ARITHMETIC PROPERTIES OF MIRROR MAPS ASSOCIATED WITH GAUSS HYPERGEOMETRIC EQUATIONS

by

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Abstract. — We draw up the list of Gauss hypergeometric differential equations having maximal unipotent monodromy at 0 whose associated mirror map has, up to a simple rescaling, integral Taylor coefficients at 0. We also prove that these equations are characterized by much weaker integrality properties (of p-adic integrality for infinitely many primes p in suitable arithmetic progressions). It turns out that the mirror maps with the above integrality property have modular origins.

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1. Introduction

For any $\alpha, \beta \in \mathbb{C}$, we let $F(\alpha, \beta; z)$ be the hypergeometric series defined by

$$F(\alpha, \beta; z) = {}_2F_1(\alpha, \beta; 1; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k (\beta)_k}{k!^2} z^k$$

where the Pochhammer symbols $(x)_k$ are defined by $(x)_0 = 1$ and, for $k \in \mathbb{N}^*$, $(x)_k = x(x+1)\cdots(x+k-1)$. It satisfies the hypergeometric differential equation with parameters (α, β) given by

$$z(z-1)y''(z) + ((\alpha+\beta+1)z - 1)y'(z) + \alpha\beta y(z) = 0. \tag{1}$$

Assuming that $\alpha, \beta \notin -\mathbb{N}$ and setting

$$G(\alpha, \beta; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k (\beta)_k}{k!^2} \left(2H_k(1) - H_k(\alpha) - H_k(\beta)\right) z^k,$$

where $H_0(x) = 0$ and, for $k \in \mathbb{N}^*$, $H_k(x) = \sum_{k=0}^{n-1} \frac{1}{x+k}$, a basis of the 2-dimensional $\mathbb{C}$-vector space of solutions of (1) is given by

$$F(\alpha, \beta; z), G(\alpha, \beta; z) + \log(z) F(\alpha, \beta; z). \tag{2}$$

Remark 1. — *For further use, note that:*
i) $F(\alpha, \beta; z)$ is the unique solution of (1) in $1 + z\mathbb{C}[[z]]$;
ii) $G(\alpha, \beta; z)$ is the unique element G in $z\mathbb{C}[[z]]$ such that $G(z) + \log(z) F(\alpha, \beta; z)$ is a solution of (1).

In this article, we are interested in arithmetic properties of the Taylor coefficients at 0 of

$$\begin{aligned} \mathcal{Q}(\alpha, \beta; z) &= z \exp\left(\frac{G(\alpha, \beta; z)}{F(\alpha, \beta; z)}\right) \\ &= \exp\left(\frac{G(\alpha, \beta; z) + \log(z) F(\alpha, \beta; z)}{F(\alpha, \beta; z)}\right). \end{aligned}$$

The map $\mathcal{Q}(\alpha, \beta; z)$ will be called the canonical coordinate with parameters (α, β). We will identify $\mathcal{Q}(\alpha, \beta; z)$ with its Taylor expansion at 0 (which belongs to $z + z^2\mathbb{C}[[z]]$).

Before stating our main result, we introduce a notation for sets of primes in some arithmetic progressions which will play a central role in this paper.

Notation 2. — *Consider $\alpha, \beta \in \mathbb{Q}$. Let d be the least common denominator in $\mathbb{N}^*$ of α and β. Let $k_1 < \cdots < k_{\varphi(d)}$ be the integers in $\{1, ..., d-1\}$ prime to d (φ denotes Euler's totient function). For any $j \in \{1, ..., \varphi(d)\}$, we denote by $\mathcal{P}_j$ the set of primes congruent to $k_j \mod d$.*

Note that the $\bigcup_{j \in \{1,...,\varphi(d)\}} \mathcal{P}_j$ coincides with the set of primes p prime to d.

Our main result is :

Theorem 3. — *Let us consider α, β in $\mathbb{Q}\cap]0,1[$. Let d be the least common denominator in $\mathbb{N}^*$ of α and β.*

The following assertions are equivalent:

i) there exists $\kappa \in \mathbb{N}^*$ such that $\kappa^{-1}\mathcal{Q}(\alpha, \beta; \kappa z) \in \mathbb{Z}[[z]]$;

ii) for all $j \in \{1, ..., \varphi(d)\}$, for infinitely many primes p in $\mathcal{P}_j$, $\mathcal{Q}(\alpha, \beta; z) \in \mathbb{Z}_p[[z]]$ (where $\mathbb{Z}_p$ is the ring of p-adic integers);

iii) up to permuting α and β, we have $(\alpha, \beta) \in \mathscr{I}$ where

$$\begin{aligned}\mathscr{I} := \{&(1/2, 1/2), (1/2, 1/3), (1/2, 2/3), (1/2, 1/4), (1/2, 3/4),\\ &(1/2, 1/6), (1/2, 5/6), (1/3, 1/3), (1/3, 2/3), (1/3, 1/6), (1/3, 5/6),\\ &(2/3, 2/3), (2/3, 1/6), (2/3, 5/6), (1/4, 1/4), (1/4, 3/4), (3/4, 3/4),\\ &(1/6, 1/6), (1/6, 5/6), (5/6, 5/6), (1/8, 3/8), (1/8, 5/8), (3/8, 7/8),\\ &(5/8, 7/8), (1/12, 5/12), (1/12, 7/12), (5/12, 11/12), (7/12, 11/12)\}.\end{aligned}$$

The (compositional) inverse of $\mathcal{Q}(\alpha, \beta; z) \in z + z^2\mathbb{C}[[z]]$, will be denoted by

$$\mathcal{Z}(\alpha, \beta; q) \in q + q^2\mathbb{C}[[q]]$$

and will be called the mirror map with parameters (α, β). Note that, for all $\kappa \in \mathbb{N}^*$,

$$\kappa^{-1}\mathcal{Q}(\alpha, \beta; \kappa z) \in \mathbb{Z}[[z]] \Leftrightarrow \kappa^{-1}\mathcal{Z}(\alpha, \beta; \kappa q) \in \mathbb{Z}[[q]]; \tag{3}$$

for a proof see for instance [**14**, Lemma 2]. In particular, Theorem 3 also holds if we replace canonical coordinates by mirror maps.

It is worth mentioning that the canonical coordinates with parameters in $\mathscr{I}$ have modular origins.

Our approach for proving Theorem 3 is based on the work of Dwork in [**5**]. The proof of Theorem 3 is given in § 4 whereas in § 2 and § 3 we give preliminary results.

In § 5, we prove that the hypothesis "$\alpha, \beta \in \mathbb{Q}\cap]0,1[$" is necessary in order to get integrality properties of the Taylor coefficients of $\mathcal{Q}(\alpha, \beta; z)$ as in Theorem 3.

For results concerning the arithmetic properties of mirror maps associated with hypergeometric series whose coefficients are quotients of factorials, we refer to the work of Lian and Yau [**10, 11, 12**], Zudilin [**14**], Krattenthaler and Rivoal [**8, 7**] and Delaygue [**3, 4, 2**]. In our case, the hypothesis "quotient of factorials" would mean that there exist $C > 0$ and integers $e_1, ..., e_r$ and $f_1, ..., f_s$ such that

$$F(\alpha, \beta; z) = \sum_{k=0}^{+\infty} C^k \frac{(e_1 k)! \cdots (e_r k)!}{(f_1 k)! \cdots (f_s k)!} z^k.$$

Using Proposition 2 in Chapter 4 of [**3**], we see that this holds in a finite number of cases, namely, if and only if, up to permuting α and β,

$$(\alpha,\beta) \in \{(1/2,1/2),(1/3,2/3),(2/3,1/3),(1/4,3/4),(3/4,1/4),(1/6,5/6)\}.$$

Note that the (well-known) integrality property of $\mathcal{Z}(1/2,1/2;z)$ (namely, $16^{-1}\mathcal{Z}(1/2,1/2;16z) \in \mathbb{Z}[[z]]$) is used by Y. André in [**1**].

2. A preliminary hypergeometric result

Lemma 4. — *Let us consider $\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbb{Q}\setminus\mathbb{Z}$.*
The following assertions are equivalent:
i) $F(\alpha_1,\beta_1;z) = F(\alpha_2,\beta_2;z)$;
ii) $(\alpha_2,\beta_2) \in \{(\alpha_1,\beta_1),(\beta_1,\alpha_1)\}$.

Proof. — One can for instance apply Proposition 1 in Chapter 4 of [**3**]. □

Proposition 5. — *Let us consider $\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbb{Q}\setminus\mathbb{Z}$.*
The following assertions are equivalent:
i) $\frac{G(\alpha_1,\beta_1;z)}{F(\alpha_1,\beta_1;z)} = \frac{G(\alpha_2,\beta_2;z)}{F(\alpha_2,\beta_2;z)}$;
ii) $(\alpha_2,\beta_2) \in \{(\alpha_1,\beta_1),(\beta_1,\alpha_1),(1-\alpha_1,1-\beta_1),(1-\beta_1,1-\alpha_1)\}$.

Proof. — We denote by $w(\alpha,\beta;z)$ the wronskian determinant of the hypergeometric equation (1) with respect to the basis of solutions (2). It satisfies the first order differential equation

$$y'(z) = -\frac{(\alpha+\beta+1)z-1}{z(z-1)}y(z)$$

so there exists $C_{\alpha,\beta} \in \mathbb{C}^*$ such that

$$w(\alpha,\beta;z) = C_{\alpha,\beta}z^{-1}(1-z)^{-\alpha-\beta}. \tag{4}$$

Assume that i) holds. Then

$$\frac{G(\alpha_1,\beta_1;z)+\log(z)F(\alpha_1,\beta_1;z)}{F(\alpha_1,\beta_1;z)} = \frac{G(\alpha_2,\beta_2;z)+\log(z)F(\alpha_2,\beta_2;z)}{F(\alpha_2,\beta_2;z)}.$$

Differentiating this equation, we get

$$-\frac{w(\alpha_1,\beta_1;z)}{F(\alpha_1,\beta_1;z)^2} = -\frac{w(\alpha_2,\beta_2;z)}{F(\alpha_2,\beta_2;z)^2}$$

so, in virtue of formula (4), there exist $C_1, C_2 \in \mathbb{C}^*$ such that

$$-\frac{C_1 z^{-1}(1-z)^{-\alpha_1-\beta_1}}{F(\alpha_1,\beta_1;z)^2} = -\frac{C_2 z^{-1}(1-z)^{-\alpha_2-\beta_2}}{F(\alpha_2,\beta_2;z)^2}.$$

It follows that there exists $\gamma \in \mathbb{Q}$ such that

$$F(\alpha_1,\beta_1;z) = (1-z)^{\gamma}F(\alpha_2,\beta_2;z).$$

A short calculation then shows that $F(\alpha_2,\beta_2;z)$ is solution of some linear differential equation with rational coefficients of the form

$$z(z-1)y''(z)+*y'(z) + \left(\frac{z\gamma(\gamma-1)+((\alpha_1+\beta_1+1)z-1)\gamma}{1-z}+\alpha_1\beta_1\right)y(z)=0. \tag{5}$$

But it is also solution of the hypergeometric differential equation

$$z(z-1)y''(z)+((\alpha_2+\beta_2+1)z-1)y'(z)+\alpha_2\beta_2 y(z)=0. \tag{6}$$

This equation being irreducible over $\mathbb{C}(z)$ ([**6**]), the coefficients of equations (5) and (6) must be the same. In particular, $\frac{z\gamma(\gamma-1)+((\alpha_1+\beta_1+1)z-1)\gamma}{1-z}$ must be regular at $z=1$; this entails that $\gamma=0$ or $\gamma=1-(\alpha_1+\beta_1)$. If $\gamma=0$ then $F(\alpha_1,\beta_1;z)=F(\alpha_2,\beta_2;z)$ and hence, in virtue of Lemma 4, $(\alpha_2,\beta_2)\in\{(\alpha_1,\beta_1),(\beta_1,\alpha_1)\}$. If $\gamma=1-(\alpha_1+\beta_1)$ then

$$F(\alpha_1,\beta_1;z)=(1-z)^{1-(\alpha_1+\beta_1)}F(\alpha_2,\beta_2;z).$$

Since (formula (1.3.15) in [**13**])

$$F(\alpha_1,\beta_1;z)=(1-z)^{1-(\alpha_1+\beta_1)}F(1-\alpha_1,1-\beta_1;z),$$

we get $F(\alpha_2,\beta_2;z) = F(1-\alpha_1,1-\beta_1;z)$ and Lemma 4 ensures that $(\alpha_2,\beta_2)\in\{(1-\alpha_1,1-\beta_1),(1-\beta_1,1-\alpha_1)\}$.

We leave the converse statement to the reader. □

3. Dwork's map $\alpha\mapsto\alpha' =: \mathfrak{D}_p(a)$: remainder and complements

For any prime number p, for any p-adic integer α in $\mathbb{Q}$, we denote by $\mathfrak{D}_p(\alpha)$ the unique p-adic integer in $\mathbb{Q}$ such that

$$p\mathfrak{D}_p(\alpha)-\alpha\in\{0,...,p-1\}.$$

The operator $\alpha\mapsto\mathfrak{D}_p(\alpha)$ was used by Dwork in [**5**] (and denoted by $\alpha\mapsto\alpha'$).

Proposition 6. — *Assume that $\alpha\in\mathbb{Q}\cap]0,1[$. Let $a,m\in\mathbb{N}^*$ be such that $\alpha=a/m$ and* $\gcd(a,m)=1$ *(so* $\gcd(m,p)=1$*). Then*

$$\mathfrak{D}_p(\alpha)=\frac{x}{m}\in\mathbb{Q}\cap]0,1[$$

where x is the unique integer in $\{1,...,m-1\}$ such that $px\equiv a \mod m$.

In particular, $\mathfrak{D}_p(\alpha)$ does not depend on the prime p coprime to m in a fixed arithmetic progression $k+\mathbb{N}m$.

Proof. — Indeed, we have $p\frac{x}{m}-\alpha=\frac{px-a}{m}\in\mathbb{Z}$. Moreover, we have

$$-1<-\alpha<p\frac{x}{m}-\alpha=\frac{px-a}{m}\le\frac{p(m-1)-a}{m}=p-\frac{p+a}{m}<p.$$

Therefore, $\mathfrak{D}_p(\alpha) = \frac{x}{m}$. □

We will need the following properties.

Lemma 7. — *Let p be a prime number and consider p-adic integers α, β in $\mathbb{Q}$ such that*

$$(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) \in \{(\alpha,\beta), (\beta,\alpha), (1-\alpha, 1-\beta), (1-\beta, 1-\alpha)\}.$$

Let $m, n \in \mathbb{N}^*$ and $a, b \in \mathbb{Z}$ be such that $\alpha = a/m$ and $\beta = b/n$ with $\gcd(a,m) = \gcd(b,n) = 1$. Let $d = \operatorname{lcm}(m,n)$ be the least common denominator in $\mathbb{N}^*$ of α and β. Then $p^2 = 1 \mod d$. Moreover, if $m \neq n$ then $p = \pm 1 \mod d$.

Proof. — Let us first assume that $(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) = (\alpha, \beta)$. This implies that $p\alpha - \alpha = (p-1)\alpha$ belongs to $\mathbb{Z}$. Therefore, $p = 1 \mod m$. Similarly, $p = 1 \mod n$.

Assume that $(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) = (\beta, \alpha)$. Then $p\beta - \alpha$ and $p\alpha - \beta$ belong to $\mathbb{Z}$. This implies $m = n$, $a = pb \mod m$ and $b = pa \mod m$, so $b = p^2 b \mod m$ and hence $p^2 = 1 \mod m$.

Assume that $(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) = (1-\alpha, 1-\beta)$. Then $p(1-\alpha) - \alpha = -(p+1)\alpha + p$ belongs to $\mathbb{Z}$. This implies that $p = -1 \mod m$. Similarly, $p = -1 \mod n$.

Assume that $(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) = (1-\beta, 1-\alpha)$. Then $p(1-\beta) - \alpha$ and $p(1-\alpha) - \beta$ belong to $\mathbb{Z}$. It follows that $m = n$, $bp = -a \mod m$ and $ap = -b \mod m$, so $b = p^2 b \mod m$ and hence $p^2 = 1 \mod m$. □

Proposition 8. — *Let us consider α, β in $\mathbb{Q} \cap]0,1[$. Let d be the least common denominator in $\mathbb{N}^*$ of α and β.*

The following assertions are equivalent:

i) for all $j \in \{1, ..., \varphi(d)\}$, there exists a prime p in $\mathcal{P}_j$ such that

$$(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) \in \{(\alpha,\beta), (\beta,\alpha), (1-\alpha, 1-\beta), (1-\beta, 1-\alpha)\};$$

ii) for all prime p prime to d,

$$(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) \in \{(\alpha,\beta), (\beta,\alpha), (1-\alpha, 1-\beta), (1-\beta, 1-\alpha)\};$$

iii) up to permuting α and β, $(\alpha, \beta) \in \mathscr{I}$ (defined in Theorem 3).

Proof. — The equivalence between assertions i) and ii) follows from the fact that $\mathfrak{D}_p(\alpha)$ and $\mathfrak{D}_p(\beta)$ do not depend on $p \in \mathcal{P}_j$.

We now prove that ii) implies iii). So we consider (α, β) satisfying ii). Let $m, n \in \mathbb{N}^*$ and $a, b \in \mathbb{Z}$ be such that $\alpha = a/m$ and $\beta = b/n$ with $\gcd(a,m) = \gcd(b,n) = 1$. So $d = \operatorname{lcm}(m,n)$.

Let us first assume that $m \neq n$. Lemma 7 ensures that, for all prime p prime to d, we have $p = \pm 1 \mod d$. Using Dirichlet's theorem, we get $\varphi(d) \in \{1, 2\}$ and hence $d \in \{2, 3, 4, 6\}$. Therefore, up to permuting m and n, we see that

(m,n) belongs to $\{(2,3),(2,4),(2,6),(3,6)\}$. Up to permuting α and β, we get that (α,β) belongs to

$$\{(1/2,1/3),(1/2,2/3),(1/2,1/4),(1/2,3/4),\\ (1/2,1/6),(1/2,5/6),(1/3,1/6),(1/3,5/6),(2/3,1/6),(2/3,5/6)\}.$$

Assume that $m=n$. Lemma 7 ensures that, for all prime p prime to m, we have $p^2=1 \mod m$. Hence, any element of the group $(\mathbb{Z}/m\mathbb{Z})^\times$ has order 1 or 2. The well known structure of $(\mathbb{Z}/m\mathbb{Z})^\times$ yields $m\in\{2,4,8,3,6,12,24\}$. Now, the fact that iii) is satisfied follows from the following observations:

- if $\alpha=1/8$ then $\beta\in\{3/8,5/8\}$ because $\mathfrak{D}_3(1/8)=3/8\neq\alpha,1-\alpha$;
- if $\alpha=3/8$ then $\beta\in\{1/8,7/8\}$ because $\mathfrak{D}_3(3/8)=1/8\neq\alpha,1-\alpha$;
- if $\alpha=5/8$ then $\beta\in\{1/8,7/8\}$ because $\mathfrak{D}_5(5/8)=1/8\neq\alpha,1-\alpha$;
- if $\alpha=7/8$ then $\beta\in\{5/8,3/8\}$ because $\mathfrak{D}_3(7/8)=5/8\neq\alpha,1-\alpha$;
- iff $\alpha=1/12$ then $\beta\in\{5/12,7/12\}$ because $\mathfrak{D}_5(1/12)=5/12\neq\alpha,1-\alpha$;
- if $\alpha=5/12$ then $\beta\in\{1/12,11/12\}$ because $\mathfrak{D}_5(5/12)=1/12\neq\alpha,1-\alpha$;
- if $\alpha=7/12$ then $\beta\in\{1/12,11/12\}$ because $\mathfrak{D}_7(7/12)=1/12\neq\alpha,1-\alpha$;
- if $\alpha=11/12$ then $\beta\in\{5/12,7/12\}$ because $\mathfrak{D}_5(11/12)=7/12\neq\alpha,1-\alpha$;
- direct calculations show that $m=n=24$ is excluded.

We leave the proof of iii) $\Rightarrow$ i) to the reader (direct calculations). □

4. Proof of Theorem 3

The fact that i) implies ii) is obvious (using Dirichlet theorem).

4.1. Proof of ii) ⇒ iii). — Assume that ii) holds. On the one hand, Dieudonné-Dwork's Lemma (Lemma 5 in [**14**] for instance) ensures that, for all $j\in\{1,...,\varphi(d)\}$, for infinitely many primes p in $\mathcal{P}_j$,

$$\frac{G\left(\alpha,\beta;z^p\right)}{F\left(\alpha,\beta;z^p\right)}=p\frac{G\left(\alpha,\beta;z\right)}{F\left(\alpha,\beta;z\right)} \mod p\mathbb{Z}_p[[z]].$$

On the other hand, Dwork's Theorem 4.1 in [**5**] ensures that, for all prime p prime to d,

$$\frac{G\left(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta);z^p\right)}{F\left(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta);z^p\right)}=p\frac{G\left(\alpha,\beta;z\right)}{F\left(\alpha,\beta;z\right)} \mod p\mathbb{Z}_p[[z]].$$

Consequently, for all $j\in\{1,...,\varphi(d)\}$, for infinitely many primes p in $\mathcal{P}_j$,

$$\frac{G\left(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta);z\right)}{F\left(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta);z\right)}=\frac{G\left(\alpha,\beta;z\right)}{F\left(\alpha,\beta;z\right)} \mod p\mathbb{Z}_p[[z]].$$

But $\mathfrak{D}_p(\alpha)$ and $\mathfrak{D}_p(\beta)$ do not depend on $p \in \mathcal{P}_j$. So, for all $j \in \{1, ..., \varphi(d)\}$, for infinitely many primes $p \in \mathcal{P}_j$,

$$\frac{G\left(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta); z\right)}{F\left(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta); z\right)} = \frac{G\left(\alpha, \beta; z\right)}{F\left(\alpha, \beta; z\right)}.$$

In virtue of Proposition 5, we get that, for all $j \in \{1, ..., \varphi(d)\}$, for infinitely many primes $p \in \mathcal{P}_j$,

$$(\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)) \in \{(\alpha, \beta), (\beta, \alpha), (1-\alpha, 1-\beta), (1-\beta, 1-\alpha)\}.$$

Proposition 8 ensures that iii) holds.

4.2. Proof of iii) $\Rightarrow$ i). — The proof of iii) $\Rightarrow$ i) follows easily form Dieudonné-Dwork's Lemma and from Dwork's congruences already used at the beginning of § 4.1. (Indeed, it is easily seen that, for all prime p, the growth of the p-adic valuations of the coefficients of $\mathcal{Q}\left(\alpha, \beta; z\right)$ is at most linear. Therefore, iii) $\Rightarrow$ i) is a consequence of Dieudonné-Dwork's Lemma and Dwork's congruences which show that $\mathcal{Q}\left(\alpha, \beta; z\right)$ belongs to $\mathbb{Z}_p[[z]]$ for almost all primes p if $(\alpha, \beta) \in \mathscr{I}$.) However, we shall give another proof which also shows the modular origin of the canonical coordinates with parameters $(\alpha, \beta) \in \mathscr{I}$.

The following lemma shows that it is sufficient to treat the cases that

$$(\alpha, \beta) \in \{(1/2, 1/2), (1/2, 2/3), (1/2, 1/4), (1/2, 1/6), (1/3, 2/3), \\ (1/3, 1/6), (1/4, 3/4), (1/6, 5/6), (1/8, 3/8), (1/12, 5/12)\}.$$

Lemma 9. — *We have*

$$\mathcal{Q}\left(\alpha, \beta; z\right) = -\mathcal{Q}\left(\alpha, 1-\beta; \frac{z}{z-1}\right)$$

and hence

$$\mathcal{Z}\left(\alpha, \beta; q\right) = \frac{\mathcal{Z}\left(\alpha, 1-\beta; -q\right)}{\mathcal{Z}\left(\alpha, 1-\beta; -q\right) - 1}.$$

Proof. — A direct calculation shows that $y(z)$ is a solution of the hypergeometric equation with parameters $(\alpha, 1-\beta)$ if and only if $(1-z)^{-\alpha} y(\frac{z}{z-1})$ is solution of the hypergeometric equation with parameters (α, β). It follows that

$$(1-z)^{-\alpha} F\left(\alpha, 1-\beta; \frac{z}{z-1}\right)$$

and

$$(1-z)^{-\alpha}\left(G\left(\alpha, 1-\beta; \frac{z}{z-1}\right) + \log\left(\frac{z}{1-z}\right) F\left(\alpha, 1-\beta; \frac{z}{z-1}\right)\right)$$

form a basis of the $\mathbb{C}$-vector space of solutions of the hypergeometric equation with parameters (α, β). Using Remark 1, it is easily seen that:

$$F(\alpha, \beta; z) = (1-z)^{-\alpha} F\left(\alpha, 1-\beta; \frac{z}{z-1}\right) \tag{7}$$

and

$$G(\alpha, \beta; z) = (1-z)^{-\alpha}\left(G\left(\alpha, 1-\beta; \frac{z}{z-1}\right) - \log(1-z)\, F\left(\alpha, 1-\beta; \frac{z}{z-1}\right)\right).$$

(Note that formula (7) is classical and known as Pfaff transformation.) Therefore,

$$\mathcal{Q}(\alpha, \beta; z) = -\mathcal{Q}\left(\alpha, 1-\beta; \frac{z}{z-1}\right).$$

□

We introduce Dedekind's η function defined by

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

and Dedekind-Klein's J-invariant defined by

$$J(q) = \frac{Q^3(q)}{Q^3(q) - R^2(q)}$$

where Q and R (with Ramanujan's notations) are the Eisenstein series defined by

$$Q(q) = 1 + 240 \sum_{n=1}^{+\infty} \sigma_3(n) q^n, \qquad R(q) = 1 - 504 \sum_{n=1}^{+\infty} \sigma_5(n) q^n$$

with $\sigma_k(n) = \sum_{d|n} d^k$.

The following formulas show that the desired integrality property of $\mathcal{Z}(\alpha, \beta; q)$ holds if

$$(\alpha, \beta) \in \{(1/2, 1/2), (1/3, 2/3), (1/3, 1/6), (1/4, 3/4), (1/6, 5/6), (1/8, 3/8), (1/12, 5/12)\}.$$

We have

$$16^{-1} \mathcal{Z}(1/2, 1/2; 16q) = e^{\frac{i\pi}{3}} \frac{\eta^8(q^4)}{\eta^8(-q)} \tag{8}$$

$$64^{-1} \mathcal{Z}(1/4, 3/4; 64q) = \frac{1}{64 + \frac{\eta^{24}(q)}{\eta^{24}(q^2)}} \tag{9}$$

$$432^{-1}\mathcal{Z}\left(1/6,5/6;432q\right)=\frac{1}{864}\left(1-\sqrt{\frac{J(q)-1}{J(q)}}\right) \tag{10}$$

$$108^{-1}\mathcal{Z}\left(1/3,1/6;108q\right)=\frac{\eta^{12}(q)}{\eta^{12}(q^3)}\frac{1}{\left(27+\frac{\eta^{12}(q)}{\eta^{12}(q^3)}\right)^2} \tag{11}$$

$$256^{-1}\mathcal{Z}\left(1/8,3/8;256q\right)=\frac{\eta^{24}(q)}{\eta^{24}(q^2)}\frac{1}{\left(64+\frac{\eta^{24}(q)}{\eta^{24}(q^2)}\right)^2} \tag{12}$$

$$1728^{-1}\mathcal{Z}\left(1/12,5/12;1728q\right)=\frac{1}{1728J(q)} \tag{13}$$

$$27^{-1}\mathcal{Z}\left(1/3,2/3;27q\right)=\frac{1}{27+\frac{\eta^{12}(q)}{\eta^{12}(q^3)}}. \tag{14}$$

For (8) see [**9**, §9, formula (9.8)], for (9) see [**9**, §9, formula (9.6)], for (10) see [**9**, §9, after formula (9.7)], for (11) see [**9**, §9, after formula (9.10)], for (12) see [**9**, §9, formula (9.13) together with (9.6)], for (13) see [**9**, §9, Case $N=(2,6)$], the proof of (14) is similar to the proof of the case $(1/8,3/8)$ in loc. cit. for instance.

The fact that the expected integrality property of $\mathcal{Z}\left(\alpha,\beta;q\right)$ also holds in the remaining cases, i.e. for $(\alpha,\beta)\in\{(1/2,2/3),(1/2,1/4),(1/2,1/6)\}$, is a direct consequence of the following lemma applied to $\beta\in\{2/3,1/4,1/6\}$ combined with the previous formulas (11), (12) and (13); the details are left to the reader.

Lemma 10. — *We have*

$$\mathcal{Q}\left(1/2,\beta;z\right)=2\sqrt{-\mathcal{Q}\left(\frac{1-\beta}{2},\frac{\beta}{2};\frac{z^2}{4z-4}\right)}$$

and hence

$$\mathcal{Z}\left(1/2,\beta;q\right)=2\mathcal{Z}\left(\frac{1-\beta}{2},\frac{\beta}{2};-q^2/4\right) + 2\sqrt{\mathcal{Z}\left(\frac{1-\beta}{2},\frac{\beta}{2};-q^2/4\right)^2-\mathcal{Z}\left(\frac{1-\beta}{2},\frac{\beta}{2};-q^2/4\right)}.$$

Proof. — A direct calculation shows that $y(z)$ is a solution of the hypergeometric equation with parameters $((1-\beta)/2,\beta/2)$ if and only if $(1-z)^{\beta/2}y(\frac{z^2}{4z-4})$

is solution of the hypergeometric equation with parameters $(1/2, \beta)$. It follows that

$$(1-z)^{\frac{\beta}{2}} F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)$$

and

$$(1-z)^{\frac{\beta}{2}} G\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right) \\ + \log\left(\frac{z^2}{1-z}\right)(1-z)^{\frac{\beta}{2}} F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)$$

form a basis of the $\mathbb{C}$-vector space of solutions of the hypergeometric equation with parameters $(1/2, \beta)$. Consequently:

$$F\left(1/2, \beta; z\right) = (1-z)^{\frac{\beta}{2}} F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right) \tag{15}$$

and

$$G\left(1/2, \beta; z\right) = \frac{1}{2}(1-z)^{\frac{\beta}{2}} G\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right) \\ - \log\left(1-z\right)^{\frac{1}{2}}(1-z)^{\frac{\beta}{2}} F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right).$$

(Note that formula (15) is classical.) Therefore,

$$\begin{aligned} \mathcal{Q}\left(1/2, \beta; z\right) &= z \exp\left(\frac{G\left(1/2, \beta; z\right)}{F\left(1/2, \beta; z\right)}\right) \\ &= \frac{z}{(1-z)^{1/2}} \exp\left(\frac{\frac{1}{2} G\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)}{F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)}\right) \\ &= 2\sqrt{\frac{z^2}{4(1-z)} \exp\left(\frac{G\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)}{F\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)}\right)} \\ &= 2\sqrt{-\mathcal{Q}\left(\frac{1-\beta}{2}, \frac{\beta}{2}; \frac{z^2}{4z-4}\right)}. \end{aligned}$$

□

5. Integrality properties of the Taylor coefficients of $\mathcal{Q}(\alpha,\beta;z)$ and the hypothesis "$\alpha,\beta \in \mathbb{Q}\cap]0,1[$"

Lemma 11. — *Consider $\alpha \in \mathbb{Q}\setminus\mathbb{Z}$. Let $a \in \mathbb{Z}$ and $m \in \mathbb{N}^*$ be such that $\alpha = a/m$. Then, for any prime $p > |a|$ prime to m, we have*

$$\mathfrak{D}_p(\alpha) = \frac{x}{m} \in \mathbb{Q}\cap]0,1[$$

where x is the unique element in $\{1,...,m-1\}$ such that $px \equiv a \mod m$.

In particular, $\mathfrak{D}_p(\alpha)$ does not depend on the prime $p > |a|$ coprime to m in a fixed arithmetic progression $k + \mathbb{N}m$.

Proof. — Indeed, we have $p\frac{x}{m} - \alpha = \frac{px-a}{m} \in \mathbb{Z}$. Moreover, we have

$$\frac{p-a}{m} \leq p\frac{x}{m} - \alpha = \frac{px-a}{m} \leq \frac{p(m-1)-a}{m} = p - \frac{p+a}{m}$$

and the fact that $p > |a|$ ensures that $0 < \frac{p-a}{m}$ and $p - \frac{p+a}{m} < p$. Therefore, $\mathfrak{D}_p(\alpha) = \frac{x}{m}$. □

Proposition 12. — *Assume that $\alpha,\beta \in \mathbb{Q}\setminus\mathbb{Z}$ are such that, for infinitely many primes p, we have $\mathcal{Q}(\alpha,\beta;z) \in \mathbb{Z}_p[[z]]$. Then $\alpha,\beta \in \mathbb{Q}\cap]0,1[$.*

Proof. — We use the notations $(d,\mathcal{P}_j,...)$ of § 1. Let $j \in \{1,...,\varphi(d)\}$ be such that, for infinitely many primes p in $\mathcal{P}_j$, we have $\mathcal{Q}(\alpha,\beta;z) \in \mathbb{Z}_p[[z]]$. Arguing as in § 4.1 (using the fact that $\mathfrak{D}_p(\alpha)$ does not depend on the prime p large enough in $\mathcal{P}_j$ in virtue of Lemma 11), we see that, for infinitely many primes p in $\mathcal{P}_j$,

$$(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta)) \in \{(\alpha,\beta),(\beta,\alpha),(1-\alpha,1-\beta),(1-\beta,1-\alpha)\}.$$

Lemma 11 ensures that, for all prime p large enough in $\mathcal{P}_j$, we have

$$(\mathfrak{D}_p(\alpha),\mathfrak{D}_p(\beta)) \in (\mathbb{Q}\cap]0,1[) \times (\mathbb{Q}\cap]0,1[),$$

whence the result. □

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