ON THE LOCAL STRUCTURE OF MAHLER MODULES

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Abstract. Despite the numerous recent works devoted or related to Mahler equations, very few is known on the classification of these equations. In this paper, we give the classification of the Mahler equations (or, better, Mahler modules), locally at 0 and ∞, over the field of Hahn series.

Contents

1. Introduction

The theory of linear ℓ-Mahler equations over $\overline{\mathbb{Q}}(z)$, *i.e.*, of functional equations of the form

$$a_n(z)f(z^{\ell^n}) + a_{n-1}(z)f(z^{\ell^{n-1}}) + \cdots + a_0(z)f(z) = 0 \tag{1}$$

for some $a_0(z), \ldots, a_n(z) \in \overline{\mathbb{Q}}(z)$ and $\ell \geq 2$, has attracted the attention of many mathematicians, from the seminal work of Mahler in [Mah29, Mah30a, Mah30b] in the early 1930s to the recent works by Nguyen [Ngu11, Ngu12], Adamczewski and Bell [AB13b], Dreyfus, Hardouin and Roques [DHR15], Adamczewski and Faverjon [AF15], Philipon [Phi], Shaëfke and Singer [SS16], through those of Mendès France [MF80], Randé [Ran92], Becker [Bec94], Nishioka [Nis96], Dumas and Flajolet [DF96], Allouche and Shallit [AS03], to name just a few. Nevertheless, at the time being, very few is known on the structure of these equations, even locally at 0. (We remind that it is natural to look for the local structure of the

Date: November 30, 2016.
2010 *Mathematics Subject Classification.* 39A06,12H10.
Key words and phrases. Linear difference equations, difference Galois theory.

ℓ-Mahler equations at $0,1$ and ∞ because they are the fixed points of the endomorphism $z \mapsto z^\ell$ of $\mathbb{P}^1(\overline{\mathbb{Q}})$. The change of variable $z = e^u$ transforms the local study of the ℓ-Mahler equations at $z = 1$ into the local study of q-difference equations (with $q = \ell$) at $u = 0$, which is well understood; see [RSZ13, Sau00, Sau04, vdPR07]. Moreover, the local study of the ℓ-Mahler equations at 0 and ∞ are equivalent : one switch between 0 and ∞ by the change of variables $z \mapsto z^{-1}$.) Roughly speaking, the aim of this note is to show that, after extension of scalars to the field $\mathscr{H}$ of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$, the local structure of these equations becomes very simple.

We shall now describe more precisely the content of this paper. We let $\mathscr{P}$ be the field of Puiseux series over $\overline{\mathbb{Q}}$ and $\mathscr{H}$ be the field of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$ (see Section 2). For $K = \mathscr{P}$ or $\mathscr{H}$, we denote by $\mathcal{D}_K$ the non-commutative algebra of ℓ-Mahler operators with coefficients in K. This means that $\mathcal{D}_K = K[\phi_\ell, \phi_\ell^{-1}]$ is the non commutative algebra of non commutative Laurent polynomials in the indeterminate ϕ_ℓ and with coefficients in K such that $\phi_\ell a = \phi_\ell(a)\phi_\ell$ for all $a \in K$, where ϕ_ℓ acts on $a = \sum a_\gamma z^\gamma \in K$ by $\phi_\ell(a) = \sum a_\gamma z^{\ell\gamma}$. By "$\ell$-Mahler module" (or, simply, "Mahler module") over $K = \mathscr{P}$ or $\mathscr{H}$, we mean "left $\mathcal{D}_K$-module of finite length" (it is equivalent to require that the K-vector space obtained by restriction of scalars has finite dimension).

The study of the formal structure at 0 of the equation (1) amounts to the study of the structure of the ℓ-Mahler module over $\mathscr{P}$ given by $\mathcal{D}_\mathscr{P}/\mathcal{D}_\mathscr{P}L$ where $L = a_n(z)\phi_\ell^n + a_{n-1}(z)\phi_\ell^{n-1} + \cdots + a_0(z) \in \mathcal{D}_\mathscr{P}$. Actually, any ℓ-Mahler module over $\mathscr{P}$ is isomorphic to $\mathcal{D}_\mathscr{P}/\mathcal{D}_\mathscr{P}L$ for some $L \in \mathcal{D}_\mathscr{P}$ (this result is known as the cyclic vector lemma), so that, in a sense, the study of the structure of the ℓ-Mahler equations with coefficients in $\mathscr{P}$ is equivalent to the study of the structure of the ℓ-Mahler modules over $\mathscr{P}$. Our main result gives a complete description of such modules after extension of scalars to $\mathscr{H}$.

Theorem 1. *Let M be a ℓ-Mahler module over $\mathscr{P}$. Then, the ℓ-Mahler module $M \otimes_\mathscr{P} \mathscr{H}$ over $\mathscr{H}$ obtained from M by extending the scalars to $\mathscr{H}$ is isomorphic to a direct sum of modules of the form $\mathcal{D}_\mathscr{H}/\mathcal{D}_\mathscr{H}(\phi_\ell - c)^k$ for some $c \in \overline{\mathbb{Q}}^\times$ and $k \in \mathbb{Z}_{\geq 0}$.*

This result can be restated in terms of difference systems as follows. For any $A \in \mathrm{GL}_n(\mathscr{P})$, there exists $F \in \mathrm{GL}_n(\mathscr{H})$ and $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that

$$AF = \phi_\ell(F)A_0.$$

In other words, the ℓ-Mahler system $\phi_\ell(Y) = AY$ is transformed into $\phi_\ell(Z) = A_0 Z$ by the linear change of variables $Y = FZ$.

This paper is organized as follows. Section 2 contains a preliminary result about inhomogeneous Mahler equations of order 1. Section 3 is concerned with the factorization of the Mahler operators according to their exponents. Section 4 is devoted to the proof of Theorem 1. In Section 5, we state a variant of Theorem 1 and outline an application to an analogue of Grothendieck's conjecture.

2. Inhomogeneous equations of order 1 with constant coefficients

We let
$$\mathscr{P} = \cup_{d\geq 1}\overline{\mathbb{Q}}((z^{\frac{1}{d}}))$$
be the field of Puiseux series.

We recall that the field
$$\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$$
of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$ is made of the formal expressions of the form $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ with $f_\gamma \in \overline{\mathbb{Q}}$ such that the support of f defined by
$$\mathrm{supp}(f) = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$
is well-ordered (as a subset of $\mathbb{Q}$ endowed with the total order $\leq$). The sum and product of $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ and $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$ are given by
$$f + g = \sum_{\gamma\in\mathbb{Q}}(f_\gamma + g_\gamma)z^\gamma$$
and
$$fg = \sum_{\gamma\in\mathbb{Q}}\left(\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'}g_{\gamma''}\right)z^\gamma.$$
(Note that there are only finitely many $(\gamma',\gamma'') \in \mathbb{Q}\times\mathbb{Q}$ such that $\gamma'+\gamma''=\gamma$ and $f_{\gamma'}g_{\gamma''} \neq 0$.)

Of course, one can see $\mathscr{P}$ as a subfield of $\mathscr{H}$.

Lemma 2. *For any subset E of $\mathbb{Q}$, we set*
$$\mathrm{Sat}_\ell(E) = \{\ell^{-k}x \mid x \in E\cap\mathbb{Q}_{\leq 0}, k \geq 0\ \} \cup \{\ell^k x \mid x \in E\cap\mathbb{Q}_{\geq 0}, k\geq 0\ \}.$$
If E is a well-ordered subset of $\mathbb{Q}$, then $\mathrm{Sat}_\ell(E)$ is a well-ordered subset of $\mathbb{Q}$.

Proof. Let F be a subset of $\mathrm{Sat}_\ell(E)$.

Assume that $F\cap\mathbb{Q}_{<0} \neq \emptyset$ and consider $\gamma \in F\cap\mathbb{Q}_{<0}$. Since E is bounded from below, there exists M such that, for all $k \geq M$, for all $x\in E$, $\gamma < \ell^{-k}x$. Therefore, in order to prove that F has a least element, it is sufficient to prove that $\{\ell^{-k}x \mid x \in E\cap\mathbb{Q}_{\leq 0}, k \in \{0,\ldots,M-1\}\ \}\cap F$ has a least element. This follows from the facts that the latter set can be rewritten has the finite union $\cup_{k=0}^{M-1}(\ell^{-k}E\cap\mathbb{Q}_{\leq 0})\cap F$ and that each $(\ell^{-k}E\cap\mathbb{Q}_{\leq 0})\cap F$ has a least element (because E and, hence, $\ell^{-k}E\cap\mathbb{Q}_{\leq 0}$ are well-ordered).

The case $F\cap\mathbb{Q}_{<0} = \emptyset$ is similar. □

Proposition 3. *For all $c,d\in\overline{\mathbb{Q}}^\times$, for all $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$ with $g_0 = 0$, there exists $f\in\mathscr{H}$ such that $g = (c\phi_\ell - d)f$.*

Moreover, if $c\neq d$, then, for all $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$, there exists $f \in \mathscr{H}$ such that $g = (c\phi_\ell - d)f$.

Proof. Dividing by c, it is clearly sufficient to consider the case $c = 1$.

We first assume that $g_0 = 0$.

We set $g^- = \sum_{\gamma\in\mathbb{Q}_{<0}} g_\gamma z^\gamma \in \mathscr{H}$ and $g^+ = \sum_{\gamma\in\mathbb{Q}_{>0}} g_\gamma z^\gamma \in \mathscr{H}$, so that $g = g^- + g^+$. We are going to prove that there exist $f^\pm \in \mathscr{H}$ such that

$g^{\pm} = (\phi_\ell - d)f^{\pm}$. This will imply the desired result because $f = f^- + f^+ \in \mathscr{H}$ satisfies $g = (\phi_\ell - d)f$.

For all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) \neq \emptyset$, we set $\gamma^- = \min \ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g)$ (it exists because $\mathrm{supp}(g)$ is well-ordered). We let $(f^-_\gamma)_{\gamma \in \mathbb{Q}_{<0}}$ be the unique element of $\overline{\mathbb{Q}}^{\mathbb{Q}_{<0}}$ such that, for all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) \neq \emptyset$,

$$\begin{cases} f^-_{\gamma^-/\ell^{i+1}} = d f^-_{\gamma^-/\ell^i} + g_{\gamma^-/\ell^i} & \text{for } i \geq 0, \\ f^-_{\gamma^-/\ell^{i+1}} = 0 & \text{for } i \leq -1 \end{cases}$$

and, for all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) = \emptyset$,

$$f^-_\gamma = 0.$$

Then, $f^- = \sum_{\gamma \in \mathbb{Q}_{<0}} f^-_\gamma z^\gamma \in \mathscr{H}$ satisfies $(\phi_\ell - d)f^- = g^-$. The fact that f^- belongs to $\mathscr{H}$ is a consequence of Lemma 2 because $\mathrm{supp}(f) \subset \mathrm{Sat}_\ell(\mathrm{supp}(g))$.

The construction of f^+ is similar.

We now assume that $c = 1 \neq d$. We set $g^- = \sum_{\gamma \in \mathbb{Q}_{<0}} g_\gamma z^\gamma \in \mathscr{H}$ and $g^+ = \sum_{\gamma \in \mathbb{Q}_{>0}} g_\gamma z^\gamma \in \mathscr{H}$, so that $g = g^- + g_0 + g^+$. We have already proved that there exist $f^{\pm} \in \mathscr{H}$ such that $g^{\pm} = (\phi_\ell - d)f^{\pm}$. Moreover, $f_0 = \frac{g_0}{1-d}$ satisfies $g_0 = (\phi_\ell - d)f_0$. So, $f = f^- + f_0 + f^+ \in \mathscr{H}$ satisfies $g = (\phi_\ell - d)f$. □

3. Factorisation of the Mahler operators by the exponents

3.1. **Exponents.** In this section, we recall the definition of the exponents introduced in [Roq15a, Section 4.2] and their basic properties.

Theorem 4 ([Roq15a, Theorem 24])**.** *Let M be a Mahler module over $\mathscr{P}$ of rank $n \geq 1$.*

(i) The module M is triangularizable, i.e., there exists a filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

by submodules of M such, for all $i \in \{0, \dots, n-1\}$, the quotient module M_{i+1}/M_i has rank 1 and, hence, is isomorphic to $\mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}(\phi_\ell - c_i)$ for an unique $c_i \in \overline{\mathbb{Q}}^\times$.

(ii) The list $c_1, \dots, c_n$ does not depend (up to permutation) on the choosen filtration.

Definition 5 ([Roq15a, Definition 25])**.** *The exponents at 0 of the Mahler module M over $\mathscr{P}$ are the non zero complex numbers $c_1, \dots, c_n$ introduced in Theorem 4. The multiplicity of an exponent of M is its number of occurrences in $c_1, \dots, c_n$.*

Definition 6 ([Roq15a, Definition 26])**.** *The exponents at 0 of $L \in \mathcal{D}_{\mathscr{P}}$ are the exponents of the Mahler module $\mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}L$ over $\mathscr{P}$. The multiplicity of an exponent of L is its multiplicity as an exponent of $\mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}L$.*

3.2. **Factorisation of the Mahler operators by the exponents.** The aim of this section is to prove the following result.

Proposition 7. *Let $c_1, \cdots, c_r$ be the pairwise distinct exponents of $L \in \mathcal{D}_{\mathscr{P}}$, with multiplicities $n_1, \cdots, n_r$ respectively. Then, L admits a factorization of the form*

$$L = aL_r \cdots L_1$$

with

$$L_i = (\phi_\ell - c_i) f_{i,n_i}^{-1} \cdots (\phi_\ell - c_i) f_{i,1}^{-1}$$

for some $f_{i,1}, \ldots, f_{i,n_i} \in \mathscr{H}^\times$.

We first prove a lemma.

Lemma 8. *Consider $c, d \in \overline{\mathbb{Q}}^\times$ with $c \neq d$ and $f \in \mathscr{H}^\times$. There exist $g, h, k \in \mathscr{H}^\times$ such that*

$$(\phi_\ell - c) f (\phi_\ell - d) = g(\phi_\ell - d) h (\phi_\ell - c) k.$$

Proof. In order to prove this lemma, it is convenient to introduce a difference ring extension of the difference field $(\mathscr{H}, \phi_\ell)$ in which any equation of the form $(\phi_\ell - c)y = 0$ with $c \in \overline{\mathbb{Q}}^\times$ has non zero solutions. Let $(X_c)_{c \in \overline{\mathbb{Q}}^\times}$ be indeterminates over $\mathscr{H}$, and consider the quotient ring

$$\mathscr{U} := \mathscr{H}[(X_c)_{c \in \overline{\mathbb{Q}}^\times}]/I$$

of the polynomial ring $\mathscr{H}[(X_c)_{c \in \overline{\mathbb{Q}}^\times}]$ by its ideal I generated by $\{X_c X_d - X_{cd} \mid c, d \in \overline{\mathbb{Q}}^\times\} \cup \{X_1 - 1\}$. Let e_c be the image of X_c in $\mathscr{U}$, so that

$$\mathscr{U} = \mathscr{H}[(e_c)_{c \in \overline{\mathbb{Q}}^\times}].$$

We endow $\mathscr{U}$ with its ring automorphism ϕ such that $\phi_{|\mathscr{H}} = \phi_\ell$ and,

$$\forall c \in \overline{\mathbb{Q}}^\times, \ \phi(e_c) = c e_c.$$

Hence, $(\mathscr{U}, \phi)$ is a difference ring extension of $(\mathscr{H}, \phi_\ell)$. We will denote ϕ by ϕ_ℓ.

Arguing as in [Roq15a, Theorem 35], it is easily seen that the ring of constants $\mathscr{U}^\phi = \{f \in \mathscr{H} \mid \phi(f) = f\}$ of $\mathscr{U}$ is equal to $\overline{\mathbb{Q}}$.

We are now ready to prove the lemma. The operator $(\phi_\ell - c) f (\phi_\ell - d)$ has a basis of solutions in $\mathscr{U}$ of the form $(e_d, a e_c)$ where $a \in \mathscr{H}^\times$ is such that $(\phi_\ell - d)(a e_c) = f^{-1} e_c$, *i.e.*, such that $c\phi_\ell(a) - da = f^{-1}$. Such a a exists in virtue of Proposition 3.

On the other hand, for any $h, k \in \mathscr{H}^\times$, the operator $(\phi_\ell - d) h (\phi_\ell - c) k$ has a basis of solutions given by $(k^{-1} e_c, k^{-1} b e_d)$ where $b \in \mathscr{H}^\times$ is such that $(\phi_\ell - c)(b e_d) = h^{-1} e_d$, *i.e.*, such that $d\phi_\ell(b) - cb = h^{-1}$. Such a b exists in virtue of Proposition 3.

So, if $k^{-1} b = 1$ and $k^{-1} = a$, the operators $(\phi_\ell - c) f (\phi_\ell - d)$ and $(\phi_\ell - d) h (\phi_\ell - c) k$ have the same spaces of solutions, and, hence, coincide up to some left factor $g \in \mathscr{H}^\times$ (indeed, let $g \in \mathscr{H}^\times$ be such that $P := (\phi_\ell - c) f (\phi_\ell - d) - g(\phi_\ell - d) h (\phi_\ell - c) k$ has order 1; then P has at least two $\overline{\mathbb{Q}}$-linearly independent solutions in $\mathscr{U}$ and, hence, is equal to 0).

In order to do so, we first choose a as above (which is necessarily non zero), we then take $k = a^{-1}$, and $h^{-1} = d\phi_\ell(b) - cb$ with $b = k$ (note that $d\phi_\ell(b) - cb$ is nonzero because $b \neq 0$ and $c \neq d$). $\square$

Proof of Proposition 7. It follows from [Roq15a, Theorem 22] that L admits a factorization of the form

$$L = a(\phi_\ell - c_n) f_n^{-1} \cdots (\phi_\ell - c_1) f_1^{-1}$$

where $c_1, \ldots, c_n \in \overline{\mathbb{Q}}^\times$ and $f_1, \ldots, f_n \in \mathscr{P}^\times$. Lemma 8 allows us to permute the factors $(\phi_\ell - c_i)$, up to changing the f_i by other elements of $\mathscr{H}^\times$. □

4. Classification of the Mahler modules over $\mathscr{H}$: Proof of Theorem 1

The proof of Theorem 1 will easily follow from the following lemma.

Lemma 9. *Consider $L \in \mathcal{D}_{\mathscr{H}}$ and $L_1, L_2 \in \mathcal{D}_{\mathscr{H}}$ such that $L = L_1 L_2$ with*

$$L_i = (\phi_\ell - c_{i,n_i}) f_{i,n_i}^{-1} \cdots (\phi_\ell - c_{i,1}) f_{i,1}^{-1}$$

where $c_{i,1}, \ldots, c_{i,n_i} \in \overline{\mathbb{Q}}^\times$ and $f_{i,1}, \ldots, f_{i,n_i} \in \mathscr{H}^\times$. We assume that $c_{1,j} \neq c_{2,k}$ for all $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, n_2\}$. Then,

$$\mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L \cong \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_1 \oplus \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_2.$$

Proof. We are going to prove that the following exact sequence splits

$$0 \to \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_1 \xrightarrow{\cdot L_2} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L \xrightarrow{\pi_1} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_2 \to 0. \tag{2}$$

Using Lemma 8, we see that we also have a decomposition $L = \widetilde{L}_2 \widetilde{L}_1$ where

$$\widetilde{L}_i = (\phi_\ell - c_{i,n_i}) \widetilde{f}_{i,n_i}^{-1} \cdots (\phi_\ell - c_{i,1}) \widetilde{f}_{i,1}^{-1}$$

for some $\widetilde{f}_{i,1}, \ldots, \widetilde{f}_{i,n_i} \in \mathscr{H}^\times$. We consider the corresponding exact sequence

$$0 \to \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} \widetilde{L_2} \xrightarrow{\cdot \widetilde{L}_1} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L \xrightarrow{\pi_2} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} \widetilde{L}_1 \to 0.$$

In order to prove that the exact sequence (2) splits, we have to prove that there exists a submodule N of $\mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L$ such that π_1 induces an isomorphism between N and $\mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_2$. We claim that $N = \mathcal{D}_{\mathscr{H}} \widetilde{L_1} / \mathcal{D}_{\mathscr{H}} L$ has the expected property. In order to prove this claim, it is sufficient to prove that

$$\psi : \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} \widetilde{L_2} \xrightarrow{\cdot \widetilde{L}_1} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L \xrightarrow{\pi_1} \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_2$$

is an isomorphism. Since $\mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} \widetilde{L_2}$ and $\mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} L_2$ have the same dimensions, it is sufficient to prove that ψ is injective. Let $P \in \mathcal{D}_{\mathscr{H}}$ be a representative of an element of the kernel of ψ; by euclidean division, we can assume that the order of P is $<$ to the degree of $\widetilde{L_2}$. So, $P\widetilde{L}_1 \in \mathcal{D}_{\mathscr{H}} L_2$, *i.e.*, $P\widetilde{L}_1 = Q L_2 =: R$ for some $Q \in \mathcal{D}_{\mathscr{H}}$. Assume that $P \neq 0$. Then, there exists a Jordan-Holder filtration of $M = \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} R = \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} P \widetilde{L}_1$ of the form

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{n_1} \subset M_{n_1+1} \subset \cdots \subset M_{n'} = M$$

by submodules of M such, for all $j \in \{0, \ldots, n_1 - 1\}$, $M_{j+1}/M_j \cong \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} (\phi_\ell - c_{1,j})$. We have another Jordan-Holder filtration of $M = \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} R = \mathcal{D}_{\mathscr{H}} / \mathcal{D}_{\mathscr{H}} Q L_2$ of the form

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_{n_2} \subset \cdots \subset M$$

by submodules of N such, for all $j \in \{0, \ldots, n_2 - 1\}$, $N_{j+1}/N_j \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi_\ell - c_{2,j})$. Using the Jordan-Holder theorem and the fact that, for all $j \in \{0, \ldots, n_1 - 1\}$ and $k \in \{0, \ldots, n_2 - 1\}$, M_{j+1}/M_j is not isomorphic to N_{k+1}/N_k, we see that up to isomorphism the list of modules $N_1/N_0, \ldots, N_{n_2}/N_{n_2-1}$, occurs (counting multiplicities) as a sublist of $M_{n_1+1}/M_{n_1}, \ldots, M_{n'}/M_{n'-1}$. It follows that the degree of P, which is equal to the dimension of M/M_{n_1}, is greater than or equal to n_2 = the degree of L_2. Whence a contradiction. Therefore, $P = 0$ as expected. □

Proof of Theorem 1. Straightforward consequence of Proposition 7 and Lemma 9. □

5. A variant and an application

5.1. **Controlling the denominators.** For $K = \mathscr{P}$ or $\mathscr{H}$, we let K_b be the subfield of K made of the $f = \sum_\gamma f_\gamma z^\gamma \in K$ whose coefficients $(f_\gamma)_\gamma$ belong to some finitely generated $\mathbb{Z}$-subalgebra of $\overline{\mathbb{Q}}$.

If we assume that M is a Mahler module over $\mathscr{P}_b$, then, one can check that, in all the previous results of the present paper, the field $\mathscr{H}$ can be replaced by $\mathscr{H}_b$, and that, in particular, the following variant of our main result holds true.

Theorem 10. *Let M be a Mahler module over $\mathscr{P}_b$. Then, the Mahler module $M \otimes_{\mathscr{P}_b} \mathscr{H}_b$ over $\mathscr{H}_b$ obtained from M by extending the scalars to $\mathscr{H}_b$ is isomorphic to a direct sum of modules of the form $\mathcal{D}_{\mathscr{H}_b}/\mathcal{D}_{\mathscr{H}_b}(\phi_\ell - c)^k$ for some $c \in \overline{\mathbb{Q}}^\times$ and $k \in \mathbb{Z}_{\geq 0}$.*

This result can be restated in terms of difference systems as follows. For any $A \in \mathrm{GL}_n(\mathscr{P}_b)$, there exists $F \in \mathrm{GL}_n(\mathscr{H}_b)$ and $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that

$$AF = \phi_\ell(F)A_0.$$

In other words, the Mahler system $\phi_\ell(Y) = AY$ is transformed into $\phi_\ell(Z) = A_0 Z$ by the linear change of variables $Y = FZ$.

5.2. **An application to an analogue of Grothendieck's conjecture.** We shall now indicate briefly how one can use Theorem 10 in order to give a variant of the proof of Theorem 11 below which was first proved in [Roq15b].

Fix $\ell \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 1}$. Consider a Mahler equation of the form

$$a_n(z)f(z^{\ell^n}) + a_{n-1}(z)f(z^{\ell^{n-1}}) + \cdots + a_0(z)f(z) = 0 \tag{3}$$

with coefficients $a_0(z), \ldots, a_n(z) \in \mathbb{Q}(z)$ such that $a_0(z)a_n(z) \neq 0$.

For almost all [1] prime numbers p, we can reduce the coefficients of equation (3) modulo p, and we obtain the equation

$$a_{n,p}(z)f(z^{\ell^n}) + a_{n-1,p}(z)f(z^{\ell^{n-1}}) + \cdots + a_{0,p}(z)f(z) = 0 \tag{4}$$

with coefficients $a_{0,p}(z), \ldots, a_{n,p}(z) \in \mathbb{F}_p(z)$, where $\mathbb{F}_p$ is the field with p elements.

1. "For almost all" means "for all but finitely many".

Theorem 11 ([Roq15b, Theorem 1])**.** *Assume that, for almost all prime p, the equation (4) has n $\mathbb{F}_p$-linearly independent solutions in $\mathbb{F}_p((z))$ algebraic over $\mathbb{F}_p(z)$. Then, the equation (3) has n $\overline{\mathbb{Q}}$-linearly independent solutions in $\overline{\mathbb{Q}}(z)$.*

Proof. We consider the difference system associated to the equation (3) :

$$\phi_\ell(Y) = AY, \text{ with } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix}. \tag{5}$$

According to Section 5.1, there exist $F \in \mathrm{GL}_n(\mathscr{H}_b)$ and $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that

$$AF = \phi_\ell(F)A_0.$$

Let K be a number field containing the entires of A_0 and the entries of the coefficients of F and A. We have, for almost all prime $\mathfrak{p}$ of K,

$$A_\mathfrak{p}F_\mathfrak{p} = \phi_\ell(F_\mathfrak{p})A_{0,\mathfrak{p}},$$

where the subscript $\mathfrak{p}$ means that we have reduced the coefficients modulo $\mathfrak{p}$. Hence, the entries of $A_{0,\mathfrak{p}}$ and of the coefficients of $A_\mathfrak{p}$ and $F_\mathfrak{p}$ belong to the residue field $\kappa_\mathfrak{p}$ of K at $\mathfrak{p}$.

On the other hand, according to [Roq15b, Theorem 2], our hypotheses imply that, for almost all prime p, the equation (4) has n $\overline{\mathbb{Q}}$-linearly independent solutions in $\mathbb{F}_p(z)$. So, for almost all prime $\mathfrak{p}$ of K, there exists $G_\mathfrak{p} \in \mathrm{GL}_n(\kappa_\mathfrak{p}(z))$ such that

$$A_\mathfrak{p}G_\mathfrak{p} = \phi_\ell(G_\mathfrak{p}).$$

Therefore, $H_\mathfrak{p} = G_\mathfrak{p}^{-1}F_\mathfrak{p}$ satisfies

$$H_\mathfrak{p} = \phi_\ell(H_\mathfrak{p})A_{0,\mathfrak{p}}.$$

Setting $H_\mathfrak{p} = \sum_{\gamma\in\mathbb{Q}} H_{\mathfrak{p},\gamma}z^\gamma$ with $H_{\mathfrak{p},\gamma} \in \mathrm{M}_n(\kappa_\mathfrak{p})$, we get $H_{\mathfrak{p},p\gamma} = H_{\mathfrak{p},\gamma}A_{0,\mathfrak{p}}$ for all $\gamma \in \mathbb{Q}$. The support of $H_\mathfrak{p}$ being well-ordered, this implies that $H_{\mathfrak{p},\gamma} = 0$ for all $\gamma \in \mathbb{Q}^\times$ (provided that $A_{0,\mathfrak{p}}$ is invertible, which is true for almost all prime $\mathfrak{p}$ of K). So, $A_{0,\mathfrak{p}} = I_n$.

It follows that $A_0 = I_n$. It follows also that, for almost all prime $\mathfrak{p}$ of K, $F_\mathfrak{p} = G_\mathfrak{p}H_\mathfrak{p} = G_\mathfrak{p}H_{\mathfrak{p},0}$ has entries in $\kappa_\mathfrak{p}(z)$. But, the first line of F is made of n $\overline{\mathbb{Q}}$-linearly independent solutions $(f_1, \ldots, f_n)$ in $\mathscr{H}_b$ of the equation (3). These f_i actually belong to $\overline{\mathbb{Q}}((z))$ because, for almost all prime $\mathfrak{p}$ of K, the reductions modulo $\mathfrak{p}$ of the f_i are elements of $\kappa_\mathfrak{p}(z) \subset \kappa_\mathfrak{p}((z))$. Then, [AB13a, Lemma 5.3] ensures that the f_i actually belong to $\overline{\mathbb{Q}}(z)$. □

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