_0)$-torsor. Thus, if we write $\widetilde{S}$ as $\mathbf{K}_0\{X,\frac{1}{\det(X)}\}_{\delta}/\mathfrak{I}$ for some δ-ideal $\mathfrak{I}$ then the following holds

- if $H=\mathrm{SL}_n(\mathbf{k})$ then $\mathfrak{I}$ equals $\{\det(X)-f\}_{\delta}$ the radical δ-ideal generated by $\det(X)-f$ for some $f\in\mathbf{K}_0$;
- if $H=\mathrm{SO}_n(\mathbf{k})$ then $\mathfrak{I}$ equals $\{X\tilde{C}\tilde{C}^tX^t-F,\det(X)-g\}_{\delta}$ the radical δ-ideal generated by $X\tilde{C}\tilde{C}^tX^t-F$ for some $F\in\mathrm{GL}_n(\mathbf{K}_0)$ and $\det(X)-g$ for some $g\in\mathbf{K}_0$;
- if $H=\mathrm{Sp}_n(\mathbf{k})$ then $\mathfrak{I}$ equals $\{X\tilde{C}J\tilde{C}^tX^t-F,\det(X)-g\}_{\delta}$ the radical δ-ideal generated by $X\tilde{C}J\tilde{C}^tX^t-F$ for some $F\in\mathrm{GL}_n(\mathbf{K}_0)$ and $\det(X)-g$ for some $g\in\mathbf{K}_0$.

Let us prove the first claim. Suppose to the contrary that $u_1, \dots, u_n$ are δ-algebraically dependent over $\mathbf{K}$. Let us denote by $(x_1, \dots, x_n)^t$ the corresponding line (resp. column) of δ-indeterminates in X. Then, there exists a non zero δ-polynomial $P \in \mathbf{K}_0\{x_1, \dots, x_n\}_\delta$ that belongs to $\mathfrak{I}$. However, it is easily seen that $\{\det(X) - f\}_\delta \cap \mathbf{K}_0\{x_1, \dots, x_n\}_\delta = \{0\}$. This is a contradiction.

Let us assume that $H = \mathrm{SO}_n(\mathbf{k})$ and set $D = \tilde{C}\tilde{C}^t$. Let us denote by $X_1 = (x_1, \dots, x_n)^t$ the line (resp. column) of δ-indeterminates in X corresponding to u and by X_i for $i \neq 1$ the other lines (resp. columns) of X. Without loss of generality, we can assume that X_1 is the first line (resp. column) of X. We claim that

$$\mathfrak{I} \cap \mathbf{K}_0\{x_1, \dots, x_{n-1}\}_\delta = \{0\}. \tag{2.2}$$

It is equivalent to prove the claim with $\mathbf{K}_0$ replaced by an algebraic closure $\overline{\mathbf{K}_0}$. Indeed, if (2.2) holds with $\mathbf{K}_0$ replaced by an algebraic closure $\overline{\mathbf{K}_0}$, some descent arguments show that (2.2) holds over $\mathbf{K}_0$. Suppose to the contrary that there exists $L(x_1, \dots, x_{n-1}) \in \mathfrak{I} \cap \overline{\mathbf{K}_0}\{x_1, \dots, x_{n-1}\}_\delta$ non zero. Let $U_0 \in \mathrm{GL}_n(\overline{\mathbf{K}_0})$ such that $U_0 D U_0^t = F$ and $\det(U_0) = g$. We can decompose U_0 as $L_0 Q$ where L_0 is lower triangular and Q is in $SO_D = \{Q \in \mathrm{GL}_n(\overline{\mathbf{K}_0}) | QDQ^t = D \text{ and } \det(Q) = 1\}$. Then, $L_0 D L_0^t = F$ and $\det(L_0) = g$. Set $Y = L_0^{-1}X$. Then, Y is a matrix of δ-indeterminates and $\overline{\mathbf{K}_0}\{X, \frac{1}{\det(X)}\}_\delta = \overline{\mathbf{K}_0}\{Y, \frac{1}{\det(Y)}\}_\delta$. Denote by $(y_1, \dots, y_n)^t$ be the first line (resp. column) of Y. Moreover, $\mathfrak{I} = \{YDY^t - D, \det(Y) - 1\}_\delta$ and since L_0 is invertible and lower triangular, $L(x_1, \dots, x_{n-1}) = \tilde{L}(y_1, \dots, y_{n-1})$ for some non-zero δ-polynomial $\tilde{L}(y_1, \dots, y_{n-1}) \in \mathfrak{I} \cap \overline{\mathbf{K}_0}\{y_1, \dots, y_{n-1}\}_\delta$. This last assertion contradicts the Gram-Schmidt process for the quadratic form YDY^t.

Let us assume that $H = \mathrm{Sp}_n(\mathbf{k})$ and set $D = \tilde{C}J\tilde{C}^t$. Let us denote by $X_1 := (x_1, \dots, x_n)^t$ the line (resp. column) of δ-indeterminates in X corresponding to u and by X_i for $i \neq 1$ the other lines of X. Without loss of generality, we can assume that X_1 is the first line (resp. column) of X. We claim that

$$\mathfrak{I} \cap \mathbf{K}_0\{x_1, \dots, x_n\}_\delta = \{0\}. \tag{2.3}$$

As above, it is equivalent to prove the claim with $\mathbf{K}_0$ replaced by an algebraic closure $\overline{\mathbf{K}_0}$. Suppose to the contrary that there exists $L(x_1, \dots, x_n) \in \mathfrak{I} \cap \overline{\mathbf{K}_0}\{x_1, \dots, x_n\}_\delta$ non zero. Let $D = \tilde{C}J\tilde{C}^t$. Let $U_0 \in \mathrm{GL}_n(\overline{\mathbf{K}_0})$ such that $U_0 D U_0^t = F$ and $\det(U_0) = g$. Let $V \in \overline{\mathbf{K}_0}^n$ be the first line vector of U_0. Since V is non zero, there exists a basis $\underline{e} = \{e_1, \dots, e_n\}$ of $\overline{\mathbf{K}_0}^n$ such that $e_0 = V$ and $\underline{e}$ is a symplectic basis for the symplectic form XDX^t. This proves that one can write $U_0 = L_0 S$ where $S \in Sp_D = \{S' \in \mathrm{GL}_n(\overline{\mathbf{K}_0}) | S'DS'^t = D\}$ and the first line of L_0 is $\begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}$. Then, $L_0 D L_0^t = F$ and $\det(L_0) = g$. Set $Y = L_0^{-1}X$. Then, Y is a matrix of δ-indeterminates and $\overline{\mathbf{K}_0}\{X, \frac{1}{\det(X)}\}_\delta = \overline{\mathbf{K}_0}\{Y, \frac{1}{\det(Y)}\}_\delta$. Denote by $(y_1, \dots, y_n)^t$ be the first line (resp. column) of Y. Moreover, $\mathfrak{I} = \{YDY^t - D, \det(Y) - 1\}_\delta$ and $L(x_1, \dots, x_n) = \tilde{L}(y_1, \dots, y_n) \in \mathfrak{I} \cap \overline{\mathbf{K}_0}\{y_1, \dots, y_n\}_\delta$ for some non-zero δ-polynomial $\tilde{L}(y_1, \dots, y_n) \in \mathfrak{I} \cap \overline{\mathbf{K}_0}\{y_1, \dots, y_n\}_\delta$. Since any non-zero vector can be completed into a symplectic basis of $\overline{\mathbf{K}_0}^n$ for the symplectic form YDY^t, we get that $\tilde{L}(y_1, \dots, y_n) = 0 = L(x_1, \dots, x_n)$. This is a contradiction. □

2.4. **Projective isomonodromy.** Let $\mathbf{K}$ be a (σ_q, δ)-field with $\mathbf{k} = \mathbf{K}^{\sigma_q}$ algebraically closed. Let $\widetilde{\mathbf{k}}$ be a δ-closure of $\mathbf{k}$. Let $\mathbf{C} = \widetilde{\mathbf{k}}^\delta = \mathbf{k}^\delta$ be the (algebraically closed) field of constants of $\widetilde{\mathbf{k}}$. Lemma 2.1 ensures that $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{K}$ is an integral domain and that $\mathbf{L} = \mathrm{Frac}(\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{K})$ is a (σ_q, δ)-field extension of $\mathbf{K}$ such that $\mathbf{L}^{\sigma_q} = \widetilde{\mathbf{k}}$. We let $\mathcal{Q}_S$ be the total ring of quotients of a (σ_q, δ)-PV ring S over $\mathbf{L}$ of the difference system

$$\sigma_q(Y) = AY$$

where $A \in \mathrm{GL}_n(\mathbf{K})$. The following proposition generalises [DHR15, Proposition 2.10].

Proposition 2.5. *The following properties are equivalent:*

(1) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is conjugate to a subgroup of* $\widetilde{\mathbf{k}}^{\times}\mathrm{SL}_n(\mathbf{C})$*;*
(2) *there exists* $B \in \mathbf{K}^{n\times n}$ *such that*

$$\sigma_q(B)A = AB + \delta(A) - \frac{1}{n}\delta(\det(A))\det(A)^{-1}A. \tag{2.4}$$

Proof. We shall first prove that (1) holds if and only if there exists $B \in \mathbf{L}^{n\times n}$ that satisfies (2.4).

Let us first assume that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\widetilde{\mathbf{k}}^{\times}\mathrm{SL}_n(\mathbf{C})$. So, there exists a fundamental matrix of solutions $U \in \mathrm{GL}_n(S)$ of $\sigma_q(Y) = AY$ such that, for all $\phi \in \mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$, there exist $\rho_\phi \in \widetilde{\mathbf{k}}^{\times}$ and $M_\phi \in \mathrm{SL}_n(\mathbf{C})$ such that $\phi(U) = U\rho_\phi M_\phi$. Let $\mathbf{d} = \det(U) \in S^{\times}$. Note that $\phi(\mathbf{d}) = \mathbf{d}\rho_\phi^n$. Easy calculations show that the matrix

$$B = \delta(U)U^{-1} - \frac{1}{n}\delta(\mathbf{d})\mathbf{d}^{-1}I_n \in S^{n\times n}$$

is left invariant by the action of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$, and, hence, belongs to $\mathbf{L}^{n\times n}$ in virtue of [HS08, Proposition 6.26], and that B satisfies equation (2.4).

Conversely, assume that there exists $B \in \mathbf{L}^{n\times n}$ satisfying equation (2.4). Consider

$$B_1 = B + \frac{1}{n}\delta(\mathbf{d})\mathbf{d}^{-1}I_n \in S^{n\times n}.$$

Note that

$$\sigma_q(B_1) = AB_1A^{-1} + \delta(A)A^{-1}.$$

Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of $\sigma_q Y = AY$. We have $\sigma_q(\delta(U) - B_1U) = A(\delta(U) - B_1U)$. So, there exists $C \in \widetilde{\mathbf{k}}^{n\times n}$ such that $\delta(U) - B_1U = UC$. Since $\widetilde{\mathbf{k}}$ is δ-closed, we can find $D \in \mathrm{GL}_n(\widetilde{\mathbf{k}})$ such that $\delta(D) + CD = 0$. Then, $V = UD$ is a fundamental matrix of solutions of $\sigma_q Y = AY$ such that $\delta(V) = B_1V$. Consider $\phi \in \mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ and let $M_\phi \in \mathrm{GL}_n(\widetilde{\mathbf{k}})$ be such that $\phi(V) = VM_\phi$; note that $\phi(\mathbf{d}) = \mathbf{d}\rho_\phi$ where $\rho_\phi = \det(M_\phi)$. On the one hand, we have $\phi(\delta(V)) = \phi(B_1V) = (B_1 + \frac{1}{n}\delta(\rho_\phi)\rho_\phi^{-1}I_n)VM_\phi$. On the other hand, we have $\phi(\delta(V)) = \delta(\phi(V)) = \delta(VM_\phi) = B_1VM_\phi + V\delta(M_\phi)$. So, $\frac{1}{n}\delta(\rho_\phi)\rho_\phi^{-1}M_\phi = \delta(M_\phi)$. So, the entries of $M_\phi = (m_{i,j})_{1\le i,j\le n}$ are solutions of $\delta(y) = \frac{1}{n}\delta(\rho_\phi)\rho_\phi^{-1}y$. Let i_0, j_0 be such that $m_{i_0,j_0} \neq 0$. Then, $M_\phi = m_{i_0,j_0}M'$ with $M' = \frac{1}{m_{i_0,j_0}}M_\phi \in \mathrm{GL}_n(\widetilde{\mathbf{k}}^{\delta}) = \mathrm{GL}_n(\mathbf{C})$. Since C is algebraically closed, we can write $M' = \lambda M''$ with $\lambda \in \mathbf{C}^{\times}$ and $M'' \in \mathrm{SL}_n(\mathbf{C})$, whence the desired result.

To conclude the proof, we have to show that if (2.4) has a solution B in $\mathbf{L}^{n\times n}$ then it has a solution in $\mathbf{K}^{n\times n}$. This can be proved by using an argument similar to the descent argument used in the proof of [DHR15, Proposition 2.6]. □

In what follows, we denote by $N_G(H)$ the normalizer of H in G.

Lemma 2.6. *Let H be an irreducible subgroup of* $\mathrm{SL}_n(\mathbf{C})$*. Then,*

$$N_{\mathrm{GL}_n(\widetilde{\mathbf{k}})}(H) = \widetilde{\mathbf{k}}^{\times}N_{\mathrm{SL}_n(\mathbf{C})}(H).$$

Proof. Let $M \in \mathrm{GL}_n(\widetilde{\mathbf{k}})$ be in the normalizer of H. Consider $N \in H$. We have $MNM^{-1} \in H$. In particular, we have $\delta(MNM^{-1}) = 0$, *i.e.*, $\delta(M)NM^{-1} - MNM^{-1}\delta(M)M^{-1} = 0$, so $M^{-1}\delta(M)$ commutes with N. It follows from Schur's lemma that $M^{-1}\delta(M) = cI_n$ for some $c \in \widetilde{\mathbf{k}}^{\times}$. So, the entries of $M = (m_{i,j})_{1\le i,j\le n}$ are solutions of $\delta(y) = cy$. Let i_0, j_0 be such that $m_{i_0,j_0} \neq 0$.

Then, $M = m_{i_0,j_0}M'$ with $M' = \frac{1}{m_{i_0,j_0}}M \in \mathrm{GL}_n(\widetilde{\mathbf{k}}^{\delta}) = \mathrm{GL}_n(\mathbf{C})$. Since $\mathbf{C}$ is algebraically closed, we can write $M = \lambda M''$ for $M'' \in \mathrm{SL}_n(\mathbf{C})$ and $\lambda \in \mathbf{C}^{\times}$. Hence, the normalizer of H in $\mathrm{GL}_n(\widetilde{\mathbf{k}})$ is included in $\mathbf{k}^{\times} N_{\mathrm{SL}_n(\mathbf{C})}(H)$. It follows that $N_{\mathrm{GL}_n(\widetilde{\mathbf{k}})}(H) \subset \widetilde{\mathbf{k}}^{\times} N_{\mathrm{SL}_n(\mathbf{C})}(H)$. The other inclusion is obvious. □

For any algebraic subgroup G of $\mathrm{GL}_n(\mathbf{k})$, let G° be the neutral component of G and $G^{\circ,der}$ be the derived subgroup of G°. We recall that a linear algebraic group G is almost simple if it is infinite, non-commutative and if every proper normal closed subgroup of G is finite. In particular, G is connected. Moreover, G equals its commutator group G^{der}.

Proposition 2.7. *Assume that the difference Galois group G of $\sigma_q(Y) = AY$ over the σ_q-field $\mathbf{K}$ satisfies the following property: the algebraic group $G^{\circ,der}$ is an irreducible almost simple algebraic subgroup of $\mathrm{GL}_n(\mathbf{k})$ defined over $\mathbf{C}$. Then, we have the following alternative:*

1. *$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\widetilde{\mathbf{k}}^{\times} N_{\mathrm{SL}_n(\mathbf{C})}(G^{\circ,der}(\mathbf{C}))$ containing $G^{\circ,der}$;*
2. *$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is equal to a subgroup of $G(\widetilde{\mathbf{k}})$ containing $G^{\circ,der}(\widetilde{\mathbf{k}})$.*

Furthermore, the first case holds if and only if there exists $B \in \mathbf{K}^{n\times n}$ such that

$$\sigma_q(B)A = AB + \delta(A) - \frac{1}{n}\delta(\det(A))\det(A)^{-1}A. \tag{2.5}$$

Proof. Let R be the $\mathbf{L}$-σ_q-algebra generated by the entries of U and by $\det(U)^{-1}$; this is a PV ring for $\sigma_q(Y) = AY$ over the σ_q-field $\mathbf{L}$. Using [CHS08, Corollary 2.5], we see that $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L}) = G(\widetilde{\mathbf{k}})$. So, $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})^{\circ,der} = G^{\circ,der}(\widetilde{\mathbf{k}})$. Since $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is Zariski-dense in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})$ (see Proposition 2.3), we have that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{der_{\delta}*}$ is Zariski-dense in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})^{der} = G^{\circ,der}(\widetilde{\mathbf{k}})$. By [Cas72, Proposition 42], $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{der_{\delta}}$ is either conjugate to $G^{\circ,der}(\mathbf{C})$ or equal to $G^{\circ,der}(\widetilde{\mathbf{k}})$. Since $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{der_{\delta}}$ is a normal subgroup of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$, Lemma 2.6 ensures that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is either conjugate to a subgroup of $\widetilde{\mathbf{k}}^{\times} N_{\mathrm{SL}_n(\mathbf{C})}(G^{\circ,der}(\mathbf{C}))$ containing $G^{\circ,der}(\mathbf{C})$ or is equal to a subgroup of $G(\widetilde{\mathbf{k}})$ containing $G^{\circ,der}(\widetilde{\mathbf{k}})$.

The remaining statement is a direct consequence of Proposition 2.5. □

3. Large (σ_q, δ)-Galois group of q-difference equations

In this section, we focus our attention on q-difference equations over $\mathbb{C}(z)$. Let us consider the field $\mathbb{C}(z)$ and the algebraic closure $\overline{\mathbb{C}(z)}$ of $\mathbb{C}(z)$ in (the algebraically closed field) $\bigcup_{j=1}^{\infty}\mathbb{C}((z^{1/j}))$. A non zero complex number q such that $|q| \neq 1$ being given, the field automorphism

$$\begin{array}{rcl} \sigma_q : \overline{\mathbb{C}(z)} & \rightarrow & \overline{\mathbb{C}(z)} \\ f(z) & \mapsto & f(qz) \end{array}$$

gives a structure of σ_q-field on $\overline{\mathbb{C}(z)}$. We have $\overline{\mathbb{C}(z)}^{\sigma_q} = \mathbb{C}$. The derivation $\delta = z\frac{d}{dz}$ endows $\overline{\mathbb{C}(z)}$ with a structure of (σ_q, δ)-field. Note also that $\mathbb{C}(z)$ is a (σ_q, δ)-subfield of $\overline{\mathbb{C}(z)}$ with $\mathbb{C}(z)^{\sigma_q} = \mathbb{C}$.

Let $(\widetilde{\mathbb{C}}, \delta)$ be a δ-field that contains $(\mathbb{C}, \delta)$ and which is δ-closed. According to Lemma 2.1, the (σ_q, δ)-field

$$\mathbf{L} = \mathrm{Frac}(\widetilde{\mathbb{C}} \otimes_{\mathbb{C}} \overline{\mathbb{C}(z)})$$

is a (σ_q, δ)-field extension of $\overline{\mathbb{C}(z)}$ such that $\mathbf{L}^{\sigma_q} = \widetilde{\mathbb{C}}$.

*This is the Kolchin-closure of the derived subgroup of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$; see [DHR15, Section 4.4.1].

Consider the q-difference system

$$\sigma_q(Y) = AY \tag{3.1}$$

with $A \in \mathrm{GL}_n(\mathbb{C}(z))$. In what follows, we let S be a (σ_q, δ)-PV ring over $\mathbf{L}$ for the equation (3.1), $\mathcal{Q}_S$ be the total ring of quotients of S, and we denote by $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ the corresponding (σ_q, δ)-Galois group over $\mathbf{L}$.

The theorem below shows that if the difference Galois group of a q-difference system is large, the same holds for the parametrized difference Galois group.

Theorem 3.1. *Let G be the difference Galois group of the q-difference system (3.1) over the σ_q-field $\mathbb{C}(z)$. Assume that $G^{\circ,der}$ is an irreducible almost simple algebraic subgroup of $\mathrm{SL}_n(\mathbb{C})$. Then, $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is a subgroup of $G(\widetilde{\mathbb{C}})$ containing $G^{\circ,der}(\widetilde{\mathbb{C}})$.*

Before giving the proof of Theorem 3.1, we state and prove some preliminary results.

Lemma 3.2. *Let G be the difference Galois group of (3.1) over the σ_q-field $\mathbb{C}(z)$. Let H be the difference Galois group of (3.1) over the σ_q-field $\mathbf{L}$. Then, $H = G^{\circ}(\widetilde{\mathbb{C}})$. Hence, if $G^{\circ,der}$ is an irreducible almost simple algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ then H^{der} equals $G^{\circ,der}(\widetilde{\mathbb{C}})$ and is an irreducible almost simple algebraic subgroup of $\mathrm{GL}_n(\widetilde{\mathbb{C}})$.*

Proof. Since $\overline{\mathbb{C}(z)}$ is an algebraic extension of $\mathbb{C}(z)$, [Roq15, Theorem 7] implies that the difference Galois group G' of (3.1) over the σ_q-field $\overline{\mathbb{C}(z)}$ has the same connected component as G. Moreover, since $\overline{\mathbb{C}(z)}$ is algebraically closed, the difference Galois correspondence implies that G' is connected and therefore coincides with G°. By [CHS08, Corollary 2.5], the group H is isomorphic to $G'(\widetilde{\mathbb{C}})$. □

Lemma 3.3. *Assume that the system (3.1), has a solution $u = (u_1, \ldots, u_n)^t$ with coefficients in $\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$. Then, there exists a (σ_q, δ)-PV ring T over $\mathbf{L}$ of (3.1) that contains the $\mathbf{L}$-δ-algebra $\mathbf{L}\{u_1, \ldots, u_n\}_{\delta}$.*

Proof. The result is obvious if $u = (0, \ldots, 0)^t$. We shall now assume that $u \neq (0, \ldots, 0)^t$. We equip $\displaystyle\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$ with the structure of (σ_q, δ)-field given by $\sigma_q(f(z)) = f(qz)$ and $\delta = z\frac{d}{dz}$. It is easily seen that we have $\displaystyle\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))^{\sigma_q} = \mathbb{C}$. We let $F = \overline{\mathbb{C}(z)}\langle u_1, \ldots, u_n\rangle_{\delta}$ be the δ-subfield of $\displaystyle\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$ generated over $\overline{\mathbb{C}(z)}$ by the series $u_1, \ldots, u_n$; this is a (σ_q, δ)-subfield of $\displaystyle\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$ such that $F^{\sigma_q} = \mathbb{C}$. By Lemma 2.1, $\widetilde{\mathbb{C}} \otimes_{\mathbb{C}} F$ is an integral domain and its field of fractions $\mathbf{L}_1 = \mathbf{L}\langle u_1, \ldots, u_n\rangle_{\delta}$ is a (σ_q, δ)-field such that ${\mathbf{L}_1}^{\sigma_q} = \widetilde{\mathbb{C}}$. We consider a (σ_q, δ)-PV ring S_1 for (3.1) over $\mathbf{L}_1$ and we let $U \in \mathrm{GL}_n(S_1)$ be a fundamental matrix of solutions of this difference system. We can assume that the first column of U is u. Then, the $\mathbf{L}$-(σ_q, δ)-algebra T generated by the entries of U and by $\det(U)^{-1}$ contains $\mathbf{L}\{u_1, \ldots, u_n\}_{\delta}$ and is a (σ_q, δ)-PV ring for (3.1) over $\mathbf{L}$. Whence the result. □

Lemma 3.4. *Let us consider a vector $u = (u_1, \ldots, u_n)^t$ with coefficients in $\displaystyle\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$ which is solution of (3.1). Assume moreover that each u_i satisfies* *some nonzero linear differential equation with coefficients in* $\mathbb{C}(z)$*. Then, the* u_i *actually belong to* $\mathbb{C}(z)$*.*

Proof. According to the cyclic vector lemma, there exists $P \in \mathrm{GL}_n(\mathbb{C}(z))$ such that $Pu = (f, \sigma_q(f), \ldots, \sigma_q^{n-1}(f))^t$ for some $f \in \bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$, which is a solution of a nonzero linear q-difference equation of order n with coefficients in $\mathbb{C}(z)$. Moreover, f satisfies a nonzero linear differential equation with coefficients in $\mathbb{C}(z)$, because it is a $\mathbb{C}(z)$-linear combination of the u_i. It follows from [Ram92, Theorem 7.6] that f belongs to $\mathbb{C}(z)$. Hence, the entries of $u = P^{-1}(Pu) = P^{-1}(f, \sigma_q(f), \ldots, \sigma_q^{n-1}(f))^t$ actually belong to $\mathbb{C}(z)$, as expected. $\square$

Proof of Theorem 3.1. Using Lemma 3.2 and Proposition 2.7, we are reduced to prove that the (σ_q, δ)-Galois group over the (σ_q, δ)-ring $\mathbf{L}$ of $\sigma_q(Y) = AY$ is not conjugate to a subgroup of $\widetilde{\mathbb{C}} \cdot N_{\mathrm{SL}_n(\mathbb{C})}(G^{\circ,der}(\mathbb{C}))$. Suppose to the contrary that it is conjugate to a subgroup of $\widetilde{\mathbb{C}} \cdot N_{\mathrm{SL}_n(\mathbb{C})}(G^{\circ,der}(\mathbb{C}))$. Let $\sqrt[n]{\det A}$ be a n-th root of $\det A$ in $\overline{\mathbb{C}(z)}$.

We consider $A' = (\sqrt[n]{\det A})^{-1} A \in \mathrm{SL}_n(\overline{\mathbb{C}(z)})$. The second part of Lemma B.2 ensures that there exist $c \in \mathbb{C}^{\times}$ and $r \in \mathbb{Q}$ such that $\sigma_q(Y) = A''Y$, with $A'' = cz^r A' \in \mathrm{GL}_n(\overline{\mathbb{C}(z)})$, has a nonzero solution $u = (u_1, \ldots, u_n)^t$ with coefficients in $\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$. We let S be a (σ_q, δ)-PV ring over the (σ_q, δ)-ring $\mathbf{L}$ for $\sigma_q(Y) = A''Y$ containing the entries of u. We let $U'' \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of $\sigma_q(Y) = A''Y$ whose first column is u. Since $A'' = hA$ for some $h \in \mathbf{L}^{\times}$, the derived groups of the difference Galois groups of the systems $\sigma_q(Y) = AY$ and $\sigma_q(Y) = A''Y$ over $\mathbf{L}$ coincides and are therefore equal to $G^{\circ,der}(\widetilde{\mathbb{C}})$. Indeed let R be a Picard-Vessiot ring over $\mathbf{L}$ for the system $\sigma_q(Y) = \begin{pmatrix} A & 0 \\ 0 & h \end{pmatrix} Y$ then there exists $U \in \mathrm{GL}_n(R)$ and $v \in R^{\times}$ such that $\sigma_q(U) = AU$ and $\sigma_q(v) = hv$. Then, $\mathbf{L}[U, \frac{1}{\det(U)}] \subset R$ (resp. $\mathbf{L}[vU, \frac{1}{v^n \det(U)}] \subset R$) is a Picard-Vessiot ring for $\sigma_q(y) = AY$ (resp $\sigma_q(Y) = A''Y$) over $\mathbf{L}$. In the representation attached to U and vU, one can easily conclude to the equality of the derived groups.

Now, since the (σ_q, δ)-Galois group of $\sigma_q(Y) = AY$ over $\mathbf{L}$ is conjugate to a subgroup of $\widetilde{\mathbb{C}} \cdot N_{\mathrm{SL}_n(\mathbb{C})}(G^{\circ,der}(\mathbb{C}))$, Proposition 2.7 ensures that there exists $B \in \overline{\mathbb{C}(z)}^{n \times n}$ such that

$$\sigma_q(B)A = AB + \delta(A) - \frac{1}{n}\delta(\det(A))\det(A)^{-1}A. \tag{3.2}$$

An easy computation shows that

$$\sigma_q(B)A'' = A''B + \delta(A'') - \frac{1}{n}\delta(\det(A''))\det(A'')^{-1}A''. \tag{3.3}$$

This equation ensures the integrability of the system of equations

$$\begin{cases} \sigma_q(Y) = A''Y \\ \delta(Y) = (B + \frac{\delta \mathbf{d}}{n\mathbf{d}})Y \end{cases}$$

where $\mathbf{d} = \det U''$ satisfies the q-difference equation $\sigma_q(\mathbf{d}) = (\det A'')\mathbf{d} = (cz^r)^n \mathbf{d}$. So, there exists $D \in \mathrm{GL}_n(\widetilde{\mathbb{C}})$ such that $V = U''D \in \mathrm{GL}_n(S)$ satisfies

$$\begin{cases} \sigma_q(V) = A''V \\ \delta(V) = (B + \frac{\delta \mathbf{d}}{n\mathbf{d}})V. \end{cases} \tag{3.4}$$

We remind that $\sigma_q \circ \delta = \delta \circ \sigma_q$. Note that $\frac{\delta \mathbf{d}}{\mathbf{d}} \in S$ is such that $\sigma_q\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right) = \frac{\delta \mathbf{d}}{\mathbf{d}} + nr$. So, $\mathbf{L}\{\frac{\delta \mathbf{d}}{\mathbf{d}}\}_\delta \subset S$ is a (σ_q, δ)-PV ring over the (σ_q, δ)-ring $\mathbf{L}$. Since, the corresponding (σ_q, δ)-Galois group is Kolchin-connected (because it is a δ-subgroup of the additive group $\mathbb{G}_a(\widetilde{\mathbb{C}})$), we get that $\mathbf{L}\{\frac{\delta \mathbf{d}}{\mathbf{d}}\}_\delta$ is an integral domain and, hence, we can consider its field of fraction $\mathbf{L}\langle\frac{\delta \mathbf{d}}{\mathbf{d}}\rangle_\delta \subset \mathcal{Q}_S$.

Note that, since $\sigma_q\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right) = \frac{\delta \mathbf{d}}{\mathbf{d}} + nr$, we have $\sigma_q\left(\delta\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right)\right) = \delta(\frac{\delta \mathbf{d}}{\mathbf{d}})$, and therefore, $\delta\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right) \in S^{\sigma_q} = \widetilde{\mathbb{C}}$. Consequently, $\mathbf{L}\langle\frac{\delta \mathbf{d}}{\mathbf{d}}\rangle_\delta = \mathbf{L}(\frac{\delta \mathbf{d}}{\mathbf{d}})$.

Using (3.4), we get $\delta(U'')D + U''\delta(D) = \delta(U''D) = \delta(V) = (B + \frac{\delta(\mathbf{d})}{n\mathbf{d}})U''D$ so

$$\delta(U'') = \left(B + \frac{\delta(\mathbf{d})}{n\mathbf{d}}\right)U'' - U''\delta(D)D^{-1}.$$

The previous formula implies that the $\mathbf{L}(\frac{\delta \mathbf{d}}{\mathbf{d}})$-vector subspace of $\mathcal{Q}_S$ generated by the entries of U'' and all their successive δ-derivatives is of finite dimension. In particular, any u_i satisfies a nonzero linear δ-equation $\mathcal{L}_i(y) = 0$ with coefficients in $\mathbf{L}[\frac{\delta \mathbf{d}}{\mathbf{d}}]$.

We claim that any u_i satisfies a nonzero linear δ-equation with coefficients in $\mathbf{L}$.

If $nr = 0$, we have $\sigma_q\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right) = \frac{\delta \mathbf{d}}{\mathbf{d}} + nr = \frac{\delta \mathbf{d}}{\mathbf{d}}$, and therefore $\frac{\delta \mathbf{d}}{\mathbf{d}} \in S^{\sigma_q} = \widetilde{\mathbb{C}}$, which proves our claim.

Assume that $nr \neq 0$. The equation $\mathcal{L}_i(y) = 0$ can be rewritten as $\sum_{j=0}^{\nu} \mathcal{L}_{i,j}(y)(\frac{\delta \mathbf{d}}{\mathbf{d}})^j = 0$ where the $\mathcal{L}_{i,j}(y)$ are linear δ-operators with coefficients in $\mathbf{L}$, not all zero.

To prove our claim, it is sufficient to prove that $\frac{\delta \mathbf{d}}{\mathbf{d}}$ is transcendent over $\mathbf{L}\langle u_1, \dots, u_n\rangle_\delta$. Indeed, assume at the contrary that there is a non zero relation

$$\sum_{k=0}^{\kappa} a_k \left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right)^k = 0 \tag{3.5}$$

with $\kappa \geq 1$ and $a_0, \dots, a_{\kappa-1}, a_\kappa = 1 \in \mathbf{L}\langle u_1, \dots, u_n\rangle_\delta$. We can and will assume that $\kappa \geq 1$ is minimal. Applying σ_q to equation (3.5), we get

$$\sum_{k=0}^{\kappa} \sigma_q(a_k) \left(\frac{\delta \mathbf{d}}{\mathbf{d}} + nr\right)^k = 0. \tag{3.6}$$

Since κ is minimal, the coefficients of any $\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right)^k$ in (3.5) and (3.6) are equal. In particular, equating the coefficients of $\left(\frac{\delta \mathbf{d}}{\mathbf{d}}\right)^{\kappa-1}$, we get

$$a_{\kappa-1} = \sigma_q(a_{\kappa-1}) + \kappa nr$$

and this is a contradiction (since $a_{\kappa-1} \in \cup_j \widetilde{\mathbb{C}}((z^{1/j}))$, the term of degree 0 in $a_{\kappa-1} - \sigma_q(a_{\kappa-1})$ is equal to 0 and, hence, is not equal to $\kappa nr \neq 0$).

It follows that all the $\mathcal{L}_{i,j}(u_i)$ are equal to zero and this proves that $\frac{\delta \mathbf{d}}{\mathbf{d}}$ is transcendent over $\mathbf{L}\langle u_1, \dots, u_n\rangle_\delta$. This proves our claim, that is, any u_i satisfies some nonzero linear δ-equations with coefficients in $\mathbf{L}$.

Since the u_i belong to $\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$, we obtain that any u_i satisfies a nonzero linear δ-equation with coefficients in $\overline{\mathbb{C}(z)}$. Since $\overline{\mathbb{C}(z)}$ is an algebraic extension of $\mathbb{C}(z)$, we get that any u_i satisfies a nonzero linear δ-equation with coefficients in $\mathbb{C}(z)$.

The vector u is a solution of $\sigma_q(Y) = A''Y$. Then, letting p be a denominator of r and considering the pn-th tensor power of this q-difference system, we get that $u^{\otimes pn}$ satisfies a linear q-difference equation with coefficients in $\mathbb{C}(z)$. Since any u_i satisfies a nonzero linear δ-equation with coefficients in $\mathbb{C}(z)$, we find that $u^{\otimes pn}$ satisfies a nonzero linear δ-equation with coefficients in $\mathbb{C}(z)$. It follows from Lemma 3.4 that the entries of $u^{\otimes pn}$ belong to $\mathbb{C}(z)$ and, hence, any u_i belongs to $\overline{\mathbb{C}(z)}$. Hence, the first column of U'' is fixed by the difference Galois group of $\sigma_q(Y) = A''Y$ over $\mathbf{L}$ and this contradicts the fact that this group contains $G^{\circ,der}(\widetilde{\mathbb{C}})$, which is irreducible by hypothesis. □

4. Applications

4.1. **User friendly criterias for transcendence.** The goal of this subsection is to use Theorem 3.1, in order to give transcendence criterias. We refer to Section 3 for the notations used in this section.

Corollary 4.1. *Let G be the difference Galois group of the q-difference system (3.1) over the σ_q-field $\mathbb{C}(z)$. Let us assume that (3.1) admits a non zero vector solution $u = (u_1, \dots, u_n)^t$ with entries in $\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$.*

- *Assume that $n \geq 2$ and $G^{\circ,der} = \mathrm{SL}_n(\mathbb{C})$. Then, the series $u_1, \dots, u_n$ are δ-algebraically independent over $\mathbb{C}(z)$. In particular, any u_i is δ-algebraically independent over $\mathbb{C}(z)$.*
- *Assume that $n \geq 3$ and $G^{\circ,der} = \mathrm{SO}_n(\mathbb{C})$. Then, the series $u_1, \dots, u_{n-1}$ are δ-algebraically independent over $\mathbb{C}(z)$.*
- *Assume that n is even and $G^{\circ,der} = \mathrm{Sp}_n(\mathbb{C})$. Then, the series $u_1, \dots, u_n$ are δ-algebraically independent over $\mathbb{C}(z)$.*

Proof. Thanks to Lemma 3.3, there exists a (σ_q, δ)-PV ring S for the system (3.1) over $\mathbf{L}$ containing $\mathbf{L}\{u_1, \dots, u_n\}_\delta$. Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of the system (3.1) whose first column is u. Since $G^{\circ,der}$ is equal to $\mathrm{SO}_n(\mathbb{C})$, (resp. $\mathrm{SL}_n(\mathbb{C})$, resp. $\mathrm{Sp}_n(\mathbb{C})$), with Theorem 3.1, we find that the (σ_q, δ)-Galois group of (3.1) contains $\mathrm{SO}_n(\widetilde{\mathbb{C}})$, (resp. $\mathrm{SL}_n(\widetilde{\mathbb{C}})$, resp. $\mathrm{Sp}_n(\widetilde{\mathbb{C}})$). The results of Section 2.3 yield the desired conclusion. □

Consider the following q-difference equation

$$a_n(z)y(q^n z) + a_{n-1}(z)y(q^{n-1}z) + \dots + a_0(z)y(z) = 0 \tag{4.1}$$

for some integer $n \geq 1$, and some $a_0(z), \dots, a_n(z) \in \mathbb{C}(z)$ with $a_0(z)a_n(z) \neq 0$. In what follows, by "difference Galois group of equation (4.1)", we mean the difference Galois group of the associated system

$$\sigma_q(Y) = AY, \text{ with } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}(z)). \tag{4.2}$$

Corollary 4.2. *Let G be the difference Galois group of the q-difference system (4.2) over the σ_q-field $\mathbb{C}(z)$. Let us assume that (4.2) admits a non zero solution $g \in \bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$.*

- *Assume that $n \geq 2$ and $G^{\circ,der} = \mathrm{SL}_n(\mathbb{C})$. Then, the series $g(z), g(qz), \dots, g(q^{n-1}z)$ are δ-algebraically independent over $\mathbb{C}(z)$.*
- *Assume that $n \geq 3$ and $G^{\circ,der} = \mathrm{SO}_n(\mathbb{C})$. Then, the series $g(z), g(qz), \dots, g(q^{n-2}z)$ are δ-algebraically independent over $\mathbb{C}(z)$.*
- *Assume that n is even and $G^{\circ,der} = \mathrm{Sp}_n(\mathbb{C})$. Then, the series $g(z), g(qz), \dots, g(q^{n-1}z)$ are δ-algebraically independent over $\mathbb{C}(z)$.*

Proof. Let us note that if $g(z) \in \bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$ is a nonzero solution of (4.1), then $u_1 = (g(z), g(qz), \dots, g(q^{n-1}z))^t$ is a nonzero solution of (4.2) with entries in $\bigcup_{j=1}^{\infty} \mathbb{C}((z^{1/j}))$. This is a direct consequance of Corollary 4.1. □

4.2. **Generalized Hypergeometric series.** In this subsection, we follow the notations of [Roq11, Roq12] and we assume that $0 < |q| < 1$. Let us fix $n, s \in \mathbb{N}^*$, let $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{R}})^n$, $\underline{b} = (b_1, \dots, b_s) \in (q^{\mathbb{R}} \setminus q^{-\mathbb{N}})^s$, $\lambda \in \mathbb{C}^\times$, and consider the q-difference operator:

$$z\lambda \prod_{i=1}^{n} (a_i \sigma_q - 1) - \prod_{j=1}^{s} \left(\frac{b_j}{q} \sigma_q - 1 \right). \tag{4.3}$$

When $b_1 = q$, this operator admits as solution the power series:

$$\begin{aligned} {}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z) &= \sum_{m=0}^{\infty} \frac{(\underline{a}; q)_m}{(\underline{b}; q)_m} \lambda^m z^m \\ &= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{n} (1-a_i)(1-a_i q) \dots (1-a_i q^{m-1})}{\prod_{j=1}^{s} (1-b_j)(1-b_j q) \dots (1-b_j q^{m-1})} \lambda^m z^m. \end{aligned}$$

Until the end of the subsection, let us assume that $s = n \geq 2$ and that $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{Q}})^n$, $\underline{b} = (b_1, \dots, b_s) \in (q^{\mathbb{Q}} \setminus q^{-\mathbb{N}})^s$.

According to [Roq11], the operator (4.3) is irreducible over $\mathbb{C}(z)$ if and only if, for all $(i,j) \in \{1, \dots, n\}^2$, $a_i \notin b_j q^{\mathbb{Z}}$. We say that (4.3) is q-Kummer induced if it is irreducible, and there exists a divisor $d \neq 1$ of n, and two permutations μ, ν of $\{1, \dots, n\}$, such that, for all $i \in \{1, \dots, n\}$, $a_i \in a_{\mu(i)} q^{1/d} q^{\mathbb{Z}}$, and $b_i \in b_{\nu(i)} q^{1/d} q^{\mathbb{Z}}$.

Theorem 4.3 ([Roq11], Theorem 6)**.** *Let us assume that (4.3) is irreducible and not q-Kummer induced. Let G be the difference Galois group of the q-difference system (4.3) over the σ_q-field $\mathbb{C}(z)$. Then, $G^{\circ,der}$ is either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ (only when n is odd), or $\mathrm{Sp}_n(\mathbb{C})$ (only when n is even). Moreover, $G^{\circ,der}$ is $\mathrm{SO}_n(\mathbb{C})$ (resp. $\mathrm{Sp}_n(\mathbb{C})$) if and only if*

- $\prod_{i=1}^{n} a_i \in q^{\mathbb{Z}} \prod_{j=1}^{n} b_j$;
- *there exists $c \in \mathbb{C}^*$, there exist two permutations μ_1, μ_2 of $\{1, \dots, n\}$, such that, for all $i, j \in \{1, \dots, n\}$, $c a_i a_{\mu_1(i)} \in q^{\mathbb{Z}}$, $c b_j b_{\mu_2(j)} \in q^{\mathbb{Z}}$;*
- *n is odd (resp. even).*

Theorem 4.3 and Corollary 4.2 yield the following result.

Corollary 4.4. *Let us assume that (4.3) is irreducible and not q-Kummer induced. Let G be the difference Galois group of the q-difference system (4.3) over the σ_q-field $\mathbb{C}(z)$ and let G^δ, be the δ-Galois group of the q-difference system (4.3) over the field* $\mathbf{L}$*.*

- *Assume that $G^{\circ,der}$ equals $\mathrm{SL}_n(\mathbb{C})$, (resp. $\mathrm{SO}_n(\mathbb{C})$, with n odd, resp. $\mathrm{Sp}_n(\mathbb{C})$, with n even). Then, G^δ contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$, (resp. $\mathrm{SO}_n(\widetilde{\mathbb{C}})$, resp. $\mathrm{Sp}_n(\widetilde{\mathbb{C}})$).*
- *Furthermore, if $b_1 = q$, then ${}_n\Phi_n(\underline{a}, \underline{b}, \lambda, q; z), \dots, \sigma_q^\kappa ({}_n\Phi_n(\underline{a}, \underline{b}, \lambda, q; z))$ with $\kappa = n-1$ (resp. $\kappa = n-2$, resp. $\kappa = n-1$) are δ-algebraically independent over $\mathbb{C}(z)$.*

Proof. The first point is a straightforward consequence of Theorems 3.1, and 4.3. We conclude with Corollary 4.2. □

4.3. **Irregular generalized Hypergeometric functions.** In this subsection, we assume that $n > s$, $n \geq 2$. Let $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{R}})^n$, $\underline{b} = (b_1, \dots, b_s) \in (q^{\mathbb{R}} \setminus q^{-\mathbb{N}})^s$, $\lambda \in \mathbb{C}^\times$, $0 < |q| < 1$.

The following result is proved in [Roq12].

Theorem 4.5. *Let G be the difference Galois group of the q-difference system (4.3) over the σ_q-field $\mathbb{C}(z)$. For $(i,j) \in \{1, \dots, n\} \times \{1, \dots, s\}$, let $\alpha_i, \beta_j \in \mathbb{R}$ such that $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_j}$. Assume that for all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, s\}$, $\alpha_i - \beta_j \notin \mathbb{Z}$, and that the algebraic group generated by $\mathrm{Diag}(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_n})$ is connected. Then, $G = \mathrm{GL}_n(\mathbb{C})$.*

Corollary 4.6. *Let G^δ, be the δ-Galois group of the q-difference system (4.3) over the field $\mathbf{L}$. Assume that for all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, s\}$, we have $\alpha_i - \beta_j \notin \mathbb{Z}$, and that the algebraic group generated by $\mathrm{Diag}(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_n})$ is connected. Then, $G^\delta = \mathrm{GL}_n(\widetilde{\mathbb{C}})$. Furthermore, if $b_1 = q$, then the series ${}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z), \dots, \sigma_q^{n-1}\left({}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z)\right)$ are δ-algebraically independent over $\mathbb{C}(z)$.*

Proof. Theorems 3.1 and 4.5 ensure that G^δ contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$. So, the group G^δ equals to $G_M \mathrm{SL}_n(\widetilde{\mathbb{C}})$, where $G_M \subset \widetilde{\mathbb{C}}^\times$ is the δ-Galois group of the q-difference equation $\sigma_q y = \det(A) y = \frac{(-1)^n z\lambda + (-1)^{s+1}}{z\lambda \prod_{i=1}^n a_i} y$, and A is the matrix associated to (4.3). It is easily seen that there do not exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$, and $f \in \mathbb{C}(z)^\times$ such that $\det(A) = cz^{-1}\frac{\sigma_q(f)}{f}$. By [HS08, Corollary 3.4], we deduce that $G_M = \widetilde{\mathbb{C}}^\times$. We conclude with Corollary 4.2. ☐

Part 2. q′-difference relations of solutions of q-difference equations

5. Parametrized difference Galois theory

5.1. **Difference algebra.** We refer to [OW15] for more details on what follows. By a $(\sigma_q, \sigma_{\mathbf{q}'})$-ring, we mean a ring equipped with two commuting automorphisms σ_q and $\sigma_{\mathbf{q}'}$. The definition of $(\sigma_q, \sigma_{\mathbf{q}'})$-fields, $\mathbf{K}$-$(\sigma_q, \sigma_{\mathbf{q}'})$-algebras for $\mathbf{K}$ a $(\sigma_q, \sigma_{\mathbf{q}'})$-field and $(\sigma_q, \sigma_{\mathbf{q}'})$-ideals are straightforward.

We say that a $\mathbf{K}$-$(\sigma_q, \sigma_{\mathbf{q}'})$-algebra R is $\sigma_{\mathbf{q}'}$-finitely generated if there exist $a_1, \dots, a_n$ such that R is generated as $\mathbf{K}$-algebra by the a_i's and their transforms via $\sigma_{\mathbf{q}'}$. We then write $R = \mathbf{K}\{a_1, \dots, a_n\}_{\sigma_{\mathbf{q}'}}$. We say that a $\mathbf{K}$-$(\sigma_q, \sigma_{\mathbf{q}'})$-field R is $\sigma_{\mathbf{q}'}$-finitely generated if there exist $a_1, \dots, a_n$ such that R is generated as $\mathbf{K}$-field by the a_i's and their transforms via $\sigma_{\mathbf{q}'}$. We then write $R = \mathbf{K}\langle a_1, \dots, a_n\rangle_{\sigma_{\mathbf{q}'}}$.

Let $(\mathbf{k}, \sigma_{\mathbf{q}'})$ be a difference field. Let R be a $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebra. If R is a field, we say that R is inversive if $\sigma_{\mathbf{q}'}$ is surjective on R. We call R $\sigma_{\mathbf{q}'}$-separable if $\sigma_{\mathbf{q}'}$ is injective on $R \otimes_{\mathbf{k}} \widetilde{\mathbf{k}}$ for every $\sigma_{\mathbf{q}'}$-field extension $\widetilde{\mathbf{k}}/\mathbf{k}$.

The ring of $\sigma_{\mathbf{q}'}$-polynomials in the differential indeterminates $y_1, \dots, y_n$ and with coefficients in $(\mathbf{k}, \sigma_{\mathbf{q}'})$, denoted by $\mathbf{k}\{y_1, \dots, y_n\}_{\sigma_{\mathbf{q}'}}$, is the ring of polynomials in the indeterminates $\{\sigma_{\mathbf{q}'}^j y_i \mid j \in \mathbb{N}, 1 \leq i \leq n\}$ with coefficients in $\mathbf{k}$. Let R be a $\mathbf{K}$-$\sigma_{\mathbf{q}'}$-algebra and let $a_1, \dots, a_n \in R$. If there exists a nonzero $\sigma_{\mathbf{q}'}$-polynomial $P \in \mathbf{K}\{y_1, \dots, y_n\}_{\sigma_{\mathbf{q}'}}$ such that $P(a_1, \dots, a_n) = 0$, then we say that $a_1, \dots, a_n$ are $\sigma_{\mathbf{q}'}$-algebraically dependent over $\mathbf{K}$. Otherwise, we say that $a_1, \dots, a_n$ are $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbf{K}$.

We would like to prove some lemmas about the extension of constants.

Lemma 5.1. *Let F be a $(\sigma_q, \sigma_{\mathbf{q}'})$-field and let $\mathbf{k} = F^{\sigma_q}$ be the field of σ_q-constants of F. We assume that $\mathbf{k}$ is an inversive $\sigma_{\mathbf{q}'}$-field. Let $\widetilde{\mathbf{k}}$ be a regular* [†] *$\sigma_{\mathbf{q}'}$-field*

[†]See [Bou03, A.V.141] for the definition.

extension of $\mathbf{k}$ *considered as a field of* σ_q*-constants. Then, the ring* $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ *is an integral domain whose fraction field* $\widetilde{F}$ *is a* $(\sigma_q, \sigma_{\mathbf{q}'})$*-field extension of* F *such that* $\widetilde{F}^{\sigma_q} = \widetilde{\mathbf{k}}$*.*

Proof. Since $\widetilde{\mathbf{k}}$ is a regular extension of $\mathbf{k}$, the ring $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ is an integral domain. Moreover since $\widetilde{\mathbf{k}}$ is a $\sigma_{\mathbf{q}'}$-separable $\sigma_{\mathbf{q}'}$-field extension of $\mathbf{k}$ by [DVHW14b, Corollary A.14], the operator $\sigma_{\mathbf{q}'}$ is injective on $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ and thus extends to $\widetilde{F}$. The rest of the proof is essentially [DHR15, Lemma 2.3]. □

Lemma 5.2. *Let* F *be a* $(\sigma_q, \sigma_{\mathbf{q}'})$*-field and let* $\mathbf{k} = F^{\sigma_q}$ *be the field of* σ_q*-constants of* F*. We assume that* $\mathbf{k}$ *is an inversive* $\sigma_{\mathbf{q}'}$*-field. Let* $\widetilde{\mathbf{k}}$ *be a regular* $\sigma_{\mathbf{q}'}$*-field extension of* $\mathbf{k}$ *considered as a field of* σ_q*-constants. By Lemma 5.1, we can consider the* $(\sigma_q, \sigma_{\mathbf{q}'})$*-field* $\widetilde{F} = \mathrm{Frac}(\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F)$*. Let* $A \in \mathrm{GL}_n(F)$ *and let* $\mathbf{V}_{\mathbf{k}}$ *(resp.* $\mathbf{V}_{\widetilde{\mathbf{k}}}$*) be the solution space of* $\sigma_q(Y) = AY$ *in* F^n *(resp. in* $\widetilde{F}^n$*). Then,* $\mathbf{V}_{\widetilde{\mathbf{k}}} = \mathbf{V}_{\mathbf{k}} \otimes_{\mathbf{k}} \widetilde{\mathbf{k}}$*.*

Proof. Obviously, we have $\mathbf{V}_{\mathbf{k}} \otimes_{\mathbf{k}} \widetilde{\mathbf{k}} \subset \mathbf{V}_{\widetilde{\mathbf{k}}}$. Let $f \in \mathbf{V}_{\widetilde{\mathbf{k}}}$ be a non zero solution. Set $S = F \otimes_{\mathbf{k}} \widetilde{\mathbf{k}}$. Let us consider

$$\mathfrak{a} = \{r \in S | rf \in S\}.$$

Since $\sigma_q(f) = Af$, the ideal $\mathfrak{a}$ is a non zero σ_q-ideal of S. By [vdPS97, Lemma 1.11], the ring S is σ_q-simple. Therefore $1 \in \mathfrak{a}$ and $f \in S$. Let $(e_i)_{i\in I}$ be a basis of $\widetilde{\mathbf{k}}$ over $\mathbf{k}$ and let us write $f = \sum_{i\in I} f_i e_i$ with $f_i \in F$. Then, $\sigma_q(f) = Af$ implies $\sigma_q(f_i) = Af_i$, which ends the proof. □

5.2. **Parametrized Difference Galois theory.** We fix $q \in \mathbb{C}^*$ with $|q| \neq 1$. Let $\mathcal{M}er(\mathbb{C}^\times)$ be the field of meromorphic functions over $\mathbb{C}^\times$. The field automorphism

$$\begin{array}{rcl} \sigma_q : \mathcal{M}er(\mathbb{C}^\times) & \to & \mathcal{M}er(\mathbb{C}^\times) \\ f(z) & \mapsto & f(qz) \end{array}$$

gives a structure of σ_q-field on $\mathcal{M}er(\mathbb{C}^\times)$ such that $\mathcal{M}er(\mathbb{C}^\times)^{\sigma_q} = C_E$, the field of elliptic functions on the elliptic curve $E_q = \mathbb{C}^\times/q^{\mathbb{Z}}$. Let us set $\mathbb{C}_E = \bigcup_{r\in\mathbb{N}^*} C_{E_r}$, where $C_{E_r} = \{f(z) \in \mathcal{M}er(\mathbb{C}^\times) | \sigma_q^r(f(z)) = f(z)\}$.

We denote by $\mathbb{C}_E(z)$ the field compositum of $\mathbb{C}(z)$ and $\mathbb{C}_E$ in $\mathcal{M}er(\mathbb{C}^\times)$. Then $\mathbb{C}_E(z)$ is a σ_q-field and $\mathbb{C}_E(z)^{\sigma_q} = C_E$.

Lemma 5.3. *The field* $\mathbb{C}_E(z)$ *is relatively algebraically closed in* $\mathcal{M}er(\mathbb{C}^\times)$*. Moreover, any finite field extension* $F \subset \mathcal{M}er(\mathbb{C}^\times)$ *of* $C_E(z)$ *stable under* σ_q *is of the form* $C_{E_r}(z)$ *for some non-negative integer* r*.*

Proof. Using a multiplicative version of the proof [DR15, Proposition 6], one can show that any finite extension stable under σ_q of C_E is of the form C_{E_r} for some non-negative integer r. This proves that the relative algebraic closure of C_E in $\mathcal{M}er(\mathbb{C}^\times)$ is contained in $\mathbb{C}_E$. Conversely, any element u of $\mathbb{C}_E$ is contained in some C_{E_r} for some non-negative integer r. This implies that the polynomial $\mu(X) = (X-u)(X-\sigma_q(u))\dots(X-\sigma_q^{r-1}(u))$ belongs to $C_E[X]$ and annihilates u. Thus $\mathbb{C}_E$ is relatively algebraically closed in $\mathcal{M}er(\mathbb{C}^\times)$.

Let $F/C_E(z)$ be a finite field extension in $\mathcal{M}er(\mathbb{C}^\times)$ stable under σ_q. The field of constants C of F, *i.e.*, the elements of F algebraic over C_E, is stable under σ_q (because C_E is stable under σ_q). Thus, there exists a non negative integer r such that $C = C_{E_r}$. Then, by [Sti09, Corollary III.5.8], the extension $F/C_{E_r}(z)$ is either ramified or $F = C_{E_r}(z)$. Since $F \subset \mathcal{M}er(\mathbb{C}^\times)$, the extension can not ramify and $F = C_{E_r}(z)$. □

Let $\mathbf{q}' \in \mathbb{C}^\times$ not a root of unity. We can consider the automorphism of $\mathcal{M}er(\mathbb{C}^\times)$ defined by $\sigma_{\mathbf{q}'}(f(z)) = f(\mathbf{q}'z)$ for all $f \in \mathcal{M}er(\mathbb{C}^\times)$.

Example 5.4. The fields $\mathcal{M}er(\mathbb{C}^\times), C_E(z)$, $\mathbb{C}_E(z)$, and $\mathbb{C}(z)$ equipped with σ_q and $\sigma_{\mathbf{q}'}$ as above are $(\sigma_q, \sigma_{\mathbf{q}'})$-fields.

Given a $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathbf{K}$ and $A \in \mathrm{GL}_n(\mathbf{K})$, the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois theory developed in [OW15] aims at understanding the algebraic relations between the solutions of $\sigma_q(Y) = AY$ and their successive transforms with respect to $\sigma_{\mathbf{q}'}$ from a Galoisian point of view. We will not recall here the theoretic aspects of this parametrized Galois theory in their greatest generality. However, we describe the definitions and results of [OW15] in the specific situation where the $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathbf{K}$ is precisely $\mathbb{C}_E(z)$. In this part of the paper, the word parametrized refers to the parametric action of the operator $\sigma_{\mathbf{q}'}$ whereas in the first part, it was related to the parametric action of the derivative. Therefore the word parametrized do not refer to the same parametric action depending on the part of the paper. Since the two parts are almost independent, this convention will not lead to confusions. It will also avoid heavy terminology.

Definition 5.5 ([OW15], Definition 2.4)**.** Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$. A $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field extension $\mathcal{Q}_S$ for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-field extension of $\mathbb{C}_E(z)$ such that:

- there exists $U \in \mathrm{GL}_n(\mathcal{Q}_S)$ such that $\sigma_q(U) = AU$,
- $\mathcal{Q}_S = \mathbb{C}_E(z)\langle U \rangle_{\sigma_{\mathbf{q}'}}$,
- ${\mathcal{Q}_S}^{\sigma_q} = \mathbb{C}_E(z)^{\sigma_q} = C_E$.

Furthermore, $S = \mathbb{C}_E(z)\{U, \frac{1}{\det(U)}\}_{\sigma_{\mathbf{q}'}}$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV ring for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$, which means that it is a $\sigma_{\mathbf{q}'}$-simple ring, *i.e.*, the $\sigma_{\mathbf{q}'}$-ideals of S are $\{0\}$ and S.

Proposition 5.6. *Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$. There exists a unique $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field $\mathcal{Q}_S$ for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ in $\mathcal{M}er(\mathbb{C}^\times)$.*

Proof. By [Pra86, Theorem 3], there exists $U \in \mathrm{GL}_n(\mathcal{M}er(\mathbb{C}^\times))$ such that $\sigma_q(U) = AU$. Consider the $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathcal{Q}_S = \mathbb{C}_E(z)\langle U \rangle_{\sigma_{\mathbf{q}'}} \subset \mathcal{M}er(\mathbb{C}^\times)$. Since $\mathbb{C}_E(z)^{\sigma_q} = C_E \subset \mathcal{Q}_S^{\sigma_q} \subset \mathcal{M}er(\mathbb{C}^\times)^{\sigma_q} = C_E$, the $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathcal{Q}_S$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$.

Let us prove the uniqueness. Let $L = \mathbb{C}_E(z)\langle V \rangle_{\sigma_{\mathbf{q}'}} \subset \mathcal{M}er(\mathbb{C}^\times)$, be a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$, and let $V \in \mathrm{GL}_n(L)$ be a fundamental solution matrix. Since $\mathcal{M}er(\mathbb{C}^\times)^{\sigma_q} = C_E$, there exists $C \in \mathrm{GL}_n(C_E)$ such that $V = U.C$. This implies $L = \mathcal{Q}_S$. □

One could be tempted to define the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of the $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field $\mathcal{Q}_S$ for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$, as the group of $(\sigma_q, \sigma_{\mathbf{q}'})$-automorphisms of $\mathcal{Q}_S$ over $\mathbb{C}_E(z)$. It appears that this approach is too naive for two reasons. The first one is that the defining equations of such a $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group would be $\sigma_{\mathbf{q}'}$-difference algebraic equations. Similarly to the continuous parameter context, we shall need to look at zeroes of these equations in a $\sigma_{\mathbf{q}'}$-difference closure of C_E. The second problem is more serious and is related to the fact that the zeroes of a difference algebraic equation in a field do not capture all the geometric information of the equation. Therefore, one has to look at zeroes in rings (see for instance the discussions in [vdPS97]).

Example 5.7. Consider the following system of difference equations $(S_1) = \{y^2 = 1\}$ and $(S_2) = \{y^2 = 1 \text{ and } \sigma_{\mathbf{q}'}(y) = y\}$ over $\mathbb{C}$. We denote by $\mathbf{V}_{S_1}(R)$ (resp. $\mathbf{V}_{S_2}(R)$) the zeroes of (S_1) (resp. (S_2)) in some $\mathbb{C}$-$\sigma_{\mathbf{q}'}$-algebra R. Then,

$\mathbf{V}_{S_1}(\mathbf{k}) = \mathbf{V}_{S_2}(\mathbf{k}) = \{1, -1\}$ for any $\sigma_{\mathbf{q}'}$-field extension $\mathbf{k}$ of $\mathbb{C}$. However, if we consider the ring of sequences $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ with the action of $\sigma_{\mathbf{q}'}$ given by the shift operator, then $\mathbf{V}_{S_1}(\mathbb{C}^{\mathbb{Z}}) = \{(a_n)_{n\in\mathbb{Z}} | a_n = 1 \text{ or } -1 \text{ for all } n \in \mathbb{Z}\}$ whereas $\mathbf{V}_{S_2}(\mathbb{C}^{\mathbb{Z}}) = \mathbb{C}$ the set of constant sequences.

Therefore, we need to adopt the following functorial approach. We denote by $\mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}$ the category of C_E-$\sigma_{\mathbf{q}'}$-algebras and by Sets the category of sets.

Definition 5.8 ([OW15], Definition 2.50)**.** Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and let $\mathcal{Q}_S = \mathbb{C}_E(z)\langle U\rangle_{\sigma_{\mathbf{q}'}} \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$. Set $S = \mathbb{C}_E(z)\{U, \frac{1}{\det(U)}\}_{\sigma_{\mathbf{q}'}}$. Then, the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\mathcal{Q}_S$ over $\mathbb{C}_E(z)$ is defined as the functor:

$$\begin{array}{llll} \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)): & \mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}} & \to & \mathrm{Sets} \\ & B & \mapsto & \mathrm{Aut}^{(\sigma_q,\sigma_{\mathbf{q}'})}(S \otimes_{C_E} B/\mathbb{C}_E(z) \otimes_{C_E} B), \end{array}$$

where, σ_q acts as the identity on B and $\mathrm{Aut}^{(\sigma_q,\sigma_{\mathbf{q}'})}(S \otimes_{C_E} B/\mathbb{C}_E(z) \otimes_{C_E} B)$ is the group of automorphisms of $S \otimes_{C_E} B$ inducing the identity on $\mathbb{C}_E(z) \otimes_{C_E} B$ and commuting with $\sigma_{\mathbf{q}'}$ and σ_q.

It is proved in [OW15, Lemma 2.51] that this functor is represented by a finitely $\sigma_{\mathbf{q}'}$-generated C_E-$\sigma_{\mathbf{q}'}$-Hopf algebra $C_E\{\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))\}$ (see Section 5.1 and A.1 for definition). Therefore, $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is a $\sigma_{\mathbf{q}'}$-algebraic group scheme (see Definition A.2). For a brief introduction to $\sigma_{\mathbf{q}'}$-group schemes, we refer to Section *A*.

In the notation of Definition 5.8, if B is a C_E-$\sigma_{\mathbf{q}'}$-algebra, then the matrix $U \otimes 1 \in \mathrm{GL}_n(\mathcal{Q}_S \otimes_{C_E} B)$ is a fundamental solution matrix of $\sigma_q(Y) = AY$ in $\mathcal{Q}_S \otimes_{C_E} B$. Then, for any $\phi \in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))(B)$, the matrix $\phi(U \otimes 1)$ is also a fundamental solution matrix of $\sigma_q(Y) = AY$ in $\mathcal{Q}_S \otimes_{C_E} B$. Thus, there exists $[\phi]_U \in \mathrm{GL}_n((\mathcal{Q}_S \otimes_{C_E} B)^{\sigma_q}) = \mathrm{GL}_{n,C_E}(B)$ such that $\phi(U \otimes 1) = (U \otimes 1)[\phi]_U$. Here GL_{n,C_E} is the $\sigma_{\mathbf{q}'}$-algebraic scheme corresponding to the general linear algebraic group of size n over C_E (see Example A.4).

Proposition 5.9. *The functor ρ_U :*

$$\begin{array}{lll} \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)) & \to & \mathrm{GL}_{n,C_E} \\ \phi \in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))(B) & \mapsto & [\phi]_U \in GL_{n,C_E}(B), \end{array}$$

where $B \in \mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}$ is a $\sigma_{\mathbf{q}'}$-closed embedding (see [DVHW14b, Definition A.3]*).*

Proof. The proof is the exact analogue of [DVHW14b, Proposition 2.5] and its proof is between the lines of [OW15, Lemma 2.51]. □

This proposition allows to identify the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group with a $\sigma_{\mathbf{q}'}$-subgroup scheme of GL_{n,C_E} via the choice of a fundamental solution matrix U. Another choice of fundamental solution matrix leads to a conjugate representation. Therefore, $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is entirely determined by a $\sigma_{\mathbf{q}'}$-Hopf ideal $\mathfrak{I}$ of $C_E\{\mathrm{GL}_{n,C_E}\} = C_E\{X, \frac{1}{det(X)}\}_{\sigma_{\mathbf{q}'}}$ (see Example A.4). The elements of $\mathfrak{I}$ are $\sigma_{\mathbf{q}'}$-polynomials and we call them the defining equations of $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ in GL_{n,C_E}.

In $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois theory, one has a complete Galois correspondence ([OW15, Theorem 2.52 and Lemma 2.53]). We only recall the following results.

Proposition 5.10. *Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and let $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6.*

Then,

$$\mathcal{Q}_S^{\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))} = \{x = \frac{r}{s} \in \mathcal{Q}_S | \forall B \in \mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}, \forall g \in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))(B), \\ g(r\otimes 1).(s\otimes 1) = (r\otimes 1).(g(s\otimes 1))\} = \mathbb{C}_E(z).$$

Moreover, we have $\sigma_{\mathbf{q}'}$-$\dim(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))) = \sigma_{\mathbf{q}'}$-$\mathrm{trdeg}(\mathcal{Q}_S/\mathbb{C}_E(z))$ *(for precise definitions see* [DVHW14b, §A.7]*).*

The last equality means that the complexity of the defining equations of $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ corresponds precisely to the complexity of the $\sigma_{\mathbf{q}'}$-difference algebraic relations satisfied by the solutions of the system $\sigma_q(Y) = AY$ in $\mathcal{Q}_S$.

The relation between the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-Vessiot theory and the non parametrized Picard-Vessiot theory as developed in [vdPS97] is explained below. We shall define the difference Galois group of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ as follows. It is a schematic version of the difference Galois group defined in Section 1 (here, the field of constants is not algebraically closed).

Proposition 5.11. *Let* $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ *and let* $\mathcal{Q}_S = \mathbb{C}_E(z)\langle U\rangle_{\sigma_{\mathbf{q}'}} \subset \mathcal{M}er(\mathbb{C}^\times)$ *the* $(\sigma_q, \sigma_{\mathbf{q}'})$*-PV field for* $\sigma_q(Y) = AY$ *over* $\mathbb{C}_E(z)$*. Set* $\mathcal{Q}_{R_{C_E}} = \mathbb{C}_E(z)(U)$ *and* $R_{C_E} = \mathbb{C}_E(z)[U, \frac{1}{\det(U)}]$*. The Galois group of* $\mathcal{Q}_{R_{C_E}}$ *over* $\mathbb{C}_E(z)$ *is the functor:*

$$\begin{array}{rccl} \mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z)) : & \mathrm{Alg}_{C_E} & \to & \mathrm{Sets}, \\ & B & \mapsto & \mathrm{Aut}^{\sigma_q}(R_{C_E} \otimes_{C_E} B/\mathbb{C}_E(z) \otimes_{C_E} B), \end{array}$$

where σ_q *acts as the identity on* B*. This functor is representable by a* $\mathbb{C}_E$*-finitely generated algebra.*

Proof. Since $C_E \subset \mathcal{Q}_{R_{\mathbb{C}_E}}^{\sigma_q} \subset \mathcal{Q}_S^{\sigma_q} = C_E$, the above functor is representable by a finitely generated C_E-algebra (see [CHS08, Proposition 2.2]). □

If $A \in \mathrm{GL}_n(\mathbb{C}(z))$, one can consider the difference Galois group of $\sigma_q(Y) = AY$ over $\mathbb{C}(z)$ as in Section 1. It is an algebraic group scheme defined over $\mathbb{C}$. The proposition below shows how this last group is related to the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of the system over $\mathbb{C}_E(z)$.

Proposition 5.12. *Let* $A \in \mathrm{GL}_n(\mathbb{C}(z))$ *and let* $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^\times)$ *be the* $(\sigma_q, \sigma_{\mathbf{q}'})$*-PV extension of* $\sigma_q(Y) = AY$ *over* $\mathbb{C}_E(z)$ *defined in Proposition 5.6. Let* $\mathrm{Gal}(\mathcal{Q}_R/\mathbb{C}(z))$ *be the difference Galois group of the system over* $\mathbb{C}(z)$*. Then,*

- *the identity component of the group* $\mathrm{Gal}(\mathcal{Q}_R/\mathbb{C}(z))$ *is isomorphic to* $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ *over an algebraic closure of* C_E*;*
- *the* $(\sigma_q, \sigma_{\mathbf{q}'})$*-Galois group* $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/C_E(z))$ *is a Zariski dense subgroup of* $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ *(see Proposition A.5).*

Proof. The second statement is a discrete analogue of [DVHW14b, Proposition 2.15].

To prove the first statement, we need to introduce another Galois group as follows. By [CHS08, Proposition 2.22 and Theorem 2.9], the group functor

$$\begin{array}{rccl} \mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/C_E(z)) : & \mathrm{Alg}_{C_E} & \to & \mathrm{Sets} \\ & B & \mapsto & \mathrm{Aut}^{\sigma_q}(R_{C_E} \otimes_{C_E} B/C_E(z) \otimes_{C_E} B), \end{array}$$

is representable and defines an algebraic group scheme over C_E, which is isomorphic to $\mathrm{Gal}(\mathcal{Q}_R/\mathbb{C}(z))$ over an algebraic closure of C_E. Since $\mathbb{C}_E(z)$ is an algebraic extension of $C_E(z)$ with same field of constants, the schematic version of [Roq15, Proposition 6] shows that the identity component of $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/C_E(z))$ is isomorphic to the identity component of $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ over an algebraic closure of C_E. By [OW15, Theorem 2.52] for a parametric operator $\sigma_{\mathbf{q}'}$ equal to the identity, we find that the Galois group $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ is connected because $\mathbb{C}_E(z)$ is relatively algebraic closed in $\mathcal{Q}_{R_{C_E}}$ by Lemma 5.3. This ends the proof. □

Remark 5.13. Thus the difference Galois group of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ is always connected. The same holds for its derived subgroup (see [Wat79, Theorem p. 74]).

5.3. **Discrete Isomonodromy.** In $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois theory, one can define a notion of discrete isomonodromy as follows.

Definition 5.14 ([OW15], Definition 2.54)**.** Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$. The system $\sigma_q(Y) = AY$ is called $\sigma_{\mathbf{q}'}$-isomonodromic if there exists $B \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and $d \in \mathbb{N}$ such that

$$\sigma_q(B)A = \sigma_{\mathbf{q}'}^d(A)B. \tag{5.1}$$

Remark 5.15. Our definition is slightly more general than Definition 2.54 in [OW15], where $\sigma_{\mathbf{q}'}$-isomonodromic means that there exists $B \in \mathrm{GL}_n(\mathbb{C}_E(z))$ such that $\sigma_q(B)A = \sigma_{\mathbf{q}'}(A)B$. However, we can apply most of the results of [OW15] by replacing $\sigma_{\mathbf{q}'}$ by $\sigma_{\mathbf{q}'}^d$.

We have the following Galoisian interpretation of $\sigma_{\mathbf{q}'}$-isomonodromy. We say that a $\sigma_{\mathbf{q}'}$-subgroup scheme $H \subset \mathrm{GL}_{n,\mathbf{k}}$ defined over a $\sigma_{\mathbf{q}'}$-field $\mathbf{k}$ is $\sigma_{\mathbf{q}'}^d$-constant if, for all $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebra S, we have $\sigma_{\mathbf{q}'}^d(g) = g$, for all $g \in H(S)$. This is equivalent to the fact that the defining ideal $\mathfrak{I}_H \subset \mathbf{k}\{X, \frac{1}{\det(X)}\}_{\sigma_{\mathbf{q}'}}$ of $H \subset \mathrm{GL}_{n,\mathbf{k}}$ contains the polynomial $\sigma_{\mathbf{q}'}^d(X) - X$ (see Example A.6).

Proposition 5.16. *Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and let $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-PV extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6. The system $\sigma_q(Y) = AY$ is $\sigma_{\mathbf{q}'}$-isomonodromic over $\mathbb{C}_E(z)$ if and only if there exists a regular $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E and an integer $d \geq 1$ such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$*[‡] *is conjugated to a $\sigma_{\mathbf{q}'}^d$-constant subgroup of $\mathrm{GL}_{n,\widetilde{C_E}}$.*

We refer to [OW15, Theorem 2.55] for an analogous result in a different setting.

Note that, since C_E is a $\sigma_{\mathbf{q}'}$-inversive field, [DVHW14b, Corollary A.14] implies that any field extension of C_E is $\sigma_{\mathbf{q}'}$-separable (see Section 5.1).

Before proving Proposition 5.16, we need some intermediate lemmas about extension of σ_q-constants. We have the following result:

Lemma 5.17. *Let $\widetilde{C_E}$ be a regular $\sigma_{\mathbf{q}'}$-field extension of C_E and let $\mathcal{Q}_S/\mathbb{C}_E(z)$ be a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV extension for $\sigma_q(Y) = AY$. By Lemma 5.1, we may consider $\widetilde{\mathbb{C}_E(z)}$ (resp. $\widetilde{\mathcal{Q}_S}$) the $(\sigma_q, \sigma_{\mathbf{q}'})$-field attached to $\mathbb{C}_E(z) \otimes_{C_E} \widetilde{C_E}$*[§] *(resp. $\mathcal{Q}_S \otimes_{C_E} \widetilde{C_E}$). Then $\widetilde{\mathcal{Q}_S}$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-Vessiot extension for $\sigma_q(Y) = AY$ over $\widetilde{\mathbb{C}_E(z)}$ and the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group $\widetilde{G}$ of $\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)}$ is obtained from the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group G of $\mathcal{Q}_S/\mathbb{C}_E(z)$ by base extension, i.e., $\widetilde{G} = G_{\widetilde{C_E}}$.*

Proof. As $\widetilde{\mathcal{Q}_S}^{\sigma_q} = \widetilde{C_E} = \widetilde{\mathbb{C}_E(z)}^{\sigma_q}$, it is clear that $\widetilde{\mathcal{Q}_S}|\widetilde{\mathbb{C}_E(z)}$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-Vessiot extension. Let $R \subset \mathcal{Q}_S$, (resp. $\widetilde{R} \subset \widetilde{\mathcal{Q}_S}$), denote the corresponding $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-Vessiot ring. Then $\widetilde{R}$ is obtained from $R \otimes_{C_E} \widetilde{C_E}$ by localizing

[‡]The subscript $\widetilde{C_E}$ means that we consider the base change of $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ over C_E to $\widetilde{C_E}$.

[§]It is worth mentioning that since $C_E \subsetneq \mathbb{C}_E$, $\widetilde{\mathbb{C}_E(z)}$ can not be identified with $\widetilde{C_E}(z)$.

at the multiplicatively closed set of all non-zero divisors of $\mathbb{C}_E(z) \otimes_{C_E} \widetilde{C_E}$. It follows that, for every $\widetilde{C_E}$-$\sigma_{\mathbf{q}'}$-algebra S,

$$\begin{aligned} G_{\widetilde{C_E}}(S) &= \mathrm{Aut}^{(\sigma_q,\sigma_{\mathbf{q}'})}(R \otimes_{C_E} S | \mathbb{C}_E(z) \otimes_{C_E} S) \\ &= \mathrm{Aut}^{(\sigma_q,\sigma_{\mathbf{q}'})}\Big((R \otimes_{C_E} \widetilde{C_E}) \otimes_{\widetilde{C_E}} S \Big| (\mathbb{C}_E(z) \otimes_{C_E} \widetilde{C_E}) \otimes_{\widetilde{C_E}} S\Big), \end{aligned}$$

i.e.,

$$G_{\widetilde{C_E}}(S) = \mathrm{Aut}^{(\sigma_q,\sigma_{\mathbf{q}'})}(\widetilde{R} \otimes_{\widetilde{C_E}} S | \widetilde{\mathbb{C}_E(z)} \otimes_{\widetilde{C_E}} S) = \widetilde{G}(S).$$

This ends the proof. ∎

Proof of Proposition 5.16. In [OW15, Theorem 2.55], it is proved that if the system is $\sigma_{\mathbf{q}'}$-isomonodromic then there exists a $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E and an integer $d \geq 1$ such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is conjugated to a $\sigma^d_{\mathbf{q}'}$-constant subgroup of $\mathrm{GL}_{n,\widetilde{C_E}}$ (see Remark 5.15). In the proof of [OW15, Theorem 2.55], we note that any $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E that contains a fundamental solution matrix of a given equation of the form $\sigma^d_{\mathbf{q}'}(Y) = DY$ for some $D \in \mathrm{GL}_n(C_E)$ is convenient. We claim that we can find among these extensions a regular one. Indeed consider $\widetilde{C_E} = C_E(X_0, \dots, X_{d-1})$ where the X_i's are $n \times n$-matrices of indeterminates. We can endow $\widetilde{C_E}$ with a structure of $\sigma_{\mathbf{q}'}$-extension of C_E by setting $\sigma_{\mathbf{q}'}(X_i) = X_{i+1}$ for $i = 0, \dots, d-1$ and $\sigma_{\mathbf{q}'}(X_{d-1}) = DX_0$. Then, $X_0 \in \mathrm{GL}_n(\widetilde{C_E})$ is a solution of $\sigma^d_{\mathbf{q}'}(X_0) = DX_0$ and since $\widetilde{C_E}$ is a pure extension of C_E, it is also a regular extension.

Conversely, let us assume that there exists a regular $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E and an integer $d \geq 1$ such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is conjugated to a $\sigma^d_{\mathbf{q}'}$-constant subgroup of $\mathrm{GL}_{n,\widetilde{C_E}}$. Endow $\widetilde{C_E}$ viewed as a field of σ_q-constants and consider the $(\sigma_q, \sigma_{\mathbf{q}'})$-fields $\widetilde{\mathcal{Q}_S}$ and $\widetilde{\mathbb{C}_E(z)}$ as in Lemma 5.17. We find that the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\widetilde{\mathcal{Q}_S}$ over $\widetilde{\mathbb{C}_E(z)}$ equals $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ and is thus conjugate to a $\sigma^d_{\mathbf{q}'}$-constant group over $\widetilde{C_E}$. By [OW15, Theorem 2.55], the system $\sigma_q(Y) = AY$ is $\sigma_{\mathbf{q}'}$-isomonodromic over $\widetilde{C_E}(z)$, *i.e.*, there exist $\widetilde{B} \in \mathrm{GL}_n(\widetilde{C_E}(z))$ and $d \in \mathbb{N}^\times$ such that $\sigma_q(\widetilde{B}) = \sigma^d_{\mathbf{q}'}(A)\widetilde{B}A^{-1}$. By Lemma 5.2, the solution space in $\widetilde{C_E(z)}^{n \times n}$ of the q-difference equation $\sigma_q(Y) = \sigma^d_{\mathbf{q}'}(A)YA^{-1}$ is generated as a $\widetilde{C_E}$-vector space by the solution space of the equation in $\mathbb{C}_E(z)^{n\times n}$. Since the condition $\det(Y) \neq 0$ is an open condition, there exists $B \in \mathrm{GL}_n(\mathbb{C}_E(z))$ such that $\sigma_q(B) = \sigma^d_{\mathbf{q}'}(A)BA^{-1}$ and the system $\sigma_q(Y) = AY$ is $\sigma_{\mathbf{q}'}$-isomonodromic over $\mathbb{C}_E(z)$. ∎

5.4. **Transcendence results.** Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and consider

$$\sigma_q Y = AY. \tag{5.2}$$

Let $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6. Let $U \in \mathrm{GL}_n(\mathcal{Q}_S)$ be a fundamental matrix of solutions of the system (5.2), and let $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ be the representation of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group associated to the fundamental matrix of solutions U.

Let SL_{n,C_E} (when $n \geq 2$), SO_{n,C_E} (when $n \geq 3$) and Sp_{n,C_E} (when n is even) be the $\sigma_{\mathbf{q}'}$-group schemes over C_E corresponding respectively to the special linear group, the special orthogonal group and the symplectic group (see Section A).

Proposition 5.18. *Assume that $n \geq 2$. Let $u = (u_1, \dots, u_n)^t$ be a line (resp. column) vector of U. If there exists $\tilde{C} \in \mathrm{GL}_n(C_E)$ such that the image of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group by the representation $\rho_{U\tilde{C}}$ associated to the fundamental matrix of solutions $U\tilde{C}$ contains*

- $\mathrm{SL}_n(C_E)$ *or* $\mathrm{Sp}_n(C_E)$*, then* $u_1, \dots, u_n$ *are* $\sigma_{\mathbf{q}'}$*-algebraically independent over* $\mathbb{C}_E(z)$*;*
- $\mathrm{SO}_n(C_E)$*, then any* $n-1$ *distinct elements among the* u_i*'s are* $\sigma_{\mathbf{q}'}$*-algebraically independent over* $\mathbb{C}_E(z)$*;*

Proof. Let $H \subset \mathrm{GL}_{n,C_E}$ be SL_{n,C_E} (resp. SO_{n,C_E} or Sp_{n,C_E}). If the image of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group by the representation associated to the fundamental matrix of solutions $U\tilde{C}$ contains H, then the image of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group by the representation associated to the fundamental matrix of solutions U contains $\widetilde{H} = \tilde{C}H\tilde{C}^{-1}$. By the parametrized Galois correspondence [OW15, Theorem 2.52], we have

- the field $\mathbf{K}_0 = \mathcal{Q}_S^{\widetilde{H}}$, the elements of $\mathcal{Q}_S$ fixed by $\widetilde{H}$ is (σ_q, δ)-field with $\mathbf{K}_0^{\sigma_q} = C_E$,
- $\mathcal{Q}_S$ is a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field extension for $\sigma_q(Y) = AY$ over $\mathbf{K}_0$,
- and in the representation attached to U, the Galois group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbf{K}_0)$ coincides with $\widetilde{H}$.

Moreover, in virtue of [OW15, Lemma 2.49], the $\mathbf{K}$-$(\sigma_q, \sigma_{\mathbf{q}'})$-algebra $\tilde{S} = \mathbf{K}_0\{U, \frac{1}{\det(U)}\}_{\sigma_{\mathbf{q}'}}$ is a $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbf{K}_0)$-torsor. Thus, if we write $\tilde{S}$ as $\mathbf{K}_0\{X, \frac{1}{\det(X)}\}_{\sigma_{\mathbf{q}'}}/\mathfrak{I}$ for some $\sigma_{\mathbf{q}'}$-ideal $\mathfrak{I}$ then the following holds

- if $H = \mathrm{SL}_{n,C_E}$ then $\mathfrak{I}$ equals $\{\det(X) - f\}_{\sigma_{\mathbf{q}'}}$ the radical $\sigma_{\mathbf{q}'}$-ideal by $\det(X) - f$ for some $f \in \mathbf{K}_0$;
- if $H = \mathrm{SO}_{n,C_E}$ then $\mathfrak{I}$ equals $\{X\tilde{C}\tilde{C}^t X^t - F, \det(X) - g\}_{\sigma_{\mathbf{q}'}}$ the radical $\sigma_{\mathbf{q}'}$-ideal by $X\tilde{C}\tilde{C}^t X^t - F$ for some $F \in \mathrm{GL}_n(\mathbf{K}_0)$ and $\det(X) - g$ for some $g \in \mathbf{K}_0$;
- if $H = \mathrm{Sp}_{n,C_E}$ then $\mathfrak{I}$ equals $\{X\tilde{C}J\tilde{C}^t X^t - F, \det(X) - g\}_{\sigma_{\mathbf{q}'}}$ the radical δ-ideal by $X\tilde{C}J\tilde{C}^t X^t - F$ for some $F \in \mathrm{GL}_n(\mathbf{K}_0)$ and $\det(X) - g$ for some $g \in \mathbf{K}_0$.

The rest of the proof follows exactly the lines of Proposition 2.4. □

6. q-DIFFERENCE EQUATIONS OF RANK ONE

We remind that $q, \mathbf{q}' \in \mathbb{C}^\times$ with $|q| \neq 1$. From now, we assume that $|q|$ and $|\mathbf{q}'|$ are multiplicatively independent. For any $a(z) \in \mathbb{C}(z)$, we denote by $\operatorname{div} a(z)$ the divisor of $a(z)$ on $\mathbb{C}^\times$, *i.e.*,

$$\operatorname{div} a(z) = \sum_{\alpha \in \mathbb{C}^\times} v_\alpha(a(z))\,[\alpha]$$

where $v_\alpha(a(z))$ denotes the valuation of $a(z)$ at α.

Let $\pi : \mathbb{C}^\times \to \mathbb{C}^\times / q^{\mathbb{Z}}$ be the natural projection. For any $a(z) \in \mathbb{C}(z)^\times$, we set

$$\operatorname{div}_q a(z) = \sum_{\alpha \in \mathbb{C}^\times} v_\alpha(a(z))\,[\pi(\alpha)]\,.$$

The proof of the following lemma is inspired by the proof of [vdPS97, Lemma 2.1].

Lemma 6.1. *Consider* $a(z) \in \mathbb{C}(z)^\times$*. Then, the following properties are equivalent:*

(i) *there exist* $c \in \mathbb{C}^\times$*,* $m \in \mathbb{Z}$ *and* $b(z) \in \mathbb{C}(z)^\times$ *such that* $a(z) = cz^m \frac{b(qz)}{b(z)}$*;*
(ii) $\operatorname{div}_q a(z) = 0$.

Proof. The proof of the fact that (i) implies (ii) is straightforward and left to the reader. Let us prove the converse implication.

We write $\operatorname{div} a(z)$ as follows:

$$\operatorname{div} a(z) = \sum_{j=1}^{s} \sum_{i=1}^{r_j} m_{i,j} \left[\alpha_{i,j}\right]$$

where the non zero complex numbers $\alpha_{i,j}$ and $\alpha_{i',j'}$ belong to the same $q^{\mathbb{Z}}$-orbit if and only if $j = j'$ and where the $m_{i,j}$ are relative integers. The hypothesis $\operatorname{div}_q a(z) = 0$ means that, for all $j \in \{1, \dots, s\}$,

$$\sum_{i=1}^{r_j} m_{i,j} = 0.$$

Up to renumbering, we can assume that, for all $j \in \{1, \dots, s\}$,

$$\alpha_{2,j} = \alpha_{1,j} q^{-k_{2,j}}, \dots, \alpha_{r_j,j} = \alpha_{1,j} q^{-k_{r_j,j}}$$

for some integers $0 < k_{2,j} < \cdots < k_{r_j,j}$. For any $j \in \{1, \dots, s\}$, we consider the rational function given by

$$\begin{aligned} b_j(z) = (z-\alpha_{1,j})^{-m_{1,j}} (z-\alpha_{1,j}q^{-1})^{-m_{1,j}} \cdots (z-\alpha_{1,j}q^{-k_{2,j}+1})^{-m_{1,j}} \\ (z-\alpha_{1,j}q^{-k_{2,j}})^{-m_{1,j}-m_{2,j}} \cdots (z-\alpha_{1,j}q^{-k_{3,j}+1})^{-m_{1,j}-m_{2,j}} \\ \vdots \\ (z-\alpha_{1,j}q^{-k_{r_j-1,j}})^{-m_{1,j}-m_{2,j}-\cdots-m_{r_j-1,j}} \cdots (z-\alpha_{1,j}q^{-k_{r_j,j}+1})^{-m_{1,j}-m_{2,j}-\cdots-m_{r_j-1,j}}. \end{aligned}$$

A straightforward calculation shows that

$$\operatorname{div} \frac{b_j(qz)}{b_j(z)} = \sum_{i=1}^{r_j} m_{i,j} \left[\alpha_{i,j}\right].$$

Letting

$$b(z) = b_1(z) \cdots b_s(z),$$

we get

$$\operatorname{div} \frac{b(qz)}{b(z)} = \sum_{j=1}^{s} \sum_{i=1}^{r_j} m_{i,j} \left[\alpha_{i,j}\right] = \operatorname{div} a(z).$$

Therefore, there exist $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}$ such that $a(z) = cz^m \frac{b(qz)}{b(z)}$. □

Proposition 6.2. *Let $a(z), b(z) \in \mathbb{C}(z)^\times$ be such that*

$$a(z)^{k_0} a(\mathbf{q}' z)^{k_1} \cdots a(\mathbf{q}'^r z)^{k_r} = \frac{b(qz)}{b(z)}$$

for some $k_0, \dots, k_r \in \mathbb{Z}$ with $k_0 k_r \neq 0$. Then, $\operatorname{div}_q a(z) = 0$, i.e., in virtue of Lemma 6.1, there exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $b_1(z) \in \mathbb{C}(z)^\times$ such that $a(z) = cz^m \frac{b_1(qz)}{b_1(z)}$.

Proof. Assume to the contrary that $\operatorname{div}_q a(z) \neq 0$. We set

$$\operatorname{div}_q a(z) = \sum_{i=1}^{m} n_i \left[\zeta_i\right]$$

where the ζ_i are pairwise distinct elements of $\mathbb{C}^\times / q^{\mathbb{Z}}$ and the n_i are non zero integers. We have

$$0 = \operatorname{div}_q \frac{b(qz)}{b(z)} = \operatorname{div}_q a(z)^{k_0} a(\mathbf{q}' z)^{k_1} \cdots a(\mathbf{q}'^r z)^{k_r} = \sum_{j=0}^{r} k_j \sum_{i=1}^{m} n_i \left[\mathbf{q}'^{-j} \zeta_i\right]. \tag{6.1}$$

Let

$$I=\{i\in\{1,\dots,m\}\mid \zeta_i\in \mathbf{q}'^{\mathbb{Z}}\zeta_1\}.$$

Let $i_1,\dots,i_s$ be pairwise distinct integers such that $I=\{i_1,\dots,i_s\}$. Up to renumbering, we can assume that

$$\zeta_{i_1}\prec\cdots\prec\zeta_{i_s}$$

where, for any $x,y\in\mathbb{C}^\times/q^{\mathbb{Z}}$, $x\prec y$ means that $y=\mathbf{q}'^{k}x$ for some $k\in\mathbb{N}^*$. Then, we have $\mathbf{q}'^{-r}\zeta_{i_1}\prec\mathbf{q}'^{-j}\zeta_{i_k}$ for all $j\in\{0,\dots,r\}$ and $k\in\{1,\dots,s\}$ such that $(j,k)\neq(r,1)$. In particular, $\mathbf{q}'^{-r}\zeta_{i_1}\neq\mathbf{q}'^{-j}\zeta_i$ for all $j\in\{0,\dots,r\}$ and $i\in I$ such that $(j,i)\neq(r,i_1)$ (indeed, if $x\prec y$ then $x\neq y$ because $|q|$ and $|\mathbf{q}'|$ are multiplicatively independent).

Moreover, for $j\in\{0,\dots,r\}$ and $i\in\{1,\dots,m\}\setminus I$, $\mathbf{q}'^{-r}\zeta_{i_1}$ and $\mathbf{q}'^{-j}\zeta_i$ are not in the same $\mathbf{q}'^{\mathbb{Z}}$-orbit and hence are not equal.

So, we have proved that $\mathbf{q}'^{-r}\zeta_{i_1}\neq\mathbf{q}'^{-j}\zeta_i$ for all $j\in\{0,\dots,r\}$ and $i\in\{1,\dots,m\}$ such that $(j,i)\neq(r,i_1)$. Therefore, the coefficient of $[\mathbf{q}'^{-r}\zeta_{i_1}]$ in equation (6.1) is equal to 0, *i.e.*, $k_r n_{i_1}=0$, whence a contradiction. ☐

The following proposition gives an example of $\sigma_{\mathbf{q}'}$-isomonodromic equation of rank one.

Proposition 6.3. *Let $a(z)\in\mathbb{C}(z)^\times$. Let $u\in\mathcal{M}er(\mathbb{C}^\times)$ be a non zero solution of $\sigma_q(y)=a(z)y$. Let $\mathcal{Q}_S=\mathbb{C}_E(z)\langle u\rangle_{\sigma_{\mathbf{q}'}}$. Then, the following statements are equivalent*

1. *u and all its transform with respect to $\sigma_{\mathbf{q}'}$ are algebraically dependent over $\mathbb{C}_E(z)$,*
2. *there exist $c\in\mathbb{C}^\times$, $m\in\mathbb{Z}$ and $b(z)\in\mathbb{C}(z)^\times$ such that $a(z)=cz^m\frac{b(qz)}{b(z)}$,*
3. *the group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ can be embedded as a subgroup of $H\subset\mathrm{GL}_{1,C_E}$ with H a $\sigma_{\mathbf{q}'}$-algebraic subgroup defined by*

$$H(S)=\left\{g\in\mathrm{GL}_{1,C_E}(S)\middle|\sigma_{\mathbf{q}'}\left(\frac{\sigma_{\mathbf{q}'}(g)}{g}\right)=\frac{\sigma_{\mathbf{q}'}(g)}{g}\right\}$$

for any $S\in\mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}$.

Moreover, the following statements are equivalent:

(a) there exist $c\in\mathbb{C}^\times$ and $b(z)\in\mathbb{C}(z)^\times$ such that $a(z)=c\frac{b(qz)}{b(z)}$,
(b) the group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is $\sigma_{\mathbf{q}'}$-constant.

Proof. Let us prove (1) $\Rightarrow$ (2). Relying on the classification of the $\sigma_{\mathbf{q}'}$-algebraic subgroups of GL_{1,C_E}, [OW15, Theorem 3.1] ensures that the first statement is equivalent to the existence of $b(z)\in\mathbb{C}(z)^\times$, $t\in\mathbb{N}$ and $n_0,\dots,n_t\in\mathbb{Z}$ not all zero, such that the following equation holds

$$a(z)^{n_0}\sigma_{\mathbf{q}'}(a(z)^{n_1})\cdots\sigma_{\mathbf{q}'}^t(a(z)^{n_t})=\frac{\sigma_q(b(z))}{b(z)}. \tag{6.2}$$

Proposition 6.2 shows then that the first statement implies the second.

Let us prove (2) $\Rightarrow$ (3). Assume that the second statement holds. By proposition 5.11, for any $S\in\mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}$ and $g\in\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))(S)$ there exists $\lambda_g\in S^\times$ such that $g(u)=\lambda_g.u$. Since $a(z)=cz^m\frac{b(qz)}{b(z)}$, an easy computation gives

$$\sigma_q\left(\frac{\sigma_{\mathbf{q}'}(\frac{\sigma_{\mathbf{q}'}(u)}{u})}{\frac{\sigma_{\mathbf{q}'}(u)}{u}}.\frac{h(z)}{\sigma_{\mathbf{q}'}(h(z))}\right)=\frac{\sigma_{\mathbf{q}'}(\frac{\sigma_{\mathbf{q}'}(u)}{u})}{\frac{\sigma_{\mathbf{q}'}(u)}{u}}.\frac{h(z)}{\sigma_{\mathbf{q}'}(h(z))},$$

where $h(z)=\frac{\sigma_{\mathbf{q}'}(b(z))}{b(z)}$. Since $\mathcal{Q}_S^{\sigma_q}=C_E$, there exists $d\in C_E$ such that we have the equality $\frac{\sigma_{\mathbf{q}'}(\frac{\sigma_{\mathbf{q}'}(u)}{u})}{\frac{\sigma_{\mathbf{q}'}(u)}{u}}=d\frac{\sigma_{\mathbf{q}'}(h(z))}{h(z)}$, *i.e.*, $\frac{\sigma_{\mathbf{q}'}(\frac{\sigma_{\mathbf{q}'}(u)}{u})}{\frac{\sigma_{\mathbf{q}'}(u)}{u}}\in\mathbb{C}_E(z)$ and is left invariant by the $(\sigma_q,\sigma_{\mathbf{q}'})$-Galois group. This implies that for any $S\in \mathrm{Alg}_{C_E,\sigma_{\mathbf{q}'}}$ and $g\in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))(S)$, we find $\sigma_{\mathbf{q}'}(\frac{\sigma_{\mathbf{q}'}(\lambda_g)}{\lambda_g})=\frac{\sigma_{\mathbf{q}'}(\lambda_g)}{\lambda_g}$ and we find that the $(\sigma_q,\sigma_{\mathbf{q}'})$-Galois group can be represented as a subgroup of H.

Let us prove $(3)\Rightarrow(1)$. If the third statement holds, then $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is a proper subgroup of GL_{1,C_E}. By proposition 5.10, this implies that u and all its transform with respect to $\sigma_{\mathbf{q}'}$ are algebraically dependent over $\mathbb{C}_E(z)$. This proves (1).

Let us prove $(a)\Rightarrow(b)$. If there exist $c\in\mathbb{C}^\times$ and $b(z)\in\mathbb{C}(z)^\times$ such that $a(z)=c\frac{b(qz)}{b(z)}$ then $\frac{\sigma_{\mathbf{q}'}(a)}{a}=\frac{\sigma_q(h)}{h}$ where $h(z)=\frac{\sigma_{\mathbf{q}'}(b(z))}{b(z)}$. Proposition 5.16 allows to conclude that the group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is $\sigma_{\mathbf{q}'}$-constant. Let us prove $(b)\Rightarrow(a)$. If the group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is $\sigma_{\mathbf{q}'}$-constant then u and all its transform with respect to $\sigma_{\mathbf{q}'}$ are algebraically dependent over $\mathbb{C}_E(z)$. By the above, there exist $c\in\mathbb{C}^\times$, $m\in\mathbb{Z}$ and $b(z)\in\mathbb{C}(z)^\times$ such that $a(z)=cz^m\frac{b(qz)}{b(z)}$. However Proposition 5.16 states that there exists $h(z)\in\mathbb{C}_E(z)$ such that $\sigma_{\mathbf{q}'}(a)/a=\sigma_q(h)/h$. An easy computation shows that $m=0$. □

7. Discrete projective isomonodromy

The following proposition allows to characterize the $(\sigma_q,\sigma_{\mathbf{q}'})$-Galois group of a q-difference system with large difference Galois group.

Proposition 7.1. *Let $A\in\mathrm{GL}_n(\mathbb{C}(z))$. Let G be the difference Galois group of $\sigma_q(Y)=AY$ over the σ_q-field $\mathbb{C}(z)$. Assume that $G^{\circ,der}$ is an irreducible almost simple algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ and has toric constant centralizer (see Definition A.16). Let $\mathcal{Q}_S\subset\mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q,\sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y)=AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6. Then, we have the following alternative:*

1. *there exist $d\in\mathbb{N}^\times$ and a regular $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is conjugate to a $\sigma_{\mathbf{q}'}$-group H such that, for all $S\in\mathrm{Alg}_{\widetilde{C_E},\sigma_{\mathbf{q}'}}$ and $g\in H(S)$, there exists $\lambda_g\in S^\times$ such that $\sigma_{\mathbf{q}'}^d(g)=\lambda_g g$;*
2. *$\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ contains $G^{\circ,der}_{C_E}$, the base change to C_E of $G^{\circ,der}$.*

Moreover, if the first case holds then there exist $\widetilde{U}\in\mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$, with $\widetilde{\mathcal{Q}_S}$ the fraction field of $\mathcal{Q}_S\otimes_{C_E}\widetilde{C_E}$, a fundamental solution matrix, $d\in\mathbb{N}^\times$ and $B\in\mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$, with $\widetilde{\mathbb{C}_E(z)}$ the fraction field of $\mathbb{C}_E(z)\otimes_{C_E}\widetilde{C_E}$, $g\in\widetilde{\mathcal{Q}_S}^\times$, such that

$$\sigma_{\mathbf{q}'}^d(\widetilde{U})=gB\widetilde{U}. \tag{7.1}$$

Proof of Proposition 7.1. Let $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ be the difference Galois group as defined in Proposition 5.11.

Since the Galois group of $\sigma_q(Y)=AY$ over $\mathbb{C}(z)$ contains $G^{\circ,der}$, Proposition 5.12 implies that $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$ contains $G^{\circ,der}_{C_E}$. We refer to Section A for the definition of the derived group of a $\sigma_{\mathbf{q}'}$-algebraic group scheme. Thus, the derived group scheme $\mathcal{D}(\mathrm{Gal}(\mathcal{Q}_{R_{\mathbb{C}_E}}/\mathbb{C}_E(z)))$ equals to $G^{\circ,der}_{C_E}$. Since $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is Zariski-dense in $\mathrm{Gal}(\mathcal{Q}_{R_{C_E}}/\mathbb{C}_E(z))$, we find that $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))$ is Zariski-dense in $\mathcal{D}(\mathrm{Gal}(\mathcal{Q}_{R_{\mathbb{C}_E}}/\mathbb{C}_E(z)))=G^{\circ,der}_{C_E}$ by Proposition A.14. By Lemma 5.3, $\mathbb{C}_E(z)$ is relatively algebraically closed in $\mathcal{Q}_S$. By straightforward analogues of [DVHW14a, Lemma 6.3] and [DVHW14b, Proposition 4.3], we find that the $\sigma_{\mathbf{q}'}$-group scheme $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is absolutely $\sigma_{\mathbf{q}'}$-integral. By Lemma A.15, the $\sigma_{\mathbf{q}'}$-algebraic group scheme $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))$ is absolutely $\sigma_{\mathbf{q}'}$-integral. Let $\overline{C_E}$ be an algebraically closure of C_E. We extend $\sigma_{\mathbf{q}'}$ from C_E to $\overline{C_E}$. Since C_E is inversive for $\sigma_{\mathbf{q}'}$, the same holds for $\overline{C_E}$. Then, $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))_{\overline{C_E}}$ is a Zariski dense $\sigma_{\mathbf{q}'}$-integral subgroup of $G^{\circ,der}_{\overline{C_E}}$. By Theorem A.10, there exists a $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of $\overline{C_E}$, such that either $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))_{\widetilde{C_E}} = G^{\circ,der}_{\widetilde{C_E}}$ or there exists an integer $d \geq 1$ such that $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))_{\widetilde{C_E}}$ is conjugate to a $\sigma^d_{\mathbf{q}'}$-constant subgroup of $G^{\circ,der}_{\widetilde{C_E}}$. The group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}} = G^{\circ,der}_{\widetilde{C_E}}$ is irreducible almost simple and has toric constant centralizer. Since $\mathcal{D}(\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z)))_{\widetilde{C_E}}$ is a normal subgroup of $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$, Lemma A.17 ensures that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is either equal to a subgroup of $\mathrm{GL}_{1,\widetilde{C_E}} \times \mathrm{SL}_{n,\widetilde{C_E}}$ containing $G^{\circ,der}_{\widetilde{C_E}}$ or conjugate to a $\sigma_{\mathbf{q}'}$-algebraic group H over $\widetilde{C_E}$ such that for all $S \in \mathrm{Alg}_{\widetilde{C_E},\sigma_{\mathbf{q}'}}$ and $g \in H(S)$ there exists $\lambda_g \in S^\times$ such that $\sigma^d_{\mathbf{q}'}(g) = \lambda_g g$.

We shall prove that if the first case holds then there there exist $\widetilde{U} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$ a fundamental solution matrix, a positive integer d and $B \in \mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$, $g \in \widetilde{\mathcal{Q}_S}^\times$, such that

$$\sigma^d_{\mathbf{q}'}(\widetilde{U}) = gB\widetilde{U}. \tag{7.2}$$

Thus, let us assume that there exists a positive integer d and a $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is conjugate to a $\sigma_{\mathbf{q}'}$-group H such that, for all $S \in \mathrm{Alg}_{\widetilde{C_E},\sigma_{\mathbf{q}'}}$ and $g \in H(S)$, there exist $\lambda_g \in \mathrm{GL}_{1,\widetilde{C_E}}(S)$ such that $\sigma^d_{\mathbf{q}'}(g) = \lambda_g g$. By Lemma 5.17, we construct a $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-Vessiot extension $\widetilde{\mathcal{Q}_S}$ for $\sigma_q(Y) = AY$ over $\widetilde{C_E(z)}$ such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}} = \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)})$. By proposition 5.9, we can choose $\widetilde{U} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$, a fundamental solution matrix, such that for any $\phi \in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)})(S)$, we have $\sigma^d_{\mathbf{q}'}([\phi]_{\widetilde{U}}) = \lambda_\phi [\phi]_{\widetilde{U}}$ and $\lambda_\phi \in \mathrm{GL}_1(S)$. Then, for any $\phi \in \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)})(S)$, we have $\phi(\sigma^d_{\mathbf{q}'}(\widetilde{U}).\widetilde{U}^{-1}) = \lambda_\phi \sigma^d_{\mathbf{q}'}(\widetilde{U}).\widetilde{U}^{-1}$. Let g be a non-zero entry of $\sigma^d_{\mathbf{q}'}(\widetilde{U}).\widetilde{U}^{-1}$. It is easy to see that the matrix $B = \frac{1}{g}\sigma^d_{\mathbf{q}'}(\widetilde{U}).\widetilde{U}^{-1} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$ is fixed by $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)})$. By Proposition 5.10, $B \in \mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$. □

8. q-DIFFERENCE EQUATIONS WITH CONVERGENT POWER SERIES SOLUTIONS

Let $A \in \mathrm{GL}_n(\mathbb{C}(z))$. We remind that $q, \mathbf{q}' \in \mathbb{C}^\times$, with $|q|$ and $|\mathbf{q}'|$ multiplicatively independant. Consider the q-difference system

$$\sigma_q(Y) = AY. \tag{8.1}$$

Let $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6. The aim of the present section is to study the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of (8.1) under the following assumption.

Assumptions 8.1. Assume that $n \geq 2$. Let G be the Galois group of (8.1) over $\mathbb{C}(z)$. We assume that $G^{\circ,der}$ is either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ (when $n \geq 3$) or $\mathrm{Sp}_n(\mathbb{C})$ (when n is even).

8.1. **$\sigma_{\mathbf{q}'}$-algebraic determinant group.** Let $\mathbb{C}(\{z\})$ be the field of germs of meromorphic functions at $z=0$. The goal of the subsection is to prove:

Theorem 8.2. *Assume that the hypothesis 8.1 holds and that there exist $b \in \mathbb{C}(z)^{\times}$ and $c \in \mathbb{C}^{\times}$ and $m \in \mathbb{Z}$ such that $\det(A) = cz^m \frac{b(qz)}{b(z)}$. Assume that the system $\sigma_q(Y) = AY$ admits a non zero solution vector $Y_0 \in \mathbb{C}(\{z\})^n$. Let $\mathcal{Q}_S \subset \mathcal{M}er(\mathbb{C}^{\times})$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ defined in Proposition 5.6. Then, the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ contains $G^{\circ,der}_{C_E}$.*

Lemma 8.3. *Let us consider a vector $u = (u_1, \dots, u_n)^t$ with coefficients in $\mathbb{C}((z))^n$ such that $\sigma_q(u) = Au$ for some $A \in \mathrm{GL}_n(\mathbb{C}(z))$. Assume moreover that each u_i satisfies some nonzero linear $\mathbf{q}'$-difference equation with coefficients in $\mathbb{C}(z)$. Then, the u_i actually belong to $\mathbb{C}(z)$.*

Proof of Lemma 8.3. Since $u = (u_1, \dots, u_n)^t$ has coefficients in $\mathbb{C}((z))^n$, and any entry of u satisfies some nonzero linear $\mathbf{q}'$-difference equation with coefficients in $\mathbb{C}(z)$, according to the cyclic vector lemma, there exists $P \in \mathrm{GL}_n(\mathbb{C}(z))$ such that $Pu = (f, \sigma_q(f), \dots, \sigma_q^{n-1}(f))^t$ for some $f \in \mathbb{C}((z))$ which is a solution of a nonzero linear q-difference equation, *i.e.*, a σ_q-difference equation, of order n with coefficients in $\mathbb{C}(z)$. Moreover, f satisfies a nonzero linear $\sigma_{\mathbf{q}'}$-equation with coefficients in $\mathbb{C}(z)$, because it is a $\mathbb{C}(z)$-linear combination of the u_i and the u_i themselves satisfy such equations. It follows from [BB92, Remark 7.5] that f belongs to $\mathbb{C}(z)$. Hence, the entries of $u = P^{-1}(Pu) = P^{-1}(f, \sigma_q(f), \dots, \sigma_q^{n-1}(f))^t$ actually belong to $\mathbb{C}(z)$, as expected. □

Remark 8.4. Let us remind that $|q| \neq 1$. Therefore, any vector solution of (8.1) in $\mathbb{C}(\{z\})^n$, belongs in fact to $(\mathbb{C}((z)) \cap \mathcal{M}er(\mathbb{C}^{\times}))^n$.

Proof of Theorem 8.2. In virtue of Remark 8.4, $Y_0 \in (\mathbb{C}((z)) \cap \mathcal{M}er(\mathbb{C}^{\times}))^n$. Let $\mathcal{Q}_S$ be the $(\sigma_q, \sigma_{\mathbf{q}'})$-Picard-vessiot extension of $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$. Since $Y_0 = (u_1, \dots, u_n)^t \in \mathcal{Q}_S^{\times}$, there exists a fundamental solution matrix $U \in \mathrm{GL}_n(\mathcal{Q}_S)$ whose first column is precisely Y_0.

We let G denotes the difference Galois group of $\sigma_q(Y) = AY$ over the field $\mathbb{C}(z)$, and we let $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ denotes the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group over the $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathbb{C}_E(z)$. By assumption, $G^{\circ,der}$ is either $\mathrm{SL}_n(\mathbb{C})$ (when $n \geq 2$), $\mathrm{SO}_n(\mathbb{C})$ (when $n \geq 3$) or $\mathrm{Sp}_n(\mathbb{C})$ (when n is even). By Proposition 7.1, we have the following alternative:

(1) there exists a positive integer d and a regular $(\sigma_q, \sigma_{\mathbf{q}'})$-field extension $\widetilde{C_E}$ of C_E such that $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ is conjugate to a $\sigma^d_{\mathbf{q}'}$-constant subgroup of $G^{\circ,der}_{\widetilde{C_E}}$;
(2) $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ contains $G^{\circ,der}_{\mathbb{C}_E}$.

Moreover, if the first case holds, then there exists $\widetilde{U} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$, with $\widetilde{\mathcal{Q}_S}$ the fraction field of $\mathcal{Q}_S \otimes_{C_E} \widetilde{C_E}$, a fundamental solution matrix, a positive integer d and $B \in \mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$, with $\widetilde{\mathbb{C}_E(z)}$, $g \in \widetilde{\mathcal{Q}_S}^{\times}$, such that

$$\sigma^d_{\mathbf{q}'}(\widetilde{U}) = gB\widetilde{U}. \tag{8.2}$$

We claim that the first case can not hold. Suppose to the contrary that there exists a regular $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E such that there exists $\widetilde{U} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$ a fundamental solution matrix, a positive integer d, $B \in \mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$ and $g \in \widetilde{\mathcal{Q}_S}^{\times}$, such that (8.2) holds. This means that there exists $D \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S}^{\sigma_q}) = \mathrm{GL}_n(\widetilde{C_E})$ such that $\sigma^d_{\mathbf{q}'}(U) = gBUD$. This formula implies that the (finite dimensional)

$\widetilde{\mathbb{C}_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}}$-vector space generated by the entries of U is stable by $\sigma_{\mathbf{q}'}^d$. In particular, any u_i (recall that the u_i are the entries of the first column Y_0 of U) satisfies a nonzero linear $\mathbf{q}'$-equation $\mathcal{L}_i(y)$ with coefficients in $\widetilde{\mathbb{C}_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}}$. We claim that $\widetilde{\mathbb{C}_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}} = \widetilde{\mathbb{C}_E(z)}(g)$. Indeed, we have $\sigma_{\mathbf{q}'}^d(\widetilde{U}) = gB\widetilde{U}$ and $\sigma_q(\widetilde{U}) = A\widetilde{U}$. Thus,

$$\sigma_q\left(\frac{\sigma_{\mathbf{q}'}^d(\det(\widetilde{U}))}{\det(\widetilde{U})}\right) = \mathbf{q}'^{md}\frac{\sigma_q(h)}{h}\frac{\sigma_{\mathbf{q}'}^d(\det(\widetilde{U}))}{\det(\widetilde{U})},$$

where $h(z) = \frac{\sigma_{\mathbf{q}'}^d(b(z))}{b(z)}$. Thus, we have $\sigma_q(g^n l) = \mathbf{q}'^{md} g^n l$ with $l(z) = \det(B)/h \in \widetilde{\mathbb{C}_E(z)}$. Thus $\sigma_q(\sigma_{\mathbf{q}'}(g^n l)) = \mathbf{q}'^{md}\sigma_{\mathbf{q}'}(g^n l)$. Therefore, there exists $c \in \widetilde{C_E}^\times$ such that $\sigma_{\mathbf{q}'}(g^n l) = cg^n l$. Then $\left(\frac{\sigma_{\mathbf{q}'}(g)}{g}\right)^n \in \widetilde{\mathbb{C}_E(z)}$. Since $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\widetilde{\mathcal{Q}_S}/\widetilde{\mathbb{C}_E(z)}) = \mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))_{\widetilde{C_E}}$ is $\sigma_{\mathbf{q}'}$-integral, $\widetilde{\mathbb{C}_E(z)}$ is relatively algebraically closed in $\widetilde{\mathcal{Q}_S}$. Thus, $\frac{\sigma_{\mathbf{q}'}(g)}{g} \in \widetilde{\mathbb{C}_E(z)}$ and $\widetilde{\mathbb{C}_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}} = \widetilde{\mathbb{C}_E(z)}(g)$.

We claim that any u_i satisfies a nonzero linear $\mathbf{q}'$-equation with coefficients in $\widetilde{\mathbb{C}_E(z)}$. If $g \in \widetilde{\mathcal{Q}_S}$ is algebraic over $\widetilde{\mathbb{C}_E(z)}$ then $g \in \widetilde{\mathbb{C}_E(z)}$, because $\widetilde{\mathbb{C}_E(z)}$ is relatively algebraically closed in $\widetilde{\mathcal{Q}_S}$. In that case, the claim is obvious. Thus, let us assume that g is transcendental over $\widetilde{\mathbb{C}_E(z)}$. By Proposition 6.3, we must have $m \neq 0$. Then, we can write the equation $\mathcal{L}_i(y) = 0$ as $\sum_{j=0}^{\nu}\mathcal{L}_{i,j}(y)g^j = 0$ where the $\mathcal{L}_{i,j}(y)$ are linear δ-operators with coefficients in $\widetilde{\mathbb{C}_E(z)}$, not all zero. To prove our claim, it is sufficient to show that g is transcendental over $\widetilde{\mathbb{C}_E(z)}\{u_1,\dots,u_n\}_{\sigma_{\mathbf{q}'}}$. It is also sufficient to prove that g^n is transcendental over $\widetilde{\mathbb{C}_E(z)}\{u_1,\dots,u_n\}_{\sigma_{\mathbf{q}'}}$. Assume that there exists a non zero relation

$$\sum_{k=0}^{\kappa} a_k g^{nk} = 0, \tag{8.3}$$

where $\kappa > 1$ and $a_0,\dots,a_{\kappa-1},a_\kappa = 1 \in \widetilde{\mathbb{C}_E(z)}\langle u_1,\dots,u_n\rangle_{\sigma_{\mathbf{q}'}}$ and κ is minimal. We remind that $\sigma_q(g^n) = g^n\mathbf{q}'^{md}l/\sigma_q(l)$. Applying σ_q to (8.3) and subtracting $\mathbf{q}'^{md\kappa}\frac{l^\kappa}{\sigma_q(l^\kappa)} *$ (8.3), we find a smaller liaison of the form

$$\sum_{k=0}^{\kappa-1}(\sigma_q(a_k/l^{\kappa-k}) - \mathbf{q}'^{md(\kappa-k)}a_k/l^{\kappa-k})g^{nk} = 0.$$

Thus, for all $k = 0,\dots,\kappa-1$, we have $\sigma_q(a_k/l^{\kappa-k}) - \mathbf{q}'^{md(\kappa-k)}a_k/l^{\kappa-k} = 0$. Let us state and prove a technical lemma.

Lemma 8.5. *Let us fix $r \in \mathbb{N}^\times$. Then, the equation $\sigma_q(y) = \mathbf{q}'^{mdr}y$ has no non zero solution in $\widetilde{\mathbb{C}_E(z)}\langle u_1,\dots,u_n\rangle_{\sigma_{\mathbf{q}'}}$.*

Proof of Lemma 8.5. We have $\widetilde{\mathbb{C}_E(z)}\langle u_1,\dots,u_n\rangle_{\sigma_{\mathbf{q}'}} \subset \widetilde{\mathbb{C}_E((z))}$, the fraction field of $\widetilde{C_E} \otimes_{C_E} \mathbb{C}_E((z))$. Suppose to the contrary that the equation has a non zero solution in $\widetilde{\mathbb{C}_E((z))}$. By Lemma 5.2, we can find a non zero solution f in $\mathbb{C}_E((z))$ Let $f = \sum_{\ell=\nu}^{\infty} y_\ell z^\ell$ with $y_\nu \neq 0$ a non zero solution of $\sigma_q(y) = \mathbf{q}'^{mdr}y$. Taking the z^ν coefficients of the two sides of $\sigma_q(y) = \mathbf{q}'^{mdr}y$, we find $\sigma_q(y_\nu)q^\nu = \mathbf{q}'^{mdr}y_\nu$. Since $y_\nu \in \mathbb{C}_E$, there exists $s \in \mathbb{N}^\times$ such that $\sigma_q^s(y_\nu) = y_\nu$. Then,

$$\sigma_q^s(y_\nu)q^{s\nu} = y_\nu q^{s\nu} = \mathbf{q}'^{smdr}y_\nu.$$

Since $|q|$ and $|\mathbf{q}'|$ are multiplicatively independent, one should have $s\nu = mdsr = 0$. We remind that $m \neq 0$, so $mdr \neq 0$. Consequently, we find a contradiction and this proves that the equation $\sigma_q(y) = \mathbf{q}'^{mdr} y$ has no non zero solution in $\widetilde{\mathbb{C}_E(z)}\langle u_1, \ldots, u_n\rangle_{\sigma_{\mathbf{q}'}}$. $\square$

Let us finish the proof of Theorem 8.2. In virtue of Lemma 8.5, for all $k \in \{0, \ldots, \kappa - 1\}$, the equation $\sigma_q(y) = \mathbf{q}'^{md(\kappa-k)} y$ has no non zero solution in $\widetilde{\mathbb{C}_E(z)}\{u_1, \ldots, u_n\}_{\sigma_{\mathbf{q}'}}$. Hence, $g^{n\kappa} = 0$. This is a contradiction with the fact that g is transcendental over $\widetilde{\mathbb{C}_E(z)}$ and proves our claim.

Therefore, the u_i satisfy a non zero linear $\sigma_{\mathbf{q}'}$-equation over $\widetilde{\mathbb{C}_E(z)}$. Since $\mathbb{C}$ is algebraically closed and $u_i \in \mathbb{C}((z))$, a descent argument shows that the u_i satisfy a non zero linear $\sigma_{\mathbf{q}'}$-equation over $\mathbb{C}(z)$.

It follows from Lemma 8.3 that the u_i belong to $\mathbb{C}(z)$. Hence, the first column of U is fixed by the Galois group G and this contradicts the hypothesis 8.1. Therefore, $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ contains $G^{\circ,der}_{C_E}$.

$\square$

Next Corollary improves Theorem 8.2 by removing the assumption that there exists a vector solution in $\mathbb{C}(\{z\})^n$.

Corollary 8.6. *Assume that the hypothesis 8.1 hold and that there exist $b(z) \in \mathbb{C}(z)^\times$ and $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ such that $\det(A) = cz^m \frac{b(qz)}{b(z)}$. Then, the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/C_E(z))$ contains $G^{\circ,der}_{C_E}$.*

Proof. By Lemma B.2, there exist $l \in \mathbb{N}^\times, d \in \mathbb{C}^\times$ and $s \in \mathbb{Z}$ such that the q^l-difference system $\sigma_{q^l}(Y) = dz^s \sigma_q^{l-1}(A) \cdots AY$ has a non zero vector solution $Y_0 \in \mathbb{C}(\{z\})^n$. Set $A_1 = dz^s \sigma_q^{l-1}(A) \cdots A$ and $A[l] = \sigma_q^{l-1}(A) \cdots A$. Let $\mathcal{Q}_S$ be a $(\sigma_q, \sigma_{\mathbf{q}'})$-PV field for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ and let $U \in \mathrm{GL}_n(\mathcal{Q}_S)$ be a fundamental solution matrix. By Lemma B.3, the ring $\mathcal{Q}_S = \mathbb{C}_E(z)\langle \sigma_q^l(U)\rangle_{\sigma_{\mathbf{q}'}}$ is a $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-Picard-Vessiot extension for $\sigma_{q^l}(Y) = A[l]Y$ and the base extension to C_{E_l} of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group (resp. the difference Galois group) of $\sigma_q(Y) = AY$ coincides with the $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-Galois group (resp. the difference Galois group) of $\sigma_q^l(Y) = A[l]Y$ over $\mathbb{C}_E(z)$.

This shows that the derived group of the difference Galois group of $\sigma_{q^l}(Y) = A[l]Y$ equals $G^{\circ,der}_{C_{E_l}}$. Moreover, the latter group equals the derived group of the difference Galois group of $\sigma_{q^l}(Y) = A_1 Y$ and $\det(A_1) = d^n z^{sn+ml} c^l \frac{b(q^l z)}{b(z)}$. Theorem 8.2 with q replaced by q^l, allows to conclude that the $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q^l(Y) = A_1 Y$ contains $G^{\circ,der}_{C_{E_l}}$. Finally, the derived group of the latter group coincides with the derived group of the $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q^l(Y) = A[l]Y$. Since the formation of derived groups commute with base extension (see Lemma A.13), we get that the derived group of the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q(Y) = AY$ contains $G^{\circ,der}_{C_E}$. This allows to conclude the proof. $\square$

8.2. **$\sigma_{\mathbf{q}'}$-transcendent determinant.** Let us remind that the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q(y) = \det(A)y$ over $\mathbb{C}_E(z)$ is a proper subroup of the multiplicative group GL_{1,C_E} if and only if there exist $b \in \mathbb{C}(z)^\times$, $m \in \mathbb{Z}$, and $c \in \mathbb{C}^\times$, such that $\det(A) = cz^m \frac{b(qz)}{b(z)}$.

The goal of the subsection is to prove:

Theorem 8.7. *Assume that the hypothesis 8.1 holds and the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q(y) = \det(A)y$ over $\mathbb{C}_E(z)$ equals to GL_{1,C_E}. Let us assume that the system $\sigma_q(Y) = AY$ admits a non zero solution vector*

$Y_0 = (f, \sigma_q(f), \ldots, \sigma_q^{n-1}(f))^t \in \mathbb{C}(\{z\})^n$. *Then, f is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$.*

Lemma 8.8. *Let L be a $\sigma_{\mathbf{q}'}$-field and let $L\langle a\rangle_{\sigma_{\mathbf{q}'}}$ and $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ be two $\sigma_{\mathbf{q}'}$-field extensions of L, both contained in a same $\sigma_{\mathbf{q}'}$-field extension of L. Assume that a is $\sigma_{\mathbf{q}'}$-algebraically independent over L and that any b_i is $\sigma_{\mathbf{q}'}$-algebraic over L. Then, the field extensions $L\langle a\rangle_{\sigma_{\mathbf{q}'}}$ and $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ are linearly disjoint over L.*

Proof of Lemma 8.8. To the contrary, suppose that $L\langle a\rangle_{\sigma_{\mathbf{q}'}}$ and $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ are not linearly disjoint over L. Then a is $\sigma_{\mathbf{q}'}$-algebraic over $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$. This implies that the $\sigma_{\mathbf{q}'}$-transcendence degree of the field $L\langle a, b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ over $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ is zero. Since the $\sigma_{\mathbf{q}'}$-transcendence degree of $L\langle b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ over L is also zero, by hypothesis, we find that the $\sigma_{\mathbf{q}'}$-transcendence degree of $L\langle a, b_1, \ldots, b_n\rangle_{\sigma_{\mathbf{q}'}}$ over L is zero by classical properties of the transcendence degree. This implies that a is $\sigma_{\mathbf{q}'}$-algebraic over L and yields a contradiction. $\square$

Proof of Theorem 8.7. In virtue of Remark 8.4, $Y_0 \in (\mathbb{C}((z)) \cap \mathcal{M}er(\mathbb{C}^\times))^n$. Since $Y_0 \in \mathcal{Q}_S^\times$, there exists a fundamental solution matrix $U \in \mathrm{GL}_n(\mathcal{Q}_S)$ whose first column is precisely Y_0. Set $u_i = \sigma_q^i(f)$.

We let G denotes the Galois group of $\sigma_q(Y) = AY$ over the field $\mathbb{C}(z)$, and we let $\mathrm{Gal}^{\sigma_{\mathbf{q}'}}(\mathcal{Q}_S/\mathbb{C}_E(z))$ denote its $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group over the $(\sigma_q, \sigma_{\mathbf{q}'})$-field $\mathbb{C}_E(z)$. By assumption, the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q(y) = \det(A)y$ over $\mathbb{C}_E(z)$ equals to GL_{1,C_E}.

We claim that at least one of the u_i is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$. Suppose to the contrary that all of them are $\sigma_{\mathbf{q}'}$-algebraic. In virtue of the results of Section 5.4, the second case of Proposition 7.1 can not hold. Then, there exist a regular $\sigma_{\mathbf{q}'}$-field extension $\widetilde{C_E}$ of C_E and $\widetilde{U} \in \mathrm{GL}_n(\widetilde{\mathcal{Q}_S})$ a fundamental solution matrix, a positive integer d and $B \in \mathrm{GL}_n(\widetilde{\mathbb{C}_E(z)})$ and such that

$$\sigma_{\mathbf{q}'}^d(\widetilde{U}) = gB\widetilde{U} \tag{8.4}$$

with $g \in \widetilde{\mathcal{Q}_S}^\times$.

But $\widetilde{U} = UC$, for some $C \in \mathrm{GL}_n(\widetilde{C_E})$. Therefore,

$$\sigma_{\mathbf{q}'}^d(U) = gBUC\sigma_{\mathbf{q}'}^{-d}(C).$$

This shows that the $\widetilde{C_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}}$-vector subspace of $\widetilde{\mathcal{Q}_S}$ generated by the entries of U and all their successive $\sigma_{\mathbf{q}'}$-transforms is of finite dimension. In particular, any u_i satisfies a nonzero linear $\sigma_{\mathbf{q}'}$-equation $\mathcal{L}_i(y) = 0$ with coefficients in $\widetilde{C_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}}$. We can assume that the coefficients of $\mathcal{L}_i(y)$ belong to $\widetilde{\mathbb{C}_E(z)}\{g\}_{\sigma_{\mathbf{q}'}}$. We write $\mathcal{L}_i(y) = \sum_\alpha L_{i,\alpha}(y)g_\alpha$ where $L_{i,\alpha}(y)$ is a linear $\sigma_{\mathbf{q}'}$-operator with coefficients in $\widetilde{\mathbb{C}_E(z)}$, and g_α is a monomial in the $\sigma_{\mathbf{q}'}^i(g)$'s.

We remind that the $(\sigma_q, \sigma_{\mathbf{q}'})$-Galois group of $\sigma_q(y) = \det(A)y$ over $\mathbb{C}_E(z)$ equals to GL_{1,C_E}. In virtue of Proposition 6.3, $\det(U)$ is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$. Since $g^n = \lambda \frac{\sigma_{\mathbf{q}'}^d(\det(U))}{\det(U)}$ for some non zero $\lambda \in \widetilde{\mathbb{C}_E(z)}$. Thus, g is $\sigma_{\mathbf{q}'}$-algebraically independent over $\widetilde{\mathbb{C}_E(z)}$.

By Lemma 8.8, the $\sigma_{\mathbf{q}'}$-fields $\widetilde{C_E(z)}\langle g\rangle_{\sigma_{\mathbf{q}'}}$ and $\widetilde{\mathbb{C}_E(z)}\langle u_1, ..., u_n\rangle_{\sigma_{\mathbf{q}'}}$ are linearly disjoint over $\widetilde{\mathbb{C}_E(z)}$. It follows easily that there exists some non zero $L_{i,\alpha}(y)$ such that $L_{i,\alpha}(u_i) = 0$. Therefore, the u_i satisfy a non zero linear $\sigma_{\mathbf{q}'}$-equation over $\widetilde{\mathbb{C}_E(z)}$. Since $\mathbb{C}$ is algebraically closed and $u_i \in \mathbb{C}((z))$, a descent argument shows that the u_i satisfy a non zero linear $\sigma_{\mathbf{q}'}$-equation over $\mathbb{C}(z)$. It follows from

Lemma 8.3 that the u_i belong to $\mathbb{C}(z)$. Hence, the first column of U is fixed by the difference Galois group G and this contradicts the hypothesis 8.1. ☐

9. Applications

9.1. **User friendly criterias for $\sigma_{\mathbf{q}'}$-transcendence.** The goal of this subsection is to use the results of Section 8 in order to give transcendence criterias. We refer to Section 8 for the notations used in this section.

Corollary 9.1. *Let G be the difference Galois group of the q-difference system (8.1) over the σ_q-field $\mathbb{C}(z)$. Assume that $n \geq 2$ and $G^{\circ,der}$ is either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ (when $n \geq 3$) or $\mathrm{Sp}_n(\mathbb{C})$ (when n is even). The following holds.*

- *Assume that there exist $b(z) \in \mathbb{C}(z)^{\times}$ and $c \in \mathbb{C}^{\times}$, $m \in \mathbb{Z}$ such that $\det(A) = cz^m \frac{b(qz)}{b(z)}$ and let $(u_1, \dots, u_n)^t \in \mathcal{M}er(\mathbb{C}^{\times})^n$ be a non zero solution vector of (8.1). If $G^{\circ,der} = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ (when n is even) (resp. $G^{\circ,der} = \mathrm{SO}_n(\mathbb{C})$ when $n \geq 3$) any n (resp. $n-1$) of the u_i's are $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$;*
- *If there exists $f \in \mathcal{M}er(\mathbb{C}^{\times})$ such that $(f, \sigma_q(f), \dots, \sigma_q^{n-1}(f))^t$ is a vector solution of (8.1), then f is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$.*

Proof. The first point is Corollary 8.6 and Section 5.4. The second point is Theorems 8.2 and 8.7. ☐

9.2. **Hypergeometric series.** In this section, we follow the notations of Section 4.2. We assume that $0 < |q| < 1$. Let us fix $n \geq 2$, let $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{Q}})^n$, $\underline{b} = (b_1, \dots, b_n) \in (q^{\mathbb{Q}} \setminus q^{-\mathbb{N}})^n$, $b_1 = q$, $\lambda \in \mathbb{C}^{\times}$.

Corollary 9.2. *Let us assume that (4.3) is irreducible and not q-Kummer induced. Then ${}_n\Phi_n(\underline{a}, \underline{b}, \lambda, q; z)$ is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$.*

Proof. Since $0 < |q| < 1$, the series ${}_n\Phi_n(\underline{a}, \underline{b}, \lambda, q; z)$ is convergent. We use Remark 8.4, to deduce that ${}_n\Phi_n(\underline{a}, \underline{b}, \lambda, q; z) \in \mathcal{M}er(\mathbb{C}^{\times})$. The conclusion is a direct application of Theorem 4.3 and Corollary 9.1. ☐

We follow the notations of Section 4.3. We assume that $0 < |q| < 1$, $n > s$, $n \geq 2$. Let $\underline{a} = (a_1, \dots, a_n) \in (q^{\mathbb{R}})^n$, $\underline{b} = (b_1, \dots, b_s) \in (q^{\mathbb{R}} \setminus q^{-\mathbb{N}})^s$, $b_1 = q$, $\lambda \in \mathbb{C}^{\times}$, $0 < |q| < 1$ and consider (4.3).

Corollary 9.3. *For $(i, j) \in \{1, \dots, n\} \times \{1, \dots, s\}$, let $\alpha_i, \beta_j \in \mathbb{R}$ such that $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_j}$. Assume that for all $(i, j) \in \{1, \dots, n\} \times \{1, \dots, s\}$, $\alpha_i - \beta_j \notin \mathbb{Z}$, and that the algebraic group generated by $\mathrm{Diag}(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_n})$ is connected. Then, ${}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z)$ is $\sigma_{\mathbf{q}'}$-algebraically independent over $\mathbb{C}_E(z)$.*

Proof. Since $0 < |q| < 1$, the series ${}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z)$ is convergent. We use Remark 8.4, to deduce that ${}_n\Phi_s(\underline{a}, \underline{b}, \lambda, q; z) \in \mathcal{M}er(\mathbb{C}^{\times})$. The conclusion is a direct application of Theorem 4.5 and Corollary 9.1. ☐

Appendix A. Difference algebraic groups

Let $(\mathbf{k}, \sigma_{\mathbf{q}'})$ be a difference field. We denote by $\mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$ the category of $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebras and by Groups the category of groups.

Definition A.1. A $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hopf algebra R is a $\mathbf{k}$-Hopf-algebra, endowed with a structure of $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebra, whose structural maps are $\sigma_{\mathbf{q}'}$-morphism. A $\sigma_{\mathbf{q}'}$-Hopf ideal of R is a Hopf ideal, which is stable under the action of $\sigma_{\mathbf{q}'}$.

We define a $\sigma_{\mathbf{q}'}$-algebraic group scheme over $\mathbf{k}$ as follows.

Definition A.2. A functor H from the category $\mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$ to the category of Groups representable by a $\sigma_{\mathbf{q}'}$-finitely generated $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hopf algebra $\mathbf{k}\{H\}$ is called a $\sigma_{\mathbf{q}'}$-algebraic group scheme. A $\sigma_{\mathbf{q}'}$-subgroup scheme G of H is a subgroup functor of H. It corresponds to a $\sigma_{\mathbf{q}'}$-Hopf ideal $\mathfrak{I}_H$ of $\mathbf{k}\{G\}$ such that $\mathbf{k}\{H\}=\mathbf{k}\{G\}/\mathfrak{I}_H$.

Remark A.3. If $\sigma_{\mathbf{q}'}$ is the identity, we find the usual definition of algebraic group scheme over $\mathbf{k}$. We adopt the following conventions. If G is an algebraic group over $\mathbf{k}$, we denote by $\mathbf{k}[G]$ its associated Hopf algebra.

The theory of $\sigma_{\mathbf{q}'}$-algebraic schemes was initiated by M.Wibmer (see for instance [Wib15]). Many of the terminology for $\sigma_{\mathbf{q}'}$-algebraic schemes is borrowed from the usual terminology of schemes, by adding a straightforward compatibility with the difference operator $\sigma_{\mathbf{q}'}$. In order to avoid too many definitions, we choose to refer often to [DVHW14b]. However, one has to take care that the $\sigma_{\mathbf{q}'}$-geometry is more subtle, even in the affine case, than the algebraic geometry.

Example A.4. Localizing $\mathbf{k}\{X\}_{\sigma_{\mathbf{q}'}}$, the $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebra of polynomials in the $n\times n$-matrix X of $\sigma_{\mathbf{q}'}$-indeterminates, with respect to $\det(X)$, we find the $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hopf algebra $\mathbf{k}\{X,\frac{1}{\det(X)}\}_{\sigma_{\mathbf{q}'}}$,that corresponds to the $\sigma_{\mathbf{q}'}$-algebraic group scheme attached to the general linear group scheme $\mathrm{GL}_{n,\mathbf{k}}$.

The following proposition shows the connection between algebraic schemes over $\mathbf{k}$ and $\sigma_{\mathbf{q}'}$-schemes.

Proposition A.5 (§A.4 and §A.5 in [DVHW14b])**.** *Let G be an algebraic group scheme over $\mathbf{k}$ represented by the finitely generated $\mathbf{k}$-Hopf algebra $\mathbf{k}[G]$. Let H be a $\sigma_{\mathbf{q}'}$-algebraic group scheme represented by the $\sigma_{\mathbf{q}'}$-finitely generated $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hopf algebra $\mathbf{k}\{H\}$. The following holds.*

- *The group functor* $\begin{array}{llll}\mathbf{G}: & \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}} & \rightarrow & \mathrm{Sets}\\ & B & \mapsto & G(B^{\#})\end{array}$, *with $B^{\#}$ the underlying $\mathbf{k}$-algebra of B, is representable by a $\sigma_{\mathbf{q}'}$-finitely generated $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hopf algebra. We call $\mathbf{G}$ the $\sigma_{\mathbf{q}'}$-algebraic group scheme attached to G.*
- *We denote by $H^{\#}$ the functor* $\begin{array}{lll}\mathrm{Alg}_{\mathbf{k}} & \rightarrow & \mathrm{Sets}\\ B & \mapsto & \mathrm{Hom}_{\mathrm{Alg}_{\mathbf{k}}}(\mathbf{k}\{H\}^{\#},B)\end{array}$. *Then,*
$$\mathrm{Hom}(H^{\#},G)\simeq\mathrm{Hom}(H,\mathbf{G}).$$
- *Assume that H is a $\sigma_{\mathbf{q}'}$-subgroup scheme of G, i.e., H is a $\sigma_{\mathbf{q}'}$-subgroup scheme of $\mathbf{G}$. The smallest $\mathbf{k}$-group scheme $\overline{H}$ such that $H^{\#}\rightarrow G$ factorizes through $\overline{H}\rightarrow G$ is called the Zariski closure of H in G.*

Example A.6. Any $\sigma_{\mathbf{q}'}$-subgroup scheme H of $\mathrm{GL}_{n,\mathbf{k}}$ is entirely determined by a $\sigma_{\mathbf{q}'}$-Hopf ideal $\mathfrak{I}_H\subset\mathbf{k}\{X,\frac{1}{det(X)}\}_{\sigma_{\mathbf{q}'}}$. The Zariski closure of H in $\mathrm{GL}_{n,\mathbf{k}}$ is defined by the Hopf ideal $\mathfrak{I}_H\cap\mathbf{k}[X,\frac{1}{det(X)}]$.

Definition A.7. Let G be a $\sigma_{\mathbf{q}'}$-algebraic group scheme over $\mathbf{k}$ and let $\widetilde{\mathbf{k}}$ be a $\sigma_{\mathbf{q}'}$-field extension of $\mathbf{k}$. The base extension of G to $\widetilde{\mathbf{k}}$ is the functor $\begin{array}{lll}\mathrm{Alg}_{\widetilde{\mathbf{k}},\sigma_{\mathbf{q}'}} & \rightarrow & \mathrm{Sets}\\ B & \mapsto & G(B)\end{array}$, where B is viewed as $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-algebra. It is represented by the $\widetilde{\mathbf{k}}$-$\sigma_{\mathbf{q}'}$-Hopf algebra $\mathbf{k}\{G\}\otimes_{\mathbf{k}}\widetilde{\mathbf{k}}$.

This allows us to define the $\sigma_{\mathbf{q}'}$-analogue of the notion of irreducibility.

Definition A.8 (Definition 4.2 and Lemma A.13 in [DVHW14b])**.** Let G be a $\sigma_{\mathbf{q}'}$-algebraic scheme over $\mathbf{k}$. Let $\widetilde{\mathbf{k}}$ be an algebraically closed, inversive field extension of $\mathbf{k}$. We say that G is absolutely $\sigma_{\mathbf{q}'}$-integral if $\widetilde{\mathbf{k}}\{G\}$, the $\sigma_{\mathbf{q}'}$-Hopf algebra $\widetilde{\mathbf{k}}\{G\}$ of $G_{\widetilde{\mathbf{k}}}$ is a $\sigma_{\mathbf{q}'}$-domain, *i.e.*, $\widetilde{\mathbf{k}}\{G\}$ is an integral domain and $\sigma_{\mathbf{q}'}$ is injective on $\widetilde{\mathbf{k}}\{G\}$.

Lemma A.9. *Let G and H be absolutely $\sigma_{\mathbf{q}'}$-integral $\sigma_{\mathbf{q}'}$-group schemes over $\mathbf{k}$. Then, the product $G \times H$ is absolutely $\sigma_{\mathbf{q}'}$-integral.*

Proof. Since the product commutes with base extension, we can directly assume that $\mathbf{k}$ is inversive and algebraically closed. Thus $\mathbf{k}\{G\}$ and $\mathbf{k}\{H\}$ are $\sigma_{\mathbf{q}'}$-domains. This means for instance that $\mathbf{k}\{H\}$ can be embedded in a $\sigma_{\mathbf{q}'}$-field L. Then, $\mathbf{k}\{G\} \otimes_{\mathbf{k}} \mathbf{k}\{H\}$ embeds as $\sigma_{\mathbf{q}'}$-ring in $\mathbf{k}\{G\} \otimes_{\mathbf{k}} L$. Since $\mathbf{k}$ is inversive and algebraically closed, [DVHW14b, Lemma A.13] shows that $\mathbf{k}\{G\}$ is $\sigma_{\mathbf{q}'}$-regular, *i.e.*, $\mathbf{k}\{G\} \otimes_{\mathbf{k}} \mathbf{k}'$ is a $\sigma_{\mathbf{q}'}$-domain for all $\sigma_{\mathbf{q}'}$-field extension $\mathbf{k}'$ of $\mathbf{k}$. Thus $\mathbf{k}\{G\} \otimes L$ is a $\sigma_{\mathbf{q}'}$-domain and the same holds for $\mathbf{k}\{G\} \otimes \mathbf{k}\{H\} = \mathbf{k}\{G \times H\}$. This ends the proof. ☐

We would like to classify some $\sigma_{\mathbf{q}'}$-subgroup schemes of $\mathrm{GL}_{n,\mathbf{k}}$. First, we state a fundamental classification theorem, which is a $\sigma_{\mathbf{q}'}$-analogue of a result of P. Cassidy.

Theorem A.10 (Theorem A.25 in [DVHW14a])**.** *Let $\mathbf{k}$ be an algebraically closed, inversive $\sigma_{\mathbf{q}'}$-field of characteristic zero and let G be a $\sigma_{\mathbf{q}'}$-integral, $\sigma_{\mathbf{q}'}$-algebraic subgroup of $\mathrm{GL}_{n,\mathbf{k}}$. Assume that the Zariski closure of G in $\mathrm{GL}_{n,\mathbf{k}}$ is an absolutely almost simple algebraic group, properly containing G. Then there exist a $\sigma_{\mathbf{q}'}$-field extension $\widetilde{\mathbf{k}}$ of $\mathbf{k}$ and an integer $d \geq 1$ such that $G_{\widetilde{\mathbf{k}}}$ is conjugate to a $\sigma^d_{\mathbf{q}'}$-constant subgroup of $\mathrm{GL}_{n,\widetilde{\mathbf{k}}}$, i.e., there exists $P \in \mathrm{GL}_n(\widetilde{k})$ such that*

$$PGP^{-1}(S) \subset \{g \in \mathrm{GL}_{n,\widetilde{k}}(S) | \sigma^d_{\mathbf{q}'}(g) = g\}$$

for all $S \in \mathrm{Alg}_{\widetilde{k},\sigma_{\mathbf{q}'}}$.

We also shall have to consider derived group. In analogy with [Wat79, §10.1], we define the derived group of a $\sigma_{\mathbf{q}'}$-algebraic group scheme as follows.

Definition A.11. Let G be a $\sigma_{\mathbf{q}'}$-algebraic group scheme defined over $\mathbf{k}$ and let $\mathbf{k}\{G\}$ be its $\sigma_{\mathbf{q}'}$-Hopf algebra. For any $n \in \mathbb{N}$, we define a natural transformation ϕ_n from G^{2n} to G as follows. For all $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in G(S)^{2n}$, we set

$$\phi_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_n y_n x_n^{-1} y_n^{-1}.$$

Let $\psi_n : \mathbf{k}\{G\} \to \otimes^{2n}\mathbf{k}\{G\}$ be the corresponding dual map by Yoneda. We denote by $\mathfrak{I}_n$ its kernel. Let $\mathfrak{I}_{\mathcal{D}(G)} = \cap_{n\in\mathbb{N}} \mathfrak{I}_n$. Then $\mathfrak{I}_{\mathcal{D}(G)}$ is a $\sigma_{\mathbf{q}'}$-Hopf ideal of $\mathbf{k}\{G\}$ and we defined the derived group $\mathcal{D}(G)$ as the $\sigma_{\mathbf{q}'}$-algebraic subgroup of G represented by $\mathbf{k}\{G\}/\mathfrak{I}_{\mathcal{D}(G)}$.

Proof. Let Δ denote the co-multiplication map of $\mathbf{k}\{G\}$. Then, it is clear that $\Delta(\mathfrak{I}_{2n}) \subset \mathfrak{I}_n \otimes \mathfrak{I}_{\mathbf{n}}$ since multiplying two products of n commutators yields a product of $2n$ commutators. This shows that $\mathfrak{I}_{\mathcal{D}(G)}$ is an Hopf ideal. For all $n \in \mathbb{N}$, the map ψ_n is a $\sigma_{\mathbf{q}'}$-morphism so that $\mathfrak{I}_n$. This proves that $\mathfrak{I}_{\mathcal{D}(G)}$ is a $\sigma_{\mathbf{q}'}$-ideal. ☐

Remark A.12. If $\sigma_{\mathbf{q}'}$ is the identity, we retrieve the definition of the derived group scheme $\mathcal{D}(H)$ of an algebraic group scheme H over $\mathbf{k}$ as in [Wat79, §10.1].

Lemma A.13. *For any $\sigma_{\mathbf{q}'}$-algebraic group scheme G over $\mathbf{k}$ and any $\sigma_{\mathbf{q}'}$-field extension $\widetilde{\mathbf{k}}$ of $\mathbf{k}$, we have $\mathcal{D}(G_{\widetilde{\mathbf{k}}}) = \mathcal{D}(G)_{\widetilde{\mathbf{k}}}$*

Proof. The definition of $\mathfrak{I}_{\mathcal{D}(G)}$ commutes with base extension. ☐

Proposition A.14. *Let H be an algebraic group scheme over $\mathbf{k}$ and let $G \subset H$ be a Zariski dense $\sigma_{\mathbf{q}'}$-algebraic subgroup of H. Then, $\mathcal{D}(G)$ is a Zariski dense subgroup of $\mathcal{D}(H)$.*

Proof. Let $\mathbf{k}\{\mathbf{H}\}$ be the $\sigma_{\mathbf{q}'}$-Hopf algebra of the $\sigma_{\mathbf{q}'}$-algebraic group scheme $\mathbf{H}$ attached to H as in Proposition A.5. Then, $\mathbf{k}[H]$ is a sub-Hopf algebra of $\mathbf{k}\{\mathbf{H}\}$. This means, in the notation above, that $\psi_n : \mathbf{k}[H] \rightarrow \otimes^{2n}\mathbf{k}[H]$ is the restriction of $\psi_n : \mathbf{k}\{\mathbf{H}\} \rightarrow \otimes^{2n}\mathbf{k}\{\mathbf{H}\}$. Thus, if $I_H \subset \mathbf{k}[H]$ denotes the Hopf ideal of $\mathcal{D}(H)$ in H, then $\mathfrak{I}_{\mathcal{D}(\mathbf{H})} \cap \mathbf{k}[H] = I_{\mathcal{D}(H)}$. Since G is a $\sigma_{\mathbf{q}'}$-algebraic subgroup of $\mathbf{H}$, we find a surjective morphism $\pi : \mathbf{k}\{\mathbf{H}\} \rightarrow \mathbf{k}\{G\}$ of $\sigma_{\mathbf{q}'}$-Hopf algebras. Since the applications ψ_n are constructed using comultiplication and co-inverse and π is a surjective morphism of $\sigma_{\mathbf{q}'}$-Hopf algebras, we get that $\pi(\mathfrak{I}_{\mathcal{D}(\mathbf{H})}) = \mathfrak{I}_{\mathcal{D}(G)}$. Let $\tilde{\mathfrak{I}}_{\mathcal{D}(G)}$ be the defining ideal of $\mathcal{D}(G)$ in H, *i.e.*, $\pi^{-1}(\mathfrak{I}_{\mathcal{D}(G)})$. Let $\mathfrak{I}_G \subset \mathbf{k}\{\mathbf{H}\}$ be the defining ideal of G in $\mathbf{H}$, *i.e.*, the kernel of π. Then, $\tilde{\mathfrak{I}}_{\mathcal{D}(G)} = \mathfrak{I}_G + \mathfrak{I}_{\mathcal{D}(\mathbf{H})}$. Since the group G is Zariski dense in H, we have $\mathfrak{I}_G \cap \mathbf{k}[H] = \{0\}$. Then,

$$\tilde{\mathfrak{I}}_{\mathcal{D}(G)} \cap \mathbf{k}[H] = (\mathfrak{I}_G + \mathfrak{I}_{\mathcal{D}(\mathbf{H})}) \cap \mathbf{k}[H] = \mathfrak{I}_{\mathcal{D}(\mathbf{H})} \cap \mathbf{k}[H] = I_{\mathcal{D}(H)}.$$

This equality means precisely that $\mathcal{D}(G)$ is Zariski dense in $\mathcal{D}(H)$. □

Lemma A.15. *The derived group of an absolutely $\sigma_{\mathbf{q}'}$-integral $\sigma_{\mathbf{q}'}$-algebraic group scheme G over $\mathbf{k}$ is absolutely $\sigma_{\mathbf{q}'}$-integral.*

Proof. Since by Lemma A.13, the formation of the derived group commutes with base extension. We can assume that $\mathbf{k}$ is algebraically closed and inversive. Since the $\mathbf{k}$-$\sigma_{\mathbf{q}'}$-Hof algebra of $\mathcal{D}(G)$ is $\mathbf{k}\{G\}/\mathfrak{I}_{\mathcal{D}(G)}$, the group $\mathcal{D}(G)$ is absolutely $\sigma_{\mathbf{q}'}$-integral if and only if $\mathfrak{I}_{\mathcal{D}(G)}$ is $\sigma_{\mathbf{q}'}$-prime, *i.e.*, prime and such that $\sigma_{\mathbf{q}'}(a) \in \mathfrak{I}_{\mathcal{D}(G)}$ implies $a \in \mathfrak{I}_{\mathcal{D}(G)}$. By Lemma A.9, we find that for all $n \in \mathbb{N}$, the group G^{2n} is absolutely $\sigma_{\mathbf{q}'}$-integral. This means that $\mathbf{k}\{G^{2n}\}$ is a $\sigma_{\mathbf{q}'}$-domain for all $n \in \mathbb{N}$. Since $\mathfrak{I}_n$ is the kernel of the $\sigma_{\mathbf{q}'}$-morphism $\psi_n : \mathbf{k}\{G\} \rightarrow \mathbf{k}\{G^{2n}\}$, the ideal $\mathfrak{I}_n$ is $\sigma_{\mathbf{q}'}$-prime for all $n \in \mathbb{N}$. This implies that $\mathfrak{I}_{\mathcal{D}(G)}$ is $\sigma_{\mathbf{q}'}$-prime. □

Definition A.16. Let $(\mathbf{k}, \sigma_{\mathbf{q}'})$ be a $\sigma_{\mathbf{q}'}$-field and let $G \subset \mathrm{GL}_{n,\mathbf{k}}$ be an algebraic group scheme defined over $\mathbf{k}$. Let $d \in \mathbb{N}^{\times}$. We consider the $\sigma_{\mathbf{q}'}$-subgroup $G^{\sigma^d_{\mathbf{q}'}}$ of G defined by $G^{\sigma^d_{\mathbf{q}'}}(S) = \{g \in G(S) | \sigma^d_{\mathbf{q}'}(g) = g\}$. We say that G has a toric constant centralizer if, for any $d \in \mathbb{N}^{\times}$, for any $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$, the following holds: if $h \in \mathrm{GL}_{n,\mathbf{k}}(S)$ centralizes $G^{\sigma^d_{\mathbf{q}'}}(S)$ then $h = \lambda I_n$ for some $\lambda \in S^{\times}$.

Lemma A.17. *Let $(\mathbf{k}, \sigma_{\mathbf{q}'})$ be a $\sigma_{\mathbf{q}'}$-field and let $G \subset \mathrm{GL}_{n,\mathbf{k}}$ be an algebraic group scheme defined over $\mathbf{k}$. Assume that G has toric constant centralizer. Then, the normalizer H of $G^{\sigma^d_{\mathbf{q}'}}$ in $\mathrm{GL}_{n,\mathbf{k}}$ is a $\sigma_{\mathbf{q}'}$-algebraic group define over $\mathbf{k}$. Moreover, for all $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$ and $g \in H(S)$ there exists $\lambda_g \in S^{\times}$ such that $\sigma^d_{\mathbf{q}'}(g) = \lambda_g g$.*

Proof of Lemma A.17. The fact that the normalizer is a $\sigma_{\mathbf{q}'}$-algebraic group comes essentially from the representability of the normalizer by [DG70, II.13.6].

If g normalizes $G^{\sigma^d_{\mathbf{q}'}}(S)$, for some $d \in \mathbb{N}^{\times}$, then $\sigma^d_{\mathbf{q}'}(g)g^{-1}$ centralizes $G^{\sigma^d_{\mathbf{q}'}}(S)$. By assumption, we conclude that $\sigma^d_{\mathbf{q}'}(g)g^{-1}$ is a scalar matrix. □

Lemma A.18. *Let $(\mathbf{k}, \sigma_{\mathbf{q}'})$ be a $\sigma_{\mathbf{q}'}$-field. The algebraic groups $\mathrm{SL}_{n,\mathbf{k}}$ (when $n \geq 2$), $\mathrm{SO}_{n,\mathbf{k}}$ (when $n \geq 3$) and $\mathrm{Sp}_{n,\mathbf{k}}$ (when n is even) have toric constant centralizer.*

Proof. The algebraic groups $\mathrm{SL}_{n,\mathbf{k}}$ (when $n \geq 2$), $\mathrm{SO}_{n,\mathbf{k}}$ (when $n \geq 3$) and $\mathrm{Sp}_{n,\mathbf{k}}$ (when n is even) are absolutely almost simple algebraic group. Let $d \in \mathbb{N}^{\times}$ and let $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$.

Let us consider $\mathrm{SL}_{n,\mathbf{k}}$ with $n \geq 2$. Let $M \in \mathrm{GL}_{n,\mathbf{k}}(S)$ that centralizes $\mathrm{SL}_n^{\sigma^d_{\mathbf{q}'}}(S)$. For $i \neq j$, the matrices $X_{i,j} = I_n + E_{i,j}$, where $E_{i,j}$ are matrices with zeroes at every entry except 1 at line i and column j, belong to $\mathrm{SL}_{n,\mathbf{k}}^{\sigma^d_{\mathbf{q}'}}(S)$ for all $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q}'}}$.

Consequently, for all $i \neq j$, $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q'}}}$, $MX_{i,j} = X_{i,j}M$. This shows that $M = \lambda I_n$ for some $\lambda \in S^{\times}$.

Let us consider $\mathrm{SO}_{n,\mathbf{k}}$ with $n \geq 3$. Let $M \in \mathrm{GL}_{n,\mathbf{k}}(S)$ that centralizes $\mathrm{SO}_n^{\sigma^d_{\mathbf{q'}}}(S)$. For all $1 \leq i < j \leq n$, $S \in \mathrm{Alg}_{\mathbf{k},\sigma_{\mathbf{q'}}}$, $MN_{i,j} = N_{i,j}M$, where $N_{i,j}$ is the diagonal matrix with 1 entry, except the diagonal entries i and j that are equal to -1. It follows that M is diagonal. To conclude that $M = \lambda I_n$ for some $\lambda \in S^{\times}$, we consider the commutation with $P_i = \mathrm{Diag}(\mathrm{I}_i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{I}_{n-i-2})$, $i \leq n-2$.

Let us consider $\mathrm{Sp}_{n,\mathbf{k}}$ with n even. Let $M \in \mathrm{GL}_{n,\mathbf{k}}(S)$ that centralizes $\mathrm{Sp}_n^{\sigma^d_{\mathbf{q'}}}(S)$. For all $N \in \mathrm{SL}_{n/2,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}(S)$, $\mathrm{Diag}(N, (N^{-1})^t) \in \mathrm{Sp}_{n,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}(S)$. Then, for all $N \in \mathrm{SL}_{n/2,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}(S)$, we have $M\mathrm{Diag}(N, (N^{-1})^t) = \mathrm{Diag}(N, (N^{-1})^t)M$. Let $M = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}$, $M_{i,j}$ are $n/2$ times $n/2$ matrices. From the commutation relation we obtain $M_{1,1}N = NM_{1,1}$. Using the fact that $\mathrm{SL}_{n/2,\mathbf{k}}$ has toric constant centralizer, we conclude that $M_{1,1} = \lambda I_{n/2}$ for some $\lambda \in S^{\times}$. Similarly, we find that $M_{2,2} = \mu I_{n/2}$ for some $\mu \in S^{\times}$. Then, $MN = NM$ with $N = \begin{pmatrix} \mathrm{I}_{n/2} & \mathrm{I}_{n/2} \\ 0 & \mathrm{I}_{n/2} \end{pmatrix} \in \mathrm{Sp}_{n,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}(S)$. We obtain $M_{2,1} = 0$. Similarly with $N = \begin{pmatrix} \mathrm{I}_{n/2} & 0 \\ \mathrm{I}_{n/2} & \mathrm{I}_{n/2} \end{pmatrix} \in \mathrm{Sp}_{n,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}(S)$, we obtain $M_{1,2} = 0$. Finally, with the commutation of M with $N = \begin{pmatrix} 0 & \mathrm{I}_{n/2} \\ -\mathrm{I}_{n/2} & 0 \end{pmatrix} \in \mathrm{Sp}_{n,\mathbf{k}}^{\sigma^d_{\mathbf{q'}}}$, we find $M = \lambda I_n$ for some $\lambda \in S^{\times}$. □

Appendix B. Convergent power series solution of q-difference equations

Let $\mathbf{K} = \mathbb{C}(\{z\})$ be the field of fraction of the ring of convergent power series $\mathbb{C}\{z\}$. Let us denote by C_E the field of elliptic functions in $\mathcal{M}er(\mathbb{C}^{\times})$ and let $\mathbb{C}_E = \{f \in \mathcal{M}er(\mathbb{C}^{\times}) | \sigma_q^r(f) = f \text{ for some } r \in \mathbb{N}\}$. Let $\mathbf{q}' \in \mathbb{C}^{\times}$ be multiplicatively independent from q and let $\sigma_{\mathbf{q'}} : \mathcal{M}er(\mathbb{C}^{\times}) \to \mathcal{M}er(\mathbb{C}^{\times}), f(z) \mapsto f(\mathbf{q}'z)$.

Let $A \in \mathrm{GL}_n(\mathbb{C}(z))$. In [Sau04], the author attaches to a q-difference system $\sigma_q(Y) = AY$, a Newton polygon $N(A)$. The slopes of the non-vertical half-lines defining the border of $N(A)$ are called the slopes of the Newton polygon and ranked in decreasing order as follows $S(A) = \{\mu_1 > \mu_2 \cdots > \mu_r\} \subset \mathbb{Q}$. The Newton polygon and the slopes of the q-difference system are invariant under gage transforms, *i.e.*, $S(A) = S(\sigma_q(P)AP^{-1})$ and $N(A) = N(\sigma_q(P)AP^{-1})$ for any $P \in \mathrm{GL}_n(\mathbf{K})$. The slopes induces a filtration of the q-difference module associated to the q-difference system $\sigma_q(Y) = AY$. One has the following proposition:

Proposition B.1 (§3.3.2, [RSZ13])**.** *Let us consider $A \in \mathrm{GL}_n(\mathbb{C}(z))$ and let $S(A) = \{\mu_1 > \mu_2 > \cdots > \mu_r\}$ be its set of slopes. Assume that the slopes of A are in $\mathbb{Z}$. Then, there exists $P \in \mathrm{GL}_n(\mathbf{K})$, $A_1, \dots A_r$ some invertible matrices* *with complex entries and $U_{i,j}$ some matrices with entries in $\mathbf{K}$ such that*

$$\sigma_q(P)AP^{-1} = \begin{pmatrix} z^{-\mu_1}A_1 & & \dots\dots & \dots & \dots & \dots & U_{1,r} \\ 0 & \ddots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \ddots & z^{-\mu_i}A_i & \dots & U_{i,j} & \dots & \vdots \\ \vdots & \dots & 0 & \ddots & \vdots & \dots & \vdots \\ \vdots & \dots & \dots & \ddots & z^{-\mu_j}A_j & \dots & \vdots \\ \vdots & \dots & \dots & \dots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & z^{-\mu_r}A_r \end{pmatrix}.$$

Lemma B.2. *Let $A \in \mathrm{GL}_n(\mathbb{C}(z))$. Let l be a minimal positive integer, such that $S(A) \subset \mathbb{Z}/l = \{a/l, a \in \mathbb{Z}\}$. Then, the following hold*

(1) *there exist a non zero integer r and a complex number $c \in \mathbb{C}^\times$ such that the system $\sigma_q(Y) = cz^r \sigma_q^{l-1}(A)\dots AY$ has a non zero vector solution $Y \in \mathbf{K}^n$;*
(2) *there exist a non zero rational integer r and a complex number $c \in \mathbb{C}^\times$ such that the system $\sigma_q(Y) = cz^r AY$ has a non zero vector solution in $\mathbb{C}((z^{1/l}))$.*

Proof. Let us begin by proving the first part of the lemma in the particular case $l = 1$. We know, by Proposition B.1, one can find $P \in \mathrm{GL}_n(\mathbf{K})$ and $A_1, \dots A_r$ some invertible constant matrices such that

$$\text{(B.1)} \quad \sigma_q(P)AP^{-1} = \begin{pmatrix} z^{-\mu_1}A_1 & & \dots\dots & \dots & \dots & \dots & U_{1,r} \\ 0 & \ddots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \ddots & z^{-\mu_i}A_i & \dots & U_{i,j} & \dots & \vdots \\ \vdots & \dots & 0 & \ddots & \vdots & \dots & \vdots \\ \vdots & \dots & \dots & \ddots & z^{-\mu_j}A_j & \dots & \vdots \\ \vdots & \dots & \dots & \dots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & z^{-\mu_r}A_r \end{pmatrix}.$$

One can also assume, up to multiply P by a constant matrix, that A_1 is upper triangular. We let $d \in \mathbb{C}^\times$ be the entry on the first row and line of A_1. An easy computation shows that the vector $Z_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution of the system $\sigma_q(Z) = \frac{z^{\mu_1}}{d}\sigma_q(P)AP^{-1}Z$. Then, the vector $Y_0 = P^{-1}Z_0 \in \mathbf{K}^n$ is a non zero solution of the system $\sigma_q(Y) = \frac{z^{\mu_1}}{d}AY$. Moreover, one can show, using the fact that $\sigma_q(Y_0) = \frac{z^{\mu_1}}{d}AY_0$ that the vector Y_0 defines a meromorphic function on $\mathbb{C}^\times$. This proves the result when $l = 1$. Let us prove the first part of the lemma in the general case. An easy computation shows that the slopes of $\sigma_q^{l-1}(A)\dots A$ are $\{l\mu_1 > l\mu_2 \dots > l\mu_r\}$ and thus are in $\mathbb{Z}$. The case $l = 1$ allows to conclude.

The second part of the lemma can be deduced from the first as follows. Le t $t = z^{1/l}$ and let q_l be a l-th root of q. We endow $\mathbb{C}(t)$ with a structure of σ_{q_l} field by sending t on $q_l t$. We can consider the q_l-difference system $\sigma_{q_l}(Z) = AZ$ over $\mathbb{C}(t)$. Its set of slopes is precisely $\{l\mu_1 > l\mu_2 \dots > l\mu_r\}$. By the above, there exist a non zero integer r and a complex number $c \in \mathbb{C}^\times$ such that the system $\sigma_{q_l}(Z) = ct^r AZ$ has a solution vector $Z_0 \in \mathbb{C}((t))^n$. Then, $Z_0 \in \mathbb{C}((z^{1/l}))^n$ is a vector solution of $\sigma_q(Y) = cz^{r/l}AY$. $\square$

Lemma B.3. *Let $A \in \mathrm{GL}_n(\mathbb{C}_E(z))$ and let l be a non negative integer. Set $A[l] = \sigma_q^{l-1}(A)\sigma_q^{l-2}(A)\cdots A$. Let $R \subset \mathcal{M}er(\mathbb{C}^\times)$ be the $\sigma_{\mathbf{q}'}$-PV extension for $\sigma_q(Y) = AY$ over $\mathbb{C}_E(z)$ and let G be the $\sigma_{\mathbf{q}'}$-Galois group of R over $\mathbb{C}_E(z)$. The following hold*

- *R is a $\sigma_{\mathbf{q}'}$-PV extension for $\sigma_q^l(Y) = A[l]Y$ over the $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-field $\mathbb{C}_E(z)$;*
- *the $\sigma_{\mathbf{q}'}$-Galois group of R over $\mathbb{C}_E(z)$ for the q^l-difference system $\sigma_q^l(Y) = A[l]Y$ coincides with the base extension from C_E to C_{E_l} of the $\sigma_{\mathbf{q}'}$-Galois group of R over $\mathbb{C}_E(z)$ for the q-difference system $\sigma_q(Y) = AY$.*

Proof. For the first assertion, let $U \in \mathrm{GL}_n(R)$ be a fundamental solution matrix for $\sigma_q(Y) = AY$. Then, U is also a fundamental solution matrix for the q^l-difference system $\sigma_{q^l}(Y) = A[l]Y$. Moreover, since

$$C_{E_l} = \mathbb{C}_E(z)^{\sigma_{q^l}} \subset R^{\sigma_{q^l}} \subset C_{E_l} = \mathcal{M}er(\mathbb{C}^\times))^{\sigma_{q^l}}.$$

This proves that R is also a $\sigma_{\mathbf{q}'}$-PV extension for $\sigma_q^l(Y) = A[l]Y$ over the $(\sigma_{q^l}, \sigma_{\mathbf{q}'})$-field $\mathbb{C}_E(z)$.

For the second assertion, let G_l be the C_{E_l}-$\sigma_{\mathbf{q}'}$-group scheme defined by

$$\begin{array}{lcl} \mathrm{Alg}_{C_{E_l}, \sigma_{\mathbf{q}'}} & \rightarrow & \text{Groups}, \\ B & \mapsto & \mathrm{Aut}^{\sigma_{q^l}, \sigma_{\mathbf{q}'}}(R \otimes_{C_{E_l}} B / \mathbb{C}_E(z) \otimes_{C_{E_l}} B). \end{array}$$

By [OW15, Lemma 2.49 and 2.51], this functor is represented by the C_{E_l}-$\sigma_{\mathbf{q}'}$-Hopf algebra $C_{E_l}\{G_l\} = (R \otimes_{\mathbb{C}_E(z)} R)^{\sigma_{q^l}}$. Moreover, $C_{E_l}\{G_l\} = C_{E_l}\{Z, \frac{1}{\det(Z)}\}_{\sigma_{\mathbf{q}'}}$ with $Z = (U \otimes 1)^{-1}(1 \otimes U) \in \mathrm{GL}_n(R \otimes_{\mathbb{C}_E(z)} R)$. The same arguments show that G is represented by the C_E-$\sigma_{\mathbf{q}'}$-Hopf algebra $C_E\{G\} = (R \otimes_{\mathbb{C}_E(z)} R)^{\sigma_q}$. Moreover, $C_E\{G\} = C_E\{Z, \frac{1}{\det(Z)}\}_{\sigma_{\mathbf{q}'}}$ with $Z = (U \otimes 1)^{-1}(1 \otimes U) \in \mathrm{GL}_n(R \otimes_{\mathbb{C}_E(z)} R)$. This proves that $C_E\{G\} \subset C_{E_l}\{G_l\}$ and that $C_E\{G\} \otimes_{C_E} C_{E_l} = C_{E_l}\{G_l\}$. This last equality means that G_l is obtained from G by base change from C_E to C_{E_l}. $\square$

References

[BB92] Jean-Paul Bézivin and Abdelbaki Boutabaa. Sur les équations fonctionelles p-adiques aux q-différences. *Collect. Math.*, 43(2):125–140, 1992.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2003. Translated from the 1981 French edition by P. M. Cohn and J. Howie, Reprint of the 1990 English edition [Springer, Berlin; MR1080964 (91h:00003)].

[Cas72] Phyllis Joan Cassidy. Differential algebraic groups. *Amer. J. Math.*, 94:891–954, 1972.

[CHS08] Zoé Chatzidakis, Charlotte Hardouin, and Michael F. Singer. On the definitions of difference Galois groups. In *Model theory with applications to algebra and analysis. Vol. 1*, volume 349 of *London Math. Soc. Lecture Note Ser.*, pages 73–109. Cambridge Univ. Press, Cambridge, 2008.

[Coh65] Richard M. Cohn. *Difference algebra.* Interscience Publishers John Wiley & Sons, New York-London-Sydeny, 1965.

[CS07] Phyllis Joan Cassidy and Michael F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. In *Differential equations and quantum groups*, volume 9 of *IRMA Lect. Math. Theor. Phys.*, pages 113–155. Eur. Math. Soc., Zürich, 2007.

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique. Généralités. Groupes commutatifs. Avec un appendice: Corps de classes local, par Michiel Hazewinkel.* 1970.

[DHR15] Thomas Dreyfus, Charlotte Hardouin, and Julien Roques. Hypertranscendance of solutions of mahler equations. 2015. To appear in J. Eur. Math. Soc. (JEMS).

[DR15] Thomas Dreyfus and Julien Roques. Galois groups of difference equations of order two on elliptic curves. *SIGMA, Symmetry Integrability Geom. Methods Appl.*, 11:paper 003, 23, 2015.

[DVHW14a] Lucia Di Vizio, Charlotte Hardouin, and Michael Wibmer. Difference algebraic relations among solutions of linear differential equations. 2014.

[DVHW14b] Lucia Di Vizio, Charlotte Hardouin, and Michael Wibmer. Difference Galois theory of linear differential equations. *Adv. Math.*, 260:1–58, 2014.

[HS08] Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.

[OW15] Alexey Ovchinnikov and Michael Wibmer. σ-Galois theory of linear difference equations. *Int. Math. Res. Not. IMRN*, (12):3962–4018, 2015.

[Pra86] Cornelis Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *J. Reine Angew. Math.*, 369:101–109, 1986.

[Ram92] Jean-Pierre Ramis. About the growth of entire functions solutions of linear algebraic q-difference equations. *Ann. Fac. Sci. Toulouse Math. (6)*, 1(1):53–94, 1992.

[Roq08] Julien Roques. Galois groups of the basic hypergeometric equations. *Pacific J. Math.*, 235(2):303–322, 2008.

[Roq11] Julien Roques. Generalized basic hypergeometric equations. *Invent. Math.*, 184(3):499–528, 2011.

[Roq12] Julien Roques. On classical irregular q-difference equations. *Compos. Math.*, 148(5):1624–1644, 2012.

[Roq15] Julien Roques. Algebraic relations between Mahler functions. *Preprint available on https://www-fourier.ujf-grenoble.fr/ jroques/mahler.pdf*, 2015.

[RSZ13] Jean-Pierre Ramis, Jacques Sauloy, and Changgui Zhang. Local analytic classification of q-difference equations. *Astérisque*, (355):vi+151, 2013.

[Sau04] Jacques Sauloy. La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. *Ann. Inst. Fourier (Grenoble)*, 54(1):181–210, 2004.

[Sti09] Henning Stichtenoth. *Algebraic function fields and codes*, volume 254 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, second edition, 2009.

[vdPS97] Marius van der Put and Michael F. Singer. *Galois theory of difference equations*, volume 1666 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

[Wib15] Michael Wibmer. Affine difference algebraic groups. 2015.

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