ON THE GALOIS GROUPS OF FAMILIES OF REGULAR SINGULAR DIFFERENCE SYSTEMS

by

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Abstract. — *We investigate the variation of the dimension of the Galois groups of families of regular singular difference systems using analytic tools.*

Contents

1. Introduction-Organization

1.1. Introduction-Main results. — In the whole paper, x will denote a *complex* variable and τ (resp. δ) will denote the difference operator acting on a function Y of the complex variable x by $\tau Y(x) = Y(x-1)$ (resp. $\delta Y(x) = (x-1)(Y(x) - Y(x-1))$).

Let us consider :

$$(\mathcal{S}_h) : \quad \tau Y = A_h Y, \ A_h \in \mathrm{GL}_n(\mathbb{C}(x,h))$$

a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$, Σ being a finite subset of $\mathbb{C}$, and let us denote by G_h the corresponding difference Galois groups over $\mathbb{C}(x)$ (see [**17**]).

In this paper we study the variation of $\dim G_h$ (dimension of the complex linear algebraic group G_h) with respect to h via an analytic approach.

Let us recall that $(\mathcal{S}_h)$ is Fuchsian if $A_h(\infty) = \mathrm{I}_n$, in which case we set $A_{h;\infty} = \lim_{x\to\infty}(x-1)(\mathrm{I}_n - A_h(x)) \in \mathrm{M}_n(\mathbb{C})$. It is moreover nonresonant if, for any pair (λ, μ) of distinct eigenvalues of $A_{h;\infty}$, we have $\lambda - \mu \notin \mathbb{Z}$. The system $(\mathcal{S}_h)$ is regular singular if there exists $F_h \in \mathrm{GL}_n(\mathbb{C}(x))$ such that the system defined by $F_h[A_h] := (F_h(x-1))^{-1}A_h(x)F_h(x)$ is Fuchsian. For details about these classical notions, we refer for instance to sections 1 to 4 of [**14**] and to chapter 9 of [**17**] and to the references therein.

The regular singular difference systems are classified by their Birkhoff connection matrices; this is in some sense similar to the classification of the regular singular differential systems by means of their monodromy representations. For any $h \in \mathbb{C} \setminus \Sigma$, we associate to $(\mathcal{S}_h)$ its Birkhoff connection matrix $P_h \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$ with $\mathbf{x} = e^{2\pi i x}$ (see section 2). These give rise, for any $h \in \mathbb{C}\setminus\Sigma$, to a family of Galoisian morphisms $\Lambda_h(a,b) := (P_h(a))^{-1}P_h(b) \in G_h$ parameterized by all $(a,b) \in \mathbb{C}^2$ such that $P_h(a)$ and $P_h(b)$ are defined and invertible (this was pointed out for the first time by P. Etingof in [**8**] in the case of regular q-difference systems and extended to regular singular (q-)difference systems by M. Van der Put and M. Singer in [**17**]; for a different, more "analytic", point of view in the q-difference case we refer to the work of J. Sauloy in [**16**]). These morphisms allow us to give a group-theoretic description of the Galois groups G_h -see Theorem 3.2 in section 3.1- (the fact that they generate Zariski-dense subgroups of the Galois groups is proved in [**17**]) and of their Lie algebras $\mathfrak{g}_h$ -see Theorem 3.3 in section 3.2-.

Using the above above-mentioned description of $\mathfrak{g}_h$, we prove, in section 5, under the hypotheses 1. to 3. stated in section 4.1, that :

Theorem. — *Let* $\kappa = \max_{h\in\mathbb{C}\setminus\Sigma}\dim(G_h)$*. Then* $\Theta = \{h \in \mathbb{C}\setminus\Sigma \mid \dim(G_h) = \kappa\}$ *is an open subset of* $\mathbb{C}\setminus\Sigma$ *with discrete complement.*

The following result is an immediate consequence of the above Theorem.

Corollary. — *Suppose that there exists* $h \in \mathbb{C}\setminus\Sigma$ *such that* $G_h = \mathrm{GL}_n(\mathbb{C})$*. Then for any* $h \in \mathbb{C}\setminus\Sigma$ *but, maybe, a discrete subset, we have* $G_h = \mathrm{GL}_n(\mathbb{C})$*.*

Note that, replacing x by $\frac{x}{h}$, we can make the parameter h be also the step of the equation. We leave the corresponding statements to the reader.

In a different context, the idea of an analytic approach for the study of the variation of Galois groups appears in the work of J. Sauloy in [**15, 16**] and is an essential motivation for A. Duval and the author's papers [**3, 5, 14, 6**];

see also L. Di Vizio and C. Zhang's paper [**7**]. The reader will find more informations about the algebraic meaning of the analytic theory of (q-)difference equations in the works of P. Etingof [**8**], of J.-P. Ramis and J. Sauloy [**12, 13**], of J. Sauloy [**16**] and of M. Singer and M. Van der Put [**17**]. Moreover, for problems and results related to the main subject of the present paper, we refer the reader to Y. André's paper [**1**]. Concerning parameterized q-difference equations, we also refer to section 5 of C. Hardouin and M. Singer's paper [**9**].

Acknowledgements. I am grateful to P. Etingof for communicating to me the proof of Theorem 3.3 and for valuable comments. I also thank A. Duval who generously gave me a copy of [**3**]. *To complete.*

1.2. Organization. — In section 2 we recall useful properties of the regular singular difference systems. In section 3 we give a group-theoretic description of the Galois group of a given regular singular difference system and of its Lie algebra in terms of a corresponding Birkhoff matrix. In section 4 we consider a family of regular singular difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$ and we study the dependence of corresponding Birkhoff matrices on the parameter h. In section 5 we prove our main theorem concerning the variation of the dimension of the Galois groups.

2. Regular singular systems : a reminder

2.1. Factorial series. — For the material presented in this section, we refer to section 2.1. of [**4**] and to [**11**].

A function a defined and holomorphic on some open subset of $\mathbb{C}$ containing some half-plane $\Pi_M^+ := \{x \in \mathbb{C} \mid \Re(x) > M\}$ is expandable into a factorial series on Π_M^+ if a admits an expansion, convergent on Π_M^+, of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[s]} = \frac{1}{x(x+1)\cdots(x+s-1)}.$$

When it exists, the factorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we also introduce the following notation :

$$x^{[s]} = x(x+1)\cdots(x+s-1).$$

The set of germs of holomorphic functions expandable into factorial series, denoted by $\mathcal{O}_{fact}$, is by definition the direct limit of the sets of holomorphic functions expandable into factorial series on the half plane Π_M^+ as M tends to $+\infty$ (in what follows, we will identify an element of the direct limit with one of its representatives). It is a subring of the ring of germs of holomorphic functions at $+\infty$ which is, by definition, the direct limit of the rings of functions holomorphic on the half plane Π_M^+ as M tends to $+\infty$. In particular $\mathcal{O}_{fact}$ is an integral domain; its field of fractions is denoted by $\mathcal{M}_{fact}$. The intersection of $\mathcal{O}_{fact}$ and $\mathcal{M}_{fact}$ with $\mathcal{M}(\mathbb{C})$, the field of meromorphic functions over $\mathbb{C}$, are respectively denoted by $\mathcal{O}_{fact}(\mathbb{C})$ and $\mathcal{M}_{fact}(\mathbb{C})$.

Replacing x by $-x$, we get the notion of function expandable into retrofactorial series. More explicitly, a function a holomorphic on some open subset of $\mathbb{C}$ containing some half-plane $\Pi_M^- := \{x \in \mathbb{C} \mid \Re(x) < M\}$ is expandable into a retrofactorial series on Π_M^- if a admits an expansion, convergent on Π_M^-, of the form :

$$\sum_{s=0}^{+\infty} a_s x^{-[-s]}$$

where, for all $s \in \mathbb{N}$,

$$x^{-[-s]} = \frac{1}{x(x-1)\cdots(x-s+1)}.$$

When it exists, the retrofactorial series expansion is unique. For later use, for all $s \in \mathbb{N}$, we introduce the notation :

$$x^{[-s]} = x(x-1)\cdots(x-s+1).$$

Moreover, we introduce the rings and fields of retrofactorial series $\mathcal{O}_{retrofact}$, $\mathcal{M}_{retrofact}$, $\mathcal{O}_{retrofact}(\mathbb{C})$ and $\mathcal{M}_{retrofact}(\mathbb{C})$ defined similarly to $\mathcal{O}_{fact}$, $\mathcal{M}_{fact}$, $\mathcal{O}_{fact}(\mathbb{C})$ and $\mathcal{M}_{fact}(\mathbb{C})$ respectively.

For instance, any function defined and analytic in a neighborhood of $\infty \in \mathbb{P}^1_{\mathbb{C}}$ is expandable into factorial series and retrofactorial series.

We will denote by $\widehat{\mathcal{O}}_{fact}$ the integral domain of formal factorial series and we denote by $\widehat{\mathcal{M}}_{fact}$ its field of fractions. The ring laws on $\widehat{\mathcal{O}}_{fact}$ are given, for all $a(x) = \sum_{s=0}^{+\infty} a_s x^{-[s]} \in \widehat{\mathcal{O}}_{fact}$ and $b(x) = \sum_{s=0}^{+\infty} b_s x^{-[s]} \in \widehat{\mathcal{O}}_{fact}$, by :

$$(a+b)(x) = \sum_{s=0}^{+\infty} (a_s + b_s) x^{-[s]} \tag{1}$$

and :

$$(ab)(x) = \sum_{s=0}^{+\infty} c_s x^{-[s]} \tag{2}$$

where :

$$c_0 = a_0 b_0 \text{ and, } \forall s \in \mathbb{N}^*, \quad c_s = a_0 b_s + a_s b_0 + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} a_j b_l$$

with :

$$\forall s \in \mathbb{N}^*, \quad J_s = \{(j,k,l) \mid j,l \geq 1,\ k \geq 0,\ j+k+l = s\} \tag{3}$$

and :

$$\forall (j,l) \in \mathbb{N}^* \times \mathbb{N}^*,\ \forall\, k \in \mathbb{N},\ \ c_{j,l}^{(k)} = \frac{(j+k-1)!(l+k-1)!}{k!(j-1)!(l-1)!}. \tag{4}$$

As above, replacing x by $-x$, we get the integral domain of formal retrofactorial series $\widehat{\mathcal{O}}_{retrofact}$; its field of fractions is denoted by $\widehat{\mathcal{M}}_{retrofact}$.

We can interpret any element A of $M_{n,m}(\mathcal{O}_{fact})$ or $M_{n,m}(\widehat{\mathcal{O}}_{fact})$ as a series $\sum_{s=0}^{+\infty} A_s x^{-[s]}$ with coefficients in $M_{n,m}(\mathbb{C})$. The above sum and product formulas (1) and (2) remain valid for factorial series with matricial coefficients, that is, for all $A(x) = \sum_{s=0}^{+\infty} A_s x^{-[s]} \in M_{n,m}(\widehat{\mathcal{O}}_{fact})$ and $B(x) = \sum_{s=0}^{+\infty} B_s x^{-[s]} \in M_{n,m}(\widehat{\mathcal{O}}_{fact})$, we have :

$$(A+B)(x) = \sum_{s=0}^{+\infty} (A_s + B_s) x^{-[s]}$$

and, for all $A(x) = \sum_{s=0}^{+\infty} A_s x^{-[s]} \in M_{n,p}(\widehat{\mathcal{O}}_{fact})$ and $B(x) = \sum_{s=0}^{+\infty} B_s x^{-[s]} \in M_{p,m}(\widehat{\mathcal{O}}_{fact})$:

$$(AB)(x) = \sum_{s=0}^{+\infty} C_s x^{-[s]} \tag{5}$$

where :

$$C_0 = A_0 B_0 \text{ and, } \forall s \in \mathbb{N}^*, \quad C_s = A_0 B_s + A_s B_0 + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} A_j B_l.$$

We denote by T_{fact} the natural injective map from $M_{n,m}(\mathcal{O}_{fact})$ to $M_{n,m}(\widehat{\mathcal{O}}_{fact})$ (it is well defined because, as noted above, the expansion into factorial series is unique).

If $m = n$, it is a monomorphism of rings.

Moreover, we define $\widehat{\delta} : M_{n,m}(\widehat{\mathcal{O}}_{fact}) \to M_{n,m}(\widehat{\mathcal{O}}_{fact})$ by $\widehat{\delta} A = \sum_{s\geq 1} -s A_s x^{-[s]}$. We denote by $\delta : M_{n,m}(\mathcal{O}_{fact}) \to M_{n,m}(\mathcal{O}_{fact})$ the map defined by $\delta A = (x-1)(A(x) - A(x-1))$. The following diagram is commutative :

$$(6)\qquad \begin{array}{ccc} M_{n,m}(\widehat{\mathcal{O}}_{fact}) & \xrightarrow{\widehat{\delta}} & M_{n,m}(\widehat{\mathcal{O}}_{fact}) \\ {\scriptstyle T_{fact}}\uparrow & & \uparrow{\scriptstyle T_{fact}} \\ M_{n,m}(\mathcal{O}_{fact}) & \xrightarrow{\delta} & M_{n,m}(\mathcal{O}_{fact}) \end{array}$$

For this reason we will simply denote $\widehat{\delta}$ by δ.

The following examples of expansions into factorial series will be used later in this article :

$$(7)\qquad \forall Y \in M_n(\mathbb{C}), \forall \lambda \in \mathbb{C}, \ \ \mathrm{I}_n - \frac{Y}{x-\lambda} = \mathrm{I}_n - Y \sum_{s=1}^{\infty} \lambda^{[s-1]} x^{-[s]},$$

$$(8)\qquad \forall Y \in M_n(\mathbb{C}), \ \ \left(\mathrm{I}_n - \frac{Y}{x-1}\right)^{-1} = \sum_{s=0}^{\infty} Y^{[s]} x^{-[s]}$$

where $Y^{[0]} = \mathrm{I}_n$ and, for $s \in \mathbb{N}^*$, $Y^{[s]} = Y(Y+\mathrm{I}_n)\cdots(Y+(s-1)\mathrm{I}_n)$.

We conclude this section with a remark on inverses of factorial series; we refer to Proposition 2.1 in [**4**] for a proof. Let us consider $(C,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $a(x) = 1 + \sum_{s=1}^{+\infty} a_s x^{-[s]} \in \mathcal{O}_{fact}$ such that, for all $s \in \mathbb{N}^*$, $|a_s| \leq C\lambda^{[s-1]}$. Then a is invertible in $\mathcal{O}_{fact}$ and if we set $a^{-1}(x) = 1 + \sum_{s=1}^{+\infty} b_s x^{-[s]}$, we have, for all $s \in \mathbb{N}^*$, $|b_s| \leq C(C+\lambda)^{[s-1]}$.

2.2. Fundamental systems of solutions and Birkhoff matrix. — We refer for instance to section 3 and section 4 of [**14**] and to the references therein for the material presented in this paragraph; see also chapter 9 of [**17**].

Denoting by Γ the classical Euler Gamma function, we set, for any $c \in \mathbb{C}$:

$$e_c^+(x) = \frac{\Gamma(x)}{\Gamma(x-c)}, \ \ l_c^{(k)}(x) = \frac{1}{k!} \frac{\partial^k}{\partial c'^k}_{|c'=c} e_{c'}^+(x).$$

Fix $A_\infty \in \mathrm{M}_n(\mathbb{C})$ and consider a Jordan decomposition of A_∞ of the form :

$$A_\infty = Q diag(c_1 \mathrm{I}_{\mu_1} + N_{\mu_1}, ..., c_m \mathrm{I}_{\mu_m} + N_{\mu_m}) Q^{-1}$$

with $Q \in \mathrm{GL}_n(\mathbb{C})$, $m \in \{1,...,n\}$, $c_1,...,c_m \in \mathbb{C}$, $\mu_1,...,\mu_m \in \{1,...,n\}$ such that $\mu_1+\cdots+\mu_m = n$ and, for all $i \in \{1,...,m\}$,

$$N_{\mu_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathrm{M}_{\mu_i}(\mathbb{C}).$$

We set :

$$e^+_{A_\infty} := Qdiag(e^+_{c_1\mathrm{I}_{\mu_1}+N_{\mu_1}},..,e^+_{c_m\mathrm{I}_{\mu_m}+N_{\mu_m}})Q^{-1}$$

with :

$$e^+_{cI_\mu+N_\mu} = \begin{pmatrix} l_c^{(0)} & l_c^{(1)} & \dots & l_c^{(\mu-1)} \\ 0 & l_c^{(0)} & \dots & l_c^{(\mu-2)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & l_c^{(0)} \end{pmatrix}.$$

It is easily seen that $e^+_{A_\infty}$ does not depend on the particular choice of the Jordan decomposition and that :

$$\tau e^+_{A_\infty} = (\mathrm{I}_n - \frac{1}{x-1}A_\infty)e^+_{A_\infty}$$

or, equivalently, that :

$$\delta e^+_{A_\infty} = A_\infty e^+_{A_\infty}.$$

Consider a difference system :

$$\tau Y = AY \tag{9}$$

with $A \in \mathrm{GL}_n(\mathbb{C}(z))$.

Definition 2.1. — The system (9) is Fuchsian if $A(\infty) = \mathrm{I}_n$, in which case we set $A_\infty = \lim_{x\to\infty}(x-1)(\mathrm{I}_n - A(x)) \in \mathrm{M}_n(\mathbb{C})$. It is moreover nonresonant if, for any pair (λ,μ) of distinct eigenvalues of A_∞, we have $\lambda-\mu \notin \mathbb{Z}$. The system (9) is regular singular if there exists $F \in \mathrm{GL}_n(\mathbb{C}(x))$ such that the system defined by $F[A] := (F(x-1))^{-1}A(x)F(x)$ is Fuchsian.

In case that (9) is Fuchsian and nonresonant, it admits a unique fundamental system of meromorphic solutions of the form $Y^+ = F^+e^+_{A_\infty}$ with $F^+ \in \mathrm{I}_n + \frac{1}{x}\mathrm{M}_n(\mathcal{O}_{fact}(\mathbb{C}))$ and where $A_\infty = \lim_{x\to\infty}(x-1)(\mathrm{I}_n - A(x)) \in \mathrm{M}_n(\mathbb{C})$; in other words, there exists a unique $F^+ \in \mathrm{I}_n + \frac{1}{x}\mathrm{M}_n(\mathcal{O}_{fact}(\mathbb{C}))$ such that :

$$\tau F^+(\mathrm{I}_n - \frac{1}{x-1}A_\infty) = AF^+. \tag{10}$$

In case that (9) is regular singular then one can prove that it admits a (nonunique) fundamental system of meromorphic solutions of the form $Y^+ = F^+e^+_{A_\infty}$ with $F^+ \in \mathrm{GL}_n(\mathcal{M}_{fact}(\mathbb{C}))$ and $A_\infty \in \mathrm{M}_n(\mathbb{C})$.

The fundamental system of solutions Y^+ is attached to $+\infty$ in the sense that it is build up from convergent factorial series which can be regarded as regular functions at $+\infty$.

Replacing x by $-x$, we obtain similar constructions at $-\infty$ involving convergent retrofactorial series which can be regarded as regular functions at $-\infty$.

So, in case that (9) is regular singular, we get two fundamental systems of solutions Y^- and Y^+. The corresponding Birkhoff matrix (also called connection matrix) is $P = (Y^+)^{-1}Y^-$ which belongs to $\mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$ with $\mathbf{x} = e^{2\pi i x}$ and satisfies $P(\pm i\infty) \in \mathrm{GL}_n(\mathbb{C})$. The Birkhoff matrix depends on some arbitrary choices but is "almost unique" and it classifies the regular singular difference systems up to rational equivalence. For details about the Birkhoff matrices we refer to section 4 of [**14**] and also to the references therein.

3. Galois groups and Birkhoff matrices

Let $\tau Y = AY$ be a regular singular difference system and let $P \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$ be a corresponding Birkhoff matrix (see section 2). It was recalled in section 1.1 that, for any $a, b \in \mathbb{C}$ not being a pole of P or a zero of $\det P$, $P(a)^{-1}P(b)$ belongs to the Galois group G of $\tau Y = AY$. In this section, we give a group-theoretic description of G and of its Lie algebra $\mathfrak{g}$ in terms of the Birkhoff matrix.

3.1. Description of the Galois groups. —

Lemma 3.1. — *Let G be an algebraic group, let X be an irreducible algebraic variety and let $f : X \to G$ be a morphism of algebraic varieties such that the neutral element of G belongs to $f(X)$. Then the abstract group generated by $f(X)$ is Zariski-closed.*

Proof. — This is a particular case of Proposition 7.5 in [**10**]. □

Theorem 3.2. — *Let us consider a regular singular difference system $\tau Y = AY$ and let P be a corresponding connection matrix. The Galois group G of $\tau Y = AY$ is generated, as an abstract group, by $\{P(a)^{-1}P(b) \mid a, b \in \mathbb{C} \setminus S\}$ where $S = \{x \in \mathbb{C} \mid x$ is a pole of P or $\det P(x) = 0\}$.*

Proof. — Let us introduce the map $\Lambda : (\mathbb{C} \setminus S) \times (\mathbb{C} \setminus S) \to G$, $(a, b) \mapsto P(a)^{-1}P(b)$. It is proved in chapter 9 of [**17**] that the algebraic group generated by $Y = \{\Lambda(a, b) \mid (a, b) \in (\mathbb{C} \setminus S) \times (\mathbb{C} \setminus S)\}$ is the whole Galois group G. So, it remains to prove that the abstract group generated by Y is Zariski-closed. The proof of this fact is analogous to that of Proposition 3.2 of [**8**]. Indeed, since P belongs to $GL_n(\mathbb{C}(\mathbf{x}))$, there exists an invertible $n \times n$ matrix $\widetilde{P}$ with coefficients in the field of rational functions on $\mathbb{P}^1_{\mathbb{C}}$ such that $P(x) = \widetilde{P}(\mathbf{x})$.

Let us denote by X the affine open subvariety of the projective variety $\mathbb{P}^1_{\mathbb{C}}$ defined by $X = \mathbb{P}^1_{\mathbb{C}} \setminus (\widetilde{S} \cup \{0, \infty\})$ where $\widetilde{S} = \{e^{2\pi i s} \mid s \in S\} = \{u \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\} \mid u \text{ is a pole of } \widetilde{P} \text{ or } \det \widetilde{P}(u) = 0\}$ and let us consider the morphism of algebraic varieties $\widetilde{\Lambda} : X \times X \to GL_n(\mathbb{C})$, $(u, v) \mapsto \widetilde{P}(u)^{-1}\widetilde{P}(v)$. Since the image of $\widetilde{\Lambda}$ coincides with Y, we have to prove that the abstract group generated by the image of $\widetilde{\Lambda}$ is Zariski-closed. This is indeed the case in virtue of Lemma 3.1. □

Note that any regular singular difference system has a connected Galois group. The above theorem was also proved by A. Duval in [**3**] in the regular case by using a different method.

3.2. Description of the Lie algebras of the Galois groups. —

***Theorem 3.3* (P. Etingof).** — *Let us consider a regular singular difference system $\tau Y = AY$ and let P be a corresponding connection matrix. The Lie algebra $\mathfrak{g}$ of the Galois group G of $\tau Y = AY$ is generated by $\{P(a)^{-1}P'(a) \mid a \in \mathbb{C} \setminus S\}$ where $S = \{x \in \mathbb{C} \mid x \text{ is a pole of } P \text{ or } \det P(x) = 0\}$.*

Proof. — Maintaining the notations introduced in the proof of theorem 3.2, we consider the analytic map $\Lambda : (\mathbb{C} \setminus S) \times (\mathbb{C} \setminus S) \to G$, $(a, b) \mapsto P(a)^{-1}P(b)$. Moreover, we introduce the analytic map $\lambda : \mathbb{C} \setminus S \to \mathfrak{g}$, $a \mapsto \partial_b \Lambda(a, a) = P(a)^{-1}P'(a)$ (the fact that λ takes its values in $\mathfrak{g}$ is a consequence of the facts that, for all $a \in \mathbb{C} \setminus S$, $\Lambda(a, \cdot)$ is an analytic map taking its values in G which evaluation at a is equal to I_n).

Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{\lambda(a) \mid a \in \mathbb{C} \setminus S\}$ and let H be the connected embedded Lie subgroup of G corresponding to $\mathfrak{h}$.

Let us consider $b \in \mathbb{C} \setminus S$. We have, for all $a \in \mathbb{C} \setminus S$, $\partial_a \Lambda(a, b) = -P(a)^{-1}P'(a)P(a)^{-1}P(b) = -\lambda(a)\Lambda(a, b)$, hence $\Lambda(\cdot, b)$ is solution of the Cauchy problem $Y' = -\lambda Y$, $Y(b) = \mathrm{I}_n$. Since, by definition, $-\lambda$ takes its values in $\mathfrak{h}$, the above Cauchy problem can be integrated by functions with values in H. By uniqueness of the solution with values in G of the above Cauchy problem, we get in particular that $\Lambda(\cdot, b)$ takes its values in H. Hence H contains $\{\Lambda(a, b) \mid (a, b) \in (\mathbb{C} \setminus S) \times (\mathbb{C} \setminus S)\}$. Theorem 3.2 implies that $H = G$, so $\mathfrak{h} = \mathfrak{g}$. □

4. Parameterized equations

4.1. Hypotheses. — Let Σ be a finite subset of $\mathbb{C}$. In this section we consider $h \in \mathbb{C} \setminus \Sigma$ as a parameter.

Let us consider a family of difference systems :

$$\tau Y = A_h Y, \quad A_h \in \mathrm{GL}_n(\mathbb{C}(x,h)). \tag{11}$$

We denote by D_+ (resp. D_-) a denominator in $\mathbb{C}[x,h]$ for A_h (resp. A_h^{-1}) and by $S_+ = \{(x,h) \in \mathbb{C} \times (\mathbb{C} \setminus \Sigma) \mid D_+(x,h) = 0\}$ (resp. $S_- = \{(x,h) \in \mathbb{C} \times (\mathbb{C} \setminus \Sigma) \mid D_-(x,h) = 0\}$) the corresponding singular locus. We set $S = S_- \cup S_+ = \{(x,h) \in \mathbb{C} \times (\mathbb{C} \setminus \Sigma) \mid D(x,h) = 0\}$ with $D = D_+ D_- \in \mathbb{C}[x,h]$.

Assumption 1. — *We assume that :*

1. *For any compact subset K of $\mathbb{C} \setminus \Sigma$, $(\mathbb{C} \times K) \cap S$ is compact (or, equivalently, bounded);*
2. *a. for all $h \in \mathbb{C} \setminus \Sigma$, $A_h(\infty) = \mathrm{I}_n$, that is, the system $\tau Y = A_h Y$ is Fuchsian;*
   b. the matrices $A_{h;\infty} := \lim_{x \to \infty}(x-1)(\mathrm{I}_n - A_h(x)) \in \mathrm{M}_n(\mathbb{C})$ have Jordan reductions (ie. Jordan normal forms and base change matrices) depending analytically on $h \in \mathbb{C} \setminus \Sigma$ and they are nonresonant in the sense that for any pair of eigenvalues λ, μ we have $\lambda - \mu \notin \mathbb{Z}^*$;
3. *S does not contain any line $\mathbb{C} \times \{h\} \subset \mathbb{C} \times (\mathbb{C} \setminus \Sigma)$.*

Let us make some remarks about these hypotheses.

- Condition 1. ensures that $(x,h) \mapsto A_h(x)$ and $(x,h) \mapsto A_h(x)^{-1}$ are analytic in a neighborhood of any point of $\{\infty\} \times (\mathbb{C} \setminus \Sigma) \subset \mathbb{P}^1_{\mathbb{C}} \times (\mathbb{C} \setminus \Sigma)$.
- Conditions 1. to 3. imply that, for any $h \in \mathbb{C} \setminus \Sigma$, $\tau Y = A_h Y$ is a nonresonant Fuchsian difference system.
- ***Definition 4.1* (Properties $\mathcal{P}^+$ and $\mathcal{P}^-$)**. — We will say that a family $(U_h = \sum_{s \geq 0} U_{h;s} x^{-[s]})_{h \in \mathbb{C} \setminus \Sigma}$ of elements of $M_{n,m}(\widehat{\mathcal{O}}_{fact})$ satisfies the property $\mathcal{P}^+$ if the following conditions hold :
  - for all $s \in \mathbb{N}$, $U_{h;s}$ depends analytically on $h \in \mathbb{C} \setminus \Sigma$;
  - for any compact subset K of $\mathbb{C} \setminus \Sigma$, there exists $(C_K, \lambda_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that, for all $s \in \mathbb{N}$, for all $h \in K$, $||U_{h;s}|| \leq C_K \lambda_K^{[s-1]}$.

  We also introduce the similar property $\mathcal{P}^-$ for families of retrofactorial series.

  In what precedes, $|| \cdot ||$ denotes a norm on $M_{n,m}(\mathbb{C})$. Since two norms over a finite dimensional $\mathbb{C}$-vector space are equivalent, the properties $\mathcal{P}^+$ and $\mathcal{P}^-$ do not depend on the norm $|| \cdot ||$. In what follows, if $n = m$, we will assume that $|| \cdot ||$ is submultiplicative.

  Note that $\tau Y = A_h Y$ is equivalent to :

$$\delta Y = B_h Y$$

with :

$$B_h = (x-1)(\mathrm{I}_n - A_h).$$

As noted at the end of section 2.1, for all $h \in \mathbb{C} \setminus \Sigma$, the analycity of B_h in the neighborhood of $x = \infty$ ensures that B_h is expandable into factorial series in some right half-plane :

$$B_h(x) = \sum_{s=0}^{+\infty} B_{h;s} x^{-[s]}.$$

Remark that :

$$B_{h;0} = A_{h;\infty}.$$

Condition 1. also entails the following.

Lemma 4.2. — *The family $(B_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the properties $\mathcal{P}^+$ and $\mathcal{P}^-$.*

Proof. — For all $h \in \mathbb{C} \setminus \Sigma$, we consider the Taylor series expansion of B_h in the neighborhood of $x = \infty$:

$$B_h(x) = \sum_{k=0}^{+\infty} \widetilde{B}_{h;k} x^{-k}.$$

In order to prove the lemma, we first relate the Taylor expansion of B_h with its factorial series expansion.

Let $(\psi_{k,s})_{(k,s) \in \mathbb{N} \times \mathbb{N}^*}$ be the sequence of real numbers such that, for all $s \in \mathbb{N}^*$:

$$\sum_{k=0}^{+\infty} \psi_{k,s} X^k = X^{[s-1]} (= X(X+1)\cdots(X+s-2)).$$

We emphasize that, for all $k \in \mathbb{N}$, for all $s \in \mathbb{N}^*$, $\psi_{k,s} \geq 0$ and, for all $s \in \mathbb{N}^*$, for all $k \geq s$, $\psi_{k,s} = 0$. We have :

$$\begin{aligned} \sum_{k=0}^{+\infty} y^k x^{-(k+1)} &= \frac{1}{x-y} \\ &= \sum_{s=1}^{+\infty} y^{[s-1]} x^{-[s]} \\ &= \sum_{s=1}^{+\infty} \left(\sum_{k=0}^{+\infty} \psi_{k,s} y^k\right) x^{-[s]}. \end{aligned}$$

So, by identification, we get, for all $k \in \mathbb{N}^*$:

$$x^{-k} = \sum_{s=1}^{+\infty} \psi_{k-1,s} x^{-[s]}$$

hence, for all $s \in \mathbb{N}$:

$$\begin{aligned} B_{h;s} &= \sum_{k=1}^{+\infty} \psi_{k-1,s} \widetilde{B}_{h;k} \\ &= \sum_{k=1}^{s+1} \psi_{k-1,s} \widetilde{B}_{h;k} \end{aligned}$$

with $\psi_{0,0} = 1$ and, for all $k \in \mathbb{N}^*$, $\psi_{k,0} = 0$. This formula implies that, for all $s \in \mathbb{N}^*$, $B_{h;s}$ depends analytically on h. Moreover, let K be a compact subset of $\mathbb{C} \setminus \Sigma$. Since $(x, h) \mapsto B_h(x)$ is analytic on a neighborhood of $\{\infty\} \times K$, there exists $(C'_K, \lambda'_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$, such that, for all $h \in K$, for all $k \in \mathbb{N}$:

$$\|\widetilde{B}_{h;k}\| \leq C'_K {\lambda'_K}^k.$$

Since, for all $k \in \mathbb{N}$, for all $s \in \mathbb{N}$, $\psi_{k,s} \geq 0$, we get, for all $h \in K$, for all $s \in \mathbb{N}^*$:

$$\|B_{h;s}\| = \|\sum_{k=1}^{+\infty} \psi_{k-1,s} \widetilde{B}_{h;k}\| \leq C'_K \sum_{k=1}^{+\infty} \psi_{k-1,s} {\lambda'_K}^k = (C'_K \lambda'_K) {\lambda'_K}^{[s-1]}.$$

Hence, we have proved that $(B_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the property $\mathcal{P}^+$. The proof of the fact that $(B_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the property $\mathcal{P}^-$ is similar. □

- We can always assume that condition 3. holds up to enlarging Σ.

4.2. Birkhoff matrices of parameterized equations. — We maintain the notations and the hypotheses formulated in the previous subsection.

For any $h \in \mathbb{C} \setminus \Sigma$, we denote by :

$$Y_h^+ = F_h^+ e_{A_{h;\infty}}^+ \text{ and } Y_h^- = F_h^- e_{A_{h;\infty}}^-$$

the solutions and by :

$$P_h = (Y_h^+)^{-1} Y_h^- \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$$

the Birkhoff matrix built in section 2.2 for the Fuchsian and nonresonant difference system (11). We recall (see section 2.2) that F_h^+ is the unique element of $\mathrm{I}_n + \frac{1}{x}\mathrm{M}_n(\mathcal{O}_{fact}(\mathbb{C}))$ such that :

$$\tau F_h^+ (\mathrm{I}_n - \frac{1}{x-1} A_{h;\infty}) = A_h F_h^+. \tag{12}$$

Similarly for F_h^-. The aim of this subsection is to study the dependence of $P_h(x)$ on x and h; the difficulty is to study the dependence of $F_h^+(x)$ and $F_h^-(x)$ on x and h. This is the purpose of the following proposition (see [**4, 5, 14, 6**] for variants).

Proposition 4.3. — *The family of gauge transformations* $(F_h^+ = \sum_{s=0}^{+\infty} F_{h;s}^+ x^{-[s]})_{h\in\mathbb{C}\setminus\Sigma}$ *satisfies the property* $\mathcal{P}^+$ *(given in definition 4.1).*

Before proceeding to the proof of Proposition 4.3, we state and prove a lemma.

Lemma 4.4. — *Let* $(V_h = \sum_{s\geq 0} V_{h;s}x^{-[s]})_{h\in\mathbb{C}\setminus\Sigma}$ *be a family of elements of* $M_n(\widehat{\mathcal{O}}_{fact})$ *satisfying to the property* $\mathcal{P}^+$ *(given in definition 4.1). We assume that, for any* $h \in \mathbb{C}\setminus\Sigma$, *none of the eigenvalues of* $V_{h;0}$ *is a nonpositive integer. Then, for any* $\Xi_0 \in ker(V_{h;0})$, *for any* $h \in \mathbb{C}\setminus\Sigma$, *the equation* $\delta U = V_h U$ *has a unique solution* U_h *in* $M_{n,1}(\widehat{\mathcal{O}}_{fact})$ *with constant term* Ξ_0. *Moreover the family* $(U_h)_{h\in\mathbb{C}\setminus\Sigma}$ *satisfies the property* $\mathcal{P}^+$.

Proof. — We first prove that, for any $h \in \mathbb{C}\setminus\Sigma$, there exists a unique $U_h = \sum_{s=0}^{+\infty} U_{h;s}x^{-[s]} \in M_{n,1}(\widehat{\mathcal{O}}_{fact})$ such that :

$$(\mathcal{E}) \qquad \begin{cases} \delta U_h = V_h U_h \\ U_{h;0} = \Xi_0. \end{cases}$$

Note that formula (5) stated in section 2.1 ensures that :

$$V_h(x)U_h(x) = \sum_{s=0}^{+\infty} C_{h;s}x^{-[s]}$$

where :

$$C_{h;0} = V_{h;0}U_{h;0} \text{ and } \forall s \in \mathbb{N}^*, \quad C_{h;s} = V_{h;0}U_{h;s} + V_{h;s}U_{h;0} + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} V_{h;j}U_{h;l}$$

(for the definitions of J_s and $c_{j,l}^{(k)}$, see formulas (3) and (4) in section 2.1). Moreover, we have :

$$\delta U_h(x) = \sum_{s=0}^{+\infty} -sU_{h;s}x^{-[s]}.$$

So, the system $(\mathcal{E})$ is equivalent to :

$$\begin{cases} U_{h;0} = \Xi_0 \\ V_{h;0}U_{h;0} = 0 \\ \forall s \in \mathbb{N}^*, -(sI + V_{h;0})U_{h;s} = V_{h;s}U_{h;0} + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} V_{h;j}U_{h;l}. \end{cases}$$

Since Ξ_0 belongs to $\ker(V_{h;0})$ and since, for all $s \in \mathbb{N}^*$, $sI + V_{h;0}$ is invertible, the above system has a unique solution $U_h = \sum_{s=0}^{+\infty} U_{h;s} x^{-[s]} \in M_{n,1}(\widehat{\mathcal{O}}_{fact})$ given by the following induction formula :

$$\text{(13)} \quad \begin{cases} U_{h;0} = \Xi_0 \\ \forall s \in \mathbb{N}^*, U_{h;s} = -(sI + V_{h;0})^{-1} \left[V_{h;s} U_{h;0} + \sum_{(j,k,l) \in J_s} c_{j,l}^{(k)} V_{h;j} U_{h;l} \right]. \end{cases}$$

It remains to check that $(U_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the property $\mathcal{P}^+$. The above induction formula clearly shows that, for all $s \in \mathbb{N}$, $U_{h;s}$ depends analytically on $h \in \mathbb{C} \setminus \Sigma$ (recall that, for all $s \in \mathbb{N}$, $V_{h;s}$ depends analytically on $h \in \mathbb{C} \setminus \Sigma$ because $(V_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the property $\mathcal{P}^+$ by hypothesis). In order to prove that $(U_h)_{h \in \mathbb{C} \setminus \Sigma}$ satisfies the property $\mathcal{P}^+$, it remains to study the growth properties of the coefficients of $U_h = \sum_{s=0}^{+\infty} U_{h;s} x^{-[s]}$. We set, for all $(h, s) \in (\mathbb{C} \setminus \Sigma) \times \mathbb{N}$, $v_{h;s} = \|V_{h;s}\|$ and $u_{h;s} = \|U_{h;s}\|$. Let K be a compact subset of $\mathbb{C} \setminus \Sigma$. Using the fact that $V_{h;0}$ depends analytically on $h \in \mathbb{C} \setminus \Sigma$ and using the hypothesis relative to the eigenvalues of $V_{h;0}$, we see that $c = \sup_{(h,s) \in K \times \mathbb{N}^*} ||(sI + V_{h;0})^{-1}||$ is finite. From formula (13), we get :

$$\text{(14)} \qquad \forall (h, s) \in K \times \mathbb{N}, \quad u_{h;s} \leq c \left[v_{h;s} u_{h;0} + \sum_{(j,k,l) \in J_s} c_{j,l}^{(k)} v_{h;j} u_{h;l} \right].$$

We introduce the family of sequences $(u^{>}_{h;s})_{(h,s) \in (\mathbb{C} \setminus \Sigma) \times \mathbb{N}^*}$ inductively defined by :
(15)

$$u^{>}_{h;1} = c v_{h;1} u_{h;0} \text{ and}, \forall s \in \mathbb{N}_{\geq 2}, \quad u^{>}_{h;s} = c \left[v_{h;s} u_{h;0} + \sum_{(j,k,l) \in J_s} c_{j,l}^{(k)} v_{h;j} u^{>}_{h;l} \right]$$

and we consider the corresponding factorial series :

$$u^{>}_h(x) = \sum_{s=1}^{+\infty} u^{>}_{h;s} x^{-[s]} \in \widehat{\mathcal{O}}_{fact}.$$

Using the fact that $u_{h;1} \leq c v_{h;1} u_{h;0} = u^{>}_{h;1}$ and using the inequality (14), we see by induction that :

$$\forall (h, s) \in K \times \mathbb{N}^*, \quad u_{h;s} \leq u^{>}_{h;s}.$$

Remark that, if we set :

$$v_h(x) = \sum_{s=1}^{+\infty} v_{h;s} x^{-[s]} \in \widehat{\mathcal{O}}_{fact},$$

then the definition of $u_h^>$ together with the product formula formula (5) given in section 2.1 imply that :

$$u_h^>(x) = cv_h(x)(u_{h;0} + u_h^>(x)),$$

so :

$$u_h^>(x) = \frac{cv_h(x)u_{h;0}}{1 - cv_h(x)} == -u_{h;0}\left(1 - \frac{1}{1 - cv_h(x)}\right). \tag{16}$$

Since $(V_h)_{h\in\mathbb{C}\setminus\Sigma}$ satisfies the property $\mathcal{P}^+$, there exists $(C_K, \lambda_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that, for all $s \in \mathbb{N}^*$, for all $h \in K$, $v_{h;s} = ||V_{h;s}|| \leq C_K\lambda_K^{[s-1]}$; using formula (16) and applying the last property stated in section 2.1, we get that there exists $(C'_K, \lambda'_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that, for all $s \in \mathbb{N}^*$, for all $h \in K$, $u_{h;s} \leq u_{h;s}^> \leq C'_K \lambda'^{[s-1]}_K$. □

Proof of Proposition 4.3.. — Note that the functional equation $\tau F_h^+(\mathrm{I}_n - \frac{1}{x-1}A_{h;\infty}) = AF_h^+$ (see formula (12) at the beginning of this section) is equivalent to :

$$\delta F_h^+ = \left(B_h F_h^+ - F_h^+ B_{h;0}\right)\left(\mathrm{I}_n - \frac{1}{x-1}B_{h;0}\right)^{-1}.$$

Using formula (8) and the product formula (5) which were both stated in section 2.1 we get that the above equation is also equivalent to :

$$\delta F_h^+(x) = \sum_{s=0}^{+\infty} L_{h;s}(F_h^+(x))x^{-[s]}$$

where, the $L_{h;s}$ are the linear operators on $M_n(\mathbb{C})$ defined by :

$$L_{h;0}(M) = B_{h;0}M - MB_{h;0},$$

$$L_{h;1}(M) = B_{h;1}M + (B_{h;0}M - MB_{h;0})B_{h;0}$$

and :

$$\forall s \geq 2,\ L_{h;s}(M) = B_{h;s}M + (B_{h;0}M - MB_{h;0})(B_{h;0})^{[s]} + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} B_{h;j}M(B_{h;0})^{[l]}.$$

In order to finish the proof, it is clearly sufficient to show that this family of difference systems parameterized by $h \in \mathbb{C} \setminus \Sigma$ satisfies the hypotheses of Lemma 4.4 (with $\Xi_0 = \mathrm{I}_n$). For all $s \in \mathbb{N}$, $L_{h;s}$ depends analytically on $h \in \mathbb{C} \setminus \Sigma$ because, for all $s \in \mathbb{N}$, $B_{h;s}$ satisfies the property $\mathcal{P}^+$ (see section 4.1) and, hence, depends analytically on $h \in \mathbb{C} \setminus \Sigma$. Let K be a compact subset of $\mathbb{C} \setminus \Sigma$. Since $(B_h)_{h\in\mathbb{C}\setminus\Sigma}$ satisfies the property $\mathcal{P}^+$ (see section 4.1), there exists $(C_K, \lambda_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that, for all $s \in \mathbb{N}$, for all $h \in K$, $||B_{h;s}|| \leq C_K \lambda_K^{[s-1]}$. So, setting $b_{h;0} = \|B_{h;0}\|$, we have, for all $(h,s) \in K \times \mathbb{N}_{\geq 2}$:

$$|||L_{h;s}||| \leq C_K \lambda_K^{[s-1]} + 2b_{h;0}(b_{h;0})^{[s]} + \sum_{(j,k,l)\in J_s} c_{j,l}^{(k)} C_K \lambda_K^{[j-1]} (b_{h;0})^{[l]}$$

(here $|||\cdot|||$ denotes the operator norm associated to $||\cdot||$). Using the product formula (5) stated in section 2.1 and the formula :

$$\begin{aligned} 1 &= \left(1 - \frac{b_{h;0} - \lambda_K + 1}{x - \lambda_K}\right)\left(1 - \frac{b_{h;0} - \lambda_K + 1}{x - b_{h;0} - 1}\right) \\ &= \left(1 - (b_{h;0} - \lambda_K + 1)\sum_{s=1}^{\infty} (\lambda_K - 1)^{[s]} x^{-[s]}\right)\left(1 - (b_{h;0} - \lambda_K + 1)\sum_{s=1}^{\infty} (b_{h;0})^{[s]} x^{-[s]}\right) \end{aligned}$$

we get that there exists $(C'_K, \lambda'_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that the above expression is dominated (uniformly in $(h,s) \in K \times \mathbb{N}$) by $C'_K {\lambda'_K}^{[s-1]}$. The set of eigenvalues of $L_{h;0}$ is equal to $\{\mu - \nu \mid \mu, \nu \in Sp(B_{h;0})\}$ and, hence, do not contain any nonpositive integer because $B_{h;0}$ is nonresonant. Moreover the matrix I_n belongs to the kernel of $L_{h;0}$. Therefore, the hypotheses of Lemma 4.4 are satisfied as expected. □

In what follows we will say that a function f is analytic in a neighborhood of $\{+\infty\} \times \{h_0\}$ if, there exists $M \in \mathbb{R}^{+*}$ such that f is defined and analytic on a neighborhood of $\Pi_M^+ \times \{h_0\} = \{x \in \mathbb{C} \mid \Re(x) > M\} \times \{h_0\}$.

Proposition 4.5. — *Let us consider* $(U_h = \sum_{s\geq 0} U_{h;s} x^{-[s]})_{h \in \mathbb{C}\setminus\Sigma}$ *a family of elements of* $\mathrm{M}_n(\widehat{\mathcal{O}}_{fact})$ *satisfying the property* $\mathcal{P}^+$ *(given in definition 4.1). Then,* $U_h(x) = \sum_{s\geq 0} U_{h;s} x^{-[s]}$ *defines an analytic function of* (x,h) *in the neighborhood of any points of* $\{+\infty\} \times (\mathbb{C} \setminus \Sigma)$.

Proof. — Let K be a compact subset of $\mathbb{C} \setminus \Sigma$ and let $(C_K, \lambda_K) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ be such that, for all $s \in \mathbb{N}$, for all $h \in K$, $||Y_{h;s}|| \leq C_K \lambda_K^{[s-1]}$. Let us consider the half-plane $\Pi_{\lambda_K+1}^+ = \{x \in \mathbb{C} \mid \Re(x) > \lambda_K + 1\}$. For all $(x,h) \in \Pi_{\lambda_K+1}^+ \times K$, for all $s \in \mathbb{N}$, we have :

$$||U_{h;s} x^{-[s]}|| \leq C_K \left|\frac{\lambda_K^{[s-1]}}{x^{[s]}}\right| \leq C_K \frac{\lambda_K^{[s-1]}}{\Re(x)^{[s]}} \leq C_K \frac{\lambda_K^{[s-1]}}{(\lambda_K+1)^{[s]}} = C'_K \frac{\Gamma(\lambda_K + s - 1)}{\Gamma(\lambda_K + s + 1)}.$$

The Stirling formula ensures that the series $\sum_{s=0}^{\infty} \frac{\Gamma(\lambda_K+s-1)}{\Gamma(\lambda_K+s+1)}$, which is independent of (x,h) in $\Pi_{\lambda_K+1}^+ \times K$, is convergent. We conclude that $U_h(x) = \sum_{s\geq 0} U_{h;s} x^{-[s]}$ defines an analytic function of (x,h) in $\Pi_{\lambda_K+1}^+ \times K$. □

We define a subset Ω of $\mathbb{C} \times (\mathbb{C} \setminus \Sigma)$ by :

$$\Omega = (\mathbb{C} \times (\mathbb{C} \setminus \Sigma)) \setminus (S + (\mathbb{Z} \times \{0\})) \subset \mathbb{C} \times (\mathbb{C} \setminus \Sigma).$$

Theorem 4.6. — *The map* $(x,h) \mapsto P_h(x) = (Y_h^+(x))^{-1} Y_h^-(x)$ *is analytic on* Ω *and meromorphic on* $\mathbb{C} \times (\mathbb{C} \setminus \Sigma)$.

Proof. — Proposition 4.3 and Proposition 4.5 entail that the map $(x,h) \mapsto F_h^+(x) = \sum_{s \geq 0} F_{h;s}^+ x^{-[s]}$ is analytic in the neighborhood of any point of $\{+\infty\} \times (\mathbb{C} \setminus \Sigma)$. The functional equation $\tau Y = A_h Y$ allows us to conclude that $(x,h) \mapsto Y_h^+(x)$ can be extended into a meromorphic function over $\mathbb{C} \times (\mathbb{C} \setminus \Sigma)$, analytic on Ω. Similar arguments prove an analogous result for Y_h^-. Whence the theorem. □

5. Variation theorem

We consider a family of difference systems :

$$(\mathcal{S}_h) : \quad \tau Y = A_h Y, \quad A_h \in \mathrm{GL}_n(\mathbb{C}(x,h))$$

parameterized by $h \in \mathbb{C} \setminus \Sigma$ satisfying the hypotheses 1. to 3. given in Assumption 1 in section 4.1. We maintain the notations of section 4.1 and of section 4.2. In particular, for any $h \in \mathbb{C} \setminus \Sigma$, we denote by :

$$Y_h^+ = F_h^+ e_{A_{h;\infty}}^+ \text{ and } Y_h^- = F_h^- e_{A_{h;\infty}}^-$$

the solutions and by :

$$P_h = (Y_h^+)^{-1} Y_h^- \in \mathrm{GL}_n(\mathbb{C}(\mathbf{x}))$$

the Birkhoff matrix built in section 2.2 for the Fuchsian and nonresonant difference system $(\mathcal{S}_h)$. We recall that Ω denotes the subset of $\mathbb{C} \times (\mathbb{C} \setminus \Sigma)$ given by :

$$\Omega = (\mathbb{C} \times (\mathbb{C} \setminus \Sigma)) \setminus (S + (\mathbb{Z} \times \{0\})) \subset \mathbb{C} \times (\mathbb{C} \setminus \Sigma).$$

where S is the singular locus defined at the begining of section 4.1. Theorem 4.6 ensures that the map $(x,h) \mapsto P_h(x) = (Y_h^+(x))^{-1} Y_h^-(x)$ is analytic on Ω and meromorphic on $\mathbb{C} \times (\mathbb{C} \setminus \Sigma)$. Moreover, for any $h \in \mathbb{C} \setminus \Sigma$, we denote by G_h the Galois group of $(\mathcal{S}_h)$ and by $\mathfrak{g}_h$ its Lie algebra. We recall that, in virtue of Theorem 3.3, for all $h \in \mathbb{C} \setminus \Sigma$, $\mathfrak{g}_h$ is generated by $\{P_h(a)^{-1} P_h'(a) \mid a \in \Omega_h\}$ where $\Omega_h = \{x \in \mathbb{C} \mid (x,h) \in \Omega\}$.

Theorem 5.1. — *Let* $\kappa = \max_{h \in \mathbb{C} \setminus \Sigma} \dim(G_h)$. *Then* $\Theta = \{h \in \mathbb{C} \setminus \Sigma \mid \dim(G_h) = \kappa\}$ *is an open subset of* $\mathbb{C} \setminus \Sigma$ *with discrete complement.*

Proof. — In what follows $[\cdot, \cdot]$ denotes the usual Lie bracket on $\mathrm{M}_n(\mathbb{C})$. For any set E and for any functions $f, g : E \to \mathrm{M}_n(\mathbb{C})$, $[f,g]$ denotes the function $E \to \mathrm{M}_n(\mathbb{C})$, $x \mapsto [f(x), g(x)]$.

We denote by $p_1, ..., p_\kappa$ the κ projections from $\mathrm{M}_n(\mathbb{C})^\kappa$ to $\mathrm{M}_n(\mathbb{C})$ and we introduce $\mathcal{R} = \{p_k \mid k \in \{1, ..., \kappa\}\} \cup \{[p_k, p_l] \mid (k,l) \in \{1, ..., \kappa\}^2\} \cup$

$\{[[p_k, p_l], p_m] \mid (k, l, m) \in \{1, ..., \kappa\}^3\} \cup \cdots$ the set of all functions on $\mathrm{M}_n(\mathbb{C})^\kappa$ obtained by iterating the Lie bracket $[\cdot, \cdot]$.

Recall that, for all $h \in \mathbb{C} \setminus \Sigma$, we set $\Omega_h = \{x \in \mathbb{C} \mid (x, h) \in \Omega\}$. For all $h \in \mathbb{C} \setminus \Sigma$, we consider the map $\lambda_h : \Omega_h \to \mathfrak{g}_h$, $a \mapsto P_h(a)^{-1} P_h'(a)$ and we use the same notation for its extension to Ω_h^κ defined by $\lambda_h : \Omega_h^\kappa \to \mathfrak{g}^\kappa$, $\underline{a} = (a_1, ..., a_\kappa) \mapsto (\lambda_h(a_1), ..., \lambda_h(a_\kappa))$.

Let $h_{\max} \in \mathbb{C} \setminus \Sigma$ be such that $\dim(G_{h_{\max}}) = \dim(\mathfrak{g}_{h_{\max}}) = \kappa$. As recalled above, Theorem 3.3 ensures that $\mathfrak{g}_{h_{\max}}$ is generated as a Lie algebra by the image of $\lambda_{h_{\max}}$ so there exist $\mathfrak{f}_1, ..., \mathfrak{f}_\kappa \in \mathcal{R}$ and $\underline{a} \in \Omega^\kappa_{h_{\max}}$ such that $\mathfrak{f}_1(\lambda_{h_{\max}}(\underline{a})), ..., \mathfrak{f}_\kappa(\lambda_{h_{\max}}(\underline{a}))$ is a basis of $\mathfrak{g}_{h_{\max}}$.

Let us introduce the meromorphic function on $\mathbb{C} \setminus \Sigma$ defined by $\varphi(h) = \det(\mathfrak{f}_1(\lambda_h(\underline{a})), ..., \mathfrak{f}_\kappa(\lambda_h(\underline{a})))$. This function is nonzero since $\varphi(h_{\max}) \neq 0$.

The description of the Lie algebra of the Galois groups given in Theorem 3.3 entails that the degeneracy locus $(\mathbb{C} \setminus \Sigma) \setminus \Theta$ is included in the set of zeroes and poles of φ, which is discrete and closed in $\mathbb{C} \setminus \Sigma$; thus, the degeneracy locus is itself discrete and closed in $\mathbb{C} \setminus \Sigma$. □

This theorem has for instance the following consequence.

Corollary 5.2. — *Suppose that there exists $h \in \mathbb{C} \setminus \Sigma$ such that $G_h = \mathrm{GL}_n(\mathbb{C})$. Then for any $h \in \mathbb{C} \setminus \Sigma$ but, maybe, a discrete subset, we have $G_h = \mathrm{GL}_n(\mathbb{C})$.*

References

[1] André, Y. - *Différentielles non commutatives et théorie de Galois différentielle ou aux différences*, Ann. Sci. École Norm. Sup., Sér. 4, 34 no. 5 (2001), p. 685-739.

[2] Casale, G. & Roques, J. - *Dynamics of rational symplectic mappings and difference Galois theory*, Int. Math. Res. Not. IMRN, 2008, Art. ID rnn 103, 23.

[3] Duval, A. - *Unpublished paper.*

[4] Duval, A. - *Séries de q-factorielles, opérateurs aux q-différences et confluence*, Ann. Fac. Sci. Toulouse Math., Sér. 6, 12 no. 3 (2003), p. 335-374.

[5] Duval, A. - *Confluence q-différence vers différence pour un système fuchsien*, Pacific J. Math. 217 (2004), 221-245.

[6] Duval, A. & Roques, J. - *Familles fuchsiennes d'équations aux (q-)différences et confluence*, Bull. Soc. math. France 136, fascicule 1 (2008), 67-96 .

[7] Di Vizio, L. & Zhang, C. - *On q-summation and confluence*, Ann. Inst. Fourier, 59 no1 (2009), p. 347-392.

[8] Etingof, P. - *Galois groups and connection matrices of q-difference equations*, Electron. Res. Announc. Amer. Math. Soc., 1(1):1-9 (electronic), 1995.

[9] Hardouin, C. & Singer, M. F. - *Differential Galois theory of linear difference equations*, Math. Ann., 342 no 2 (2008), 333-377.

[10] Humphreys, J. E. - *Linear algebraic groups*, Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York, 1975, xiv+247.

[11] Nörlund, N.-E. - *Leçons sur les séries d'interpolation*, Gauthiers Villards et Cie, Paris.

[12] Ramis, J.-P. & Sauloy, J. - *The q-analogue of the wild fundamental group. I*, Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies. P. 167–193. RIMS Kôkyûroku Bessatsu, B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.

[13] Ramis, J.-P. & Sauloy, J. - *The q-analogue of the wild fundamental group. II*, Differential Equations and Singularities, 60th years of J.M. Aroca, Astérisque, Société Mathématique de France, vol. 323, 2009.

[14] Roques, J. - *Classification rationnelle et confluence des systèmes aux différences singuliers réguliers*, Ann. Inst. Fourier, 56 no. 6 (2006), p. 1663-1699.

[15] Sauloy, J. - *Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie*, Ann. Inst. Fourier (Grenoble),50 (2000), no. 4, P. 1021-1071.

[16] Sauloy, J. - *Galois theory of Fuchsian q-difference equations*, Ann. Scient. Ec. Norm. Sup., 4eme série, 36 (2003), no 6, 925-968.

[17] Van der Put, M. & Singer, M. - *Galois theory of difference equations*, Lecture Notes in Mathematics, 1666. Springer-Verlag, Berlin, 1997.

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