BIRKHOFF MATRICES, RESIDUES AND RIGIDITY FOR q-DIFFERENCE EQUATIONS

by

Julien Roques

Abstract. — We introduce and give numerical characterizations of two notions of rigidity for a class of regular singular q-difference equations. A special attention is devoted to the generalized q-hypergeometric equations : we show their rigidity and we proceed with a detailed "monodromic" study of these equations.

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In this paper, q is a complex number such that $0 < |q| < 1$.

1. Introduction - Organization of the paper

1.1. Introduction. — There is a Riemann-Hilbert equivalence, due to G. D. Birkhoff [**2**], between the category $\mathscr{E}$ of regular singular q-difference systems and a category $\mathscr{C}$ of "connection data" (see section 3). In this equivalence, a $n \times n$ regular singular q-difference system

$$Y(qz) = A(z)Y(z), \quad A \in \mathrm{GL}_n(\mathbb{C}(z))$$

corresponds to some triple

$$C_A = (A^{(0)}, M, A^{(\infty)}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^*)) \times \mathrm{GL}_n(\mathbb{C})$$

such that $M(qz)A^{(0)} = A^{(\infty)}M(z)$ where $\mathcal{M}$ denotes the sheaf over $\mathbb{P}^1_{\mathbb{C}}$ of meromorphic functions (this is not exactly Birkhoff's original equivalence, but a modified version introduced by J. Sauloy in [**14**]).

In the present paper, we introduce and study two notions of rigidity based on the residues of the Birkhoff matrices M appearing in the above equivalence. The systematic use of residues is partially motivated by our work in [**12**]. Note that the notion of residue is fundamental for the Galoisian use of the q-analogue of Stokes' phenomenon by J.-P. Ramis and J. Sauloy in [**10, 11**].

Actually, we will not work in the whole category $\mathscr{C}$ but in its full subcategory $\mathscr{C}_c$ of "completely regular singular connection data" made of the triples $(A^{(0)}, M, A^{(\infty)})$ such that M has at most simple poles on $\mathbb{C}^*$. Recall that the usual notion of "regular singular" for q-difference systems is based only on singularities at 0 and ∞, letting aside intermediate singularities; here, we also pay attention to these intermediate singularities.

We first introduce two notions of local isomorphy for the objects of $\mathscr{C}_c$. We shall only give here the heuristic ideas and we refer to section 4 for the formal definitions. We classically think about C_A as the gluing of the local data $A^{(0)}$ and $A^{(\infty)}$ via a "global connection data" namely Birkhoff matrix M; here, we extract from Birkhoff matrix itself local data (the places of localisation "live" on the complex torus $\mathbb{C}^*/q^{\mathbb{Z}}$). The first notion, referred to as *local isomorphy*, relies on the idea that C_A is the gluing of $A^{(0)}$ viewed as a local data at 0, $A^{(\infty)}$ viewed as a local data at ∞ and, for all $i \in \{1,...,m\}$, $\mathrm{Res}_{s_i} M$ viewed as a "local connection data" from $A^{(0)}$ to $A^{(\infty)}$ where we have denoted by $s_1,...,s_m$ the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$ ($(q^k,z) \mapsto q^k z$). The second notion, referred to as *weak local isomorphy*, is similar except that we consider the residues $\mathrm{Res}_{s_i} M$ independently of $A^{(0)}$ and $A^{(\infty)}$.

The corresponding notions of rigidity for the objects of $\mathscr{C}_c$ are the following : an object C of $\mathscr{C}_c$ is *rigid* (resp. *strongly rigid*) if and only if any object C' of $\mathscr{C}_c$ locally isomorphic (resp. weakly locally isomorphic) to C is actually isomorphic to C.

In sections 5.4 and 5.5, we give numerical characterizations of these notions of rigidity under the hypothesis that $q^{\mathbb{Z}}\,\mathrm{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\,\mathrm{Sp}(A^{(\infty)}) = \emptyset$. In what follows, we denote, for all $A \in \mathrm{M}_n(\mathbb{C})$, by $\mathfrak{z}(A)$ the centralizer of A in $\mathrm{M}_n(\mathbb{C})$ and, for all $R \in \mathrm{M}_n(\mathbb{C})$, by $\mathfrak{g}(R)$ and $\mathfrak{h}(R)$ the complex Lie sub-algebras of $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ defined by

$$\mathfrak{g}(R) = \{(X,Y) \in \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) \mid YR = RX\}$$

and

$$\mathfrak{h}(R) = \{(X,Y) \in \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \mid YR = RX\}.$$

Moreover, an object $(A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}_c$ is said to be normalized if the eigenvalues of both $A^{(0)}$ and $A^{(\infty)}$ belong to $\{c \in \mathbb{C}^* \mid |q| \leq |c| < 1\}$.

Theorem **(Numerical characterization of rigidity; Theorem 28)**

Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}_c$ of size n such that $q^{\mathbb{Z}}\,\mathrm{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\,\mathrm{Sp}(A^{(\infty)}) = \emptyset$. Let $s_1,...,s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$ and set, for all $i \in \{1,...,m\}$, $\mathrm{Res}_{s_i} M = R_i$. Then :

i) $\sum_{i=1}^m \dim \mathfrak{g}(R_i) \leq (m-1)(\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)})) + 1$*;*

ii) C is rigid if and only if the inequality in i) is an equality.

Theorem **(Numerical characterization of strong rigidity; Theorem 29)**

We use the same notations and hypotheses as above. Then :

i) $\sum_{i=1}^m \dim \mathfrak{h}(R_i) \leq 2mn^2 - (\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)})) + 1$*;*

ii) C is strongly rigid if and only if the inequality in i) is an equality.

In section 6.2, a special attention is devoted to the generalized q-hypergeometric equations. We prove the following results concerning the rigidity and the "monodromic" description of the generalized q-hypergeometric equations. In what follows, we will say that an object C of $\mathscr{C}$ "comes from" a generalized q-hypergeometric equation if it corresponds to a generalized q-hypergeometric equation in Birkhoff's correspondance mentionned at the beginning of the paper.

Theorem **(Rigidity and "monodromic" characterization of the q-hypergeometrics; Theorem 37)**

i) Any irreducible object of $\mathscr{C}$ coming from a generalized q-hypergeometric equation is strongly rigid.

ii) A normalized irreducible object $C = (A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}$ such that $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)}) = \emptyset$ comes from a generalized q-hypergeometric equation if and only if the following properties hold :

a) M has at most simple poles on $\mathbb{C}^*$;

b) the poles of M in $\mathbb{C}^*$ belong to some q-logarithmic spiral $q^{\mathbb{Z}}z_0 \subset \mathbb{C}^*$;

c) $\operatorname{Res}_{z_0} M$ has rank one.

We also have the following characterization :

Proposition **(Proposition 38)**. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}_c$ such that $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)}) = \emptyset$. If both $A^{(0)}$ and $A^{(\infty)}$ have n distinct eigenvalues then the following properties are equivalent :*

- C is strongly rigid;

- C comes from a generalized q-hypergeometric equation.

Moreover, for fixed $A^{(0)}$ and $A^{(\infty)}$, we describe explicitly the residues $\operatorname{Res}_{z_0} M$ occurring for some M such that $(A^{(0)}, M, A^{(\infty)})$ comes from a (normalized irreducible) generalized q-hypergeometric equation with fixed parameters; see Theorem 40 in section 6.3.

It is usually considered that the theory of q-difference equations started with the work of Euler on problems of combinatorics. It was followed by the work of Gauss, Jacobi, Heine, Ramanujan, etc. The first systematic study of the q-hypergeometric series and equations is due to Heine [**7**]. We refer the reader to the classical books of Fine [**5**], Gasper and Rahman [**6**] and Slater [**15**] for additional informations. The whole paper is influenced by the theory of differential equations. The generalized q-hypergeometric equations are quantizations of the so-called generalized hypergeometric equations. The hypergeometric theory goes back at least as far as Euler. The hypergeometric equations were studied in the 19th century by Gauss (who gave the first full systematic treatment), Riemann, Schwarz, Klein, etc. More recently, the monodromy of the generalized hypergeometric equations has been studied by Beukers and Heckman [**1**]. Recall that these equations are rigid. The usual notion of rigidity for local systems has attracted the attention of many mathematicians. We refer in particular to Katz' book [**8**]. The first chapter of this book provides a numerical characterization of rigidity (Theorem 1.1.2 in *loc. cit.*). This result was generalized to any field (instead of $\mathbb{C}$) by Strambach and Völklein in [**16**]. These results were a source of inspiration for our numerical characterizations of rigidity and of strong rigidity for q-difference equations.

We hope that this paper will serve as basis for further research. We shall now raise a couple of questions that deserve special attention.

Katz gave in [**8**] an algorithmic proof of the fact that any irreducible rigid local system can be build up from a rank one local system by applying a finite sequence of middle convolution and middle tensor operations. Is there a convenient q-analogue of the middle convolution operation and of Katz's algorithm? It is worth mentioning that a purely algebraic convolution functor was introduced by Detweiller and Reiter in [**4**]. It shares many properties with Katz's middle convolution functor and can be used to reprove many of Katz's results. After the present paper was completed, we became aware of the fact that the q-middle convolution is the subject of the ongoing PhD thesis work of Yamaguchi, to whom we refer the interested reader. Last, we refer to the work of Bloch and Esnault in [**3**] for Fourier transforms and rigidity; unfortunately, q-Fourier transform theory is still in its infancy.

Consider some q-deformations $Y(qz) = A_q(z)Y(z)$ of a given differential system $Y'(z) = A(z)Y(z)$ on $\mathbb{P}^1_{\mathbb{C}}$.[(1)] Is there is a link between rigidity of $Y(qz) = A_q(z)Y(z)$ and rigidity of $Y'(z) = A(z)Y(z)$? More precisely: Is $Y'(z) = A(z)Y(z)$ rigid if the deformations $Y(qz) = A_q(z)Y(z)$ are rigid? (The converse statement seems hopeless because rigidity is a "closed" condition.) This is connected with the problem of finding relations between the local data (residues of Birkhoff matrices) associated with $Y(qz) = A_q(z)Y(z)$ and the usual monodromies of $Y'(z) = A(z)Y(z)$. Is it possible to derive the later from the former? It is worth mentioning that, thanks to the work of Sauloy in [**13**], it is possible to recover the monodromies of $Y'(z) = A(z)Y(z)$ from the connection data associated with $Y(qz) = A_q(z)Y(z)$ as q tends to 1. We do not enter into details here; we simply emphasize that Sauloy's method relies on the values of Birkhoff matrices out of a singular locus. In our situation, one of the difficulties lies in the fact that the local data we consider precisely come from the behavior of Birkhoff matrices at some points of this singular locus.

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1. Roughly speaking, this means that $\frac{A_q(z)-I_n}{(q-1)z}$ tends to $A(z)$ as q tends to 1.

1.2. Organization of the paper. — In section 2, we introduce general notations. In section 3, we recall a Riemann-Hilbert correspondence for regular singular q-difference systems. In section 4, we introduce the categories of completely regular singular connection data and equations. We then introduce the notions of local isomorphy and weak local isomorphy and also the corresponding notions of rigidity and strong rigidity. In section 5, we give numerical characterizations of rigidity and strong rigidity. In section 6.1, we show the strong rigidity and we give simple "monodromic" characterizations of the generalized q-hypergeometric equations. We also describe in detail the residues coming from generalized q-hypergeometric connection data with fixed local data at 0 and ∞.

2. General notations

We will denote by $\mathcal{O}$ (resp. $\mathcal{M}$) the sheaf over $\mathbb{P}^1_{\mathbb{C}}$ of analytic (resp. meromorphic) functions.

We will denote by $\mathbb{C}\{z-a\}$ the local ring $\mathcal{O}_a$ of germs of analytic functions at $a \in \mathbb{C}$ and by $\mathbb{C}(\{z-a\})$ its field of fractions.

We will denote by $\mathbb{C}[[z]]$ the local ring of formal power series with coefficients in $\mathbb{C}$ and by $\mathbb{C}((z))$ its field of fractions.

We will denote by σ_q the q-dilatation operator ($\sigma_q y(z) = y(qz)$).

We will denote by $\mathrm{Sp}(M)$ (resp. $\mathrm{Sp}(f)$) the set of complex eigenvalues of a matrix $M \in \mathrm{M}_n(\mathbb{C})$ (resp. of an endomorphism f of some finite dimensional $\mathbb{C}$-vector space).

We will denote by $\mathrm{rk}(M)$ (resp. $\mathrm{rk}(f)$) the rank of a matrix $M \in \mathrm{M}_n(\mathbb{C})$ (resp. of an endomorphism f of some finite dimensional $\mathbb{C}$-vector space).

For any matrix-valued meromorphic function $M \in \mathrm{M}_n(\mathcal{M}(\Omega))$ (Ω is an open subset of $\mathbb{C}$), we denote by $\mathrm{Res}_u(M) \in \mathrm{M}_n(\mathbb{C})$ the residue of M at $u \in \Omega$.

3. Riemann-Hilbert correspondence for regular singular q-difference systems

This section follows the presentation of J. Sauloy in [**13, 14**]. We also refer the reader to M. van der Put and M. Singer's book [**9**]; especially to section 12.3.

3.1. The category of regular singular q-difference systems $\mathscr{E}$. — Let $\mathscr{F}$ be the category of q-difference systems on $\mathbb{P}^1_{\mathbb{C}}$. Its objects are the matrices $A \in \mathrm{GL}_n(\mathbb{C}(z))$ for some $n \in \mathbb{N}^*$; the integer n will be called the size of A. Its morphisms from an object A of size n to an object B of size p are the matrices $F \in \mathrm{M}_{p,n}(\mathbb{C}(z))$ such that $(\sigma_q F)A = BF$.

The category $\mathscr{E}$ of regular singular q-difference systems on $\mathbb{P}^1_{\mathbb{C}}$ is the full subcategory of $\mathscr{F}$ whose objects are the matrices $A \in \mathrm{GL}_n(\mathbb{C}(z))$ such that there exists $R \in \mathrm{GL}_n(\mathbb{C}(z))$ with the property that $(\sigma_q R)^{-1}AR$ belongs to both $\mathrm{GL}_n(\mathbb{C}\{z\})$ and $\mathrm{GL}_n(\mathbb{C}\{z^{-1}\})$. (Equivalently, $\mathscr{E}$ is the full subcategory of $\mathscr{F}$ whose objects are the matrices $A \in \mathrm{GL}_n(\mathbb{C}(z))$ such that there exists $R_0 \in \mathrm{GL}_n(\mathbb{C}(z))$ and $R_\infty \in \mathrm{GL}_n(\mathbb{C}(z))$ with the property that $(\sigma_q R_0)^{-1}AR_0$ belongs to $\mathrm{GL}_n(\mathbb{C}\{z\})$ and $(\sigma_q R_\infty)^{-1}AR_\infty$ belongs to $\mathrm{GL}_n(\mathbb{C}\{z^{-1}\})$.)

3.2. The category of connection data $\mathscr{C}$. — We denote by $\mathscr{C}$ the category of connection data. Its objects are the triples

$$(A^{(0)}, M, A^{(\infty)}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^*)) \times \mathrm{GL}_n(\mathbb{C}),$$

for some $n \in \mathbb{N}^*$, such that

$$(\sigma_q M)A^{(0)} = A^{(\infty)}M;$$

the integer n will be called the size of $(A^{(0)}, M, A^{(\infty)})$. The morphisms of $\mathscr{C}$ from an object $(A^{(0)}, M, A^{(\infty)})$ of size n to an object $(B^{(0)}, N, B^{(\infty)})$ of size p are the pairs

$$(S^{(0)}, S^{(\infty)}) \in \mathrm{M}_{p,n}(\mathbb{C}[z, z^{-1}]) \times \mathrm{M}_{p,n}(\mathbb{C}[z, z^{-1}])$$

such that

$$\begin{cases} (\sigma_q S^{(0)})A^{(0)} = B^{(0)}S^{(0)} \\ (\sigma_q S^{(\infty)})A^{(\infty)} = B^{(\infty)}S^{(\infty)} \\ S^{(\infty)}M = NS^{(0)}. \end{cases}$$

Note that it would have been more natural to require that the coefficients of $S^{(0)}$ and $S^{(\infty)}$ belong to $\mathbb{C}(\{z\})$ and $\mathbb{C}(\{z^{-1}\})$ respectively; actually, we would not get additional morphisms in this way in virtue of the following result (and its variant at ∞) which is [**14**, Lemma 2.1.3.2].

Proposition 1. — *Let us consider $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ and $B^{(0)} \in \mathrm{GL}_p(\mathbb{C})$. If $S^{(0)} \in \mathrm{M}_{p,n}(\mathbb{C}(\{z\}))$ is such that $(\sigma_q S^{(0)})A^{(0)} = B^{(0)}S^{(0)}$ then $S^{(0)} \in \mathrm{M}_{p,n}(\mathbb{C}[z, z^{-1}])$.*

We will use the following variant of Proposition 1.

Proposition 2. — *We maintain the notations and hypotheses of Proposition 1. If we assume moreover that $\mathrm{Sp}(A^{(0)}) \cap q^{\mathbb{Z}^*}\,\mathrm{Sp}(B^{(0)}) = \emptyset$ then $S^{(0)} \in \mathrm{M}_{p,n}(\mathbb{C})$.*

Before proceeding with the proof of Proposition 2, we recall without proof a classical result of linear algebra (which will be used several times).

Lemma 3. — *Let us consider $X \in \mathrm{M}_n(\mathbb{C})$, $Y \in \mathrm{M}_p(\mathbb{C})$ and the $\mathbb{C}$-linear endomorphism φ of $\mathrm{M}_{p,n}(\mathbb{C})$ given by*

$$\begin{array}{rcl}\varphi : \mathrm{M}_{p,n}(\mathbb{C}) & \to & \mathrm{M}_{p,n}(\mathbb{C}) \\ U & \mapsto & UX - YU.\end{array}$$

We have

$$\mathrm{Sp}(\varphi) = Sp(X) - Sp(Y) = \{\lambda - \mu \mid \lambda \in \mathrm{Sp}(X), \mu \in \mathrm{Sp}(Y)\}.$$

In particular, φ is a $\mathbb{C}$-linear automorphism if and only if $\mathrm{Sp}(X) \cap \mathrm{Sp}(Y) = \emptyset$.

Proof of Proposition 2. — Let $S^{(0)} = \sum_{k \in \mathbb{Z}} U_k z^k$ be the Laurent series expansion of $S^{(0)}$ at 0. The equality $(\sigma_q S^{(0)})A^{(0)} = B^{(0)} S^{(0)}$ ensures that, for all $k \in \mathbb{Z}$, $q^k U_k A^{(0)} = B^{(0)} U_k$. But, for $k \in \mathbb{Z}^*$, $\mathrm{Sp}(q^k A^{(0)}) \cap \mathrm{Sp}(B^{(0)}) = \emptyset$ so $U_k = 0$ in virtue of Lemma 3. □

***Definition 4* (Normalized connection data).** — *We say that an object $C = (A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}$ is normalized if $\mathrm{Sp}(A^{(0)}) \cup \mathrm{Sp}(A^{(\infty)}) \subset \{c \in \mathbb{C}^* \mid |q| \leq |c| < 1\}$.*

Remark 5. — *In Definition 4, we could replace $\{c \in \mathbb{C}^* \mid |q| \leq |c| < 1\}$ by any fixed fundamental domain of $\mathbb{C}^*$ for the action by multiplication of $q^{\mathbb{Z}}$.*

It is well known that the full subcategory of $\mathscr{C}$ made of the normalized objects is essential (this will not be used).

The following result will be used later.

Proposition 6. — *If $(S^{(0)}, S^{(\infty)})$ is a isomorphism between normalized objects of $\mathscr{C}$ then $(S^{(0)}, S^{(\infty)}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$*

Proof. — Immediate consequence of Proposition 2. □

3.3. The category of solutions $\mathscr{S}$. — In order to state the Riemann-Hilbert correspondence, it is convenient to introduce a category $\mathscr{S}$ of solutions. Its objects are the quadruples

$$(A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)}) \\ \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathcal{M}(\mathbb{C})) \times \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathcal{M}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0\})),$$

for some n in $\mathbb{N}^*$, such that

$$(\sigma_q M)A^{(0)} = A^{(\infty)} M$$

where

$$M = (M^{(\infty)})^{-1} M^{(0)};$$

the integer n is called the size of $(A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)})$. Its morphisms from an object of size n to an object of size p are triples

$$(F, S^{(0)}, S^{(\infty)}) \in \mathrm{M}_{p,n}(\mathbb{C}(z)) \times \mathrm{M}_{n,p}(\mathbb{C}(z)) \times \mathrm{M}_{n,p}(\mathbb{C}(z))$$

satisfying some compatibility conditions : we refer to section 3.1.1.2 of [**14**] for details.

3.4. The Riemann-Hilbert correspondence. — We now state the Riemann-Hilbert correspondence for regular singular q-difference systems; the following result is Proposition 3.1.1.3 in [**14**].

Theorem 7 **(Riemann-Hilbert correspondence)**. — *The functors $\mathcal{F}_{\mathscr{S},\mathscr{E}}$ and $\mathcal{F}_{\mathscr{S},\mathscr{C}}$ respectively defined on objects and morphisms by :*

$$\begin{aligned} \mathcal{F}_{\mathscr{S},\mathscr{E}} : (A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)}) &\rightsquigarrow (\sigma_q M^{(0)}) A^{(0)} (M^{(0)})^{-1} \\ &\quad = (\sigma_q M^{(\infty)}) A^{(\infty)} (M^{(\infty)})^{-1} =: A \\ (F, S^{(0)}, S^{(\infty)}) &\rightsquigarrow F \end{aligned}$$

and by :

$$\begin{aligned} \mathcal{F}_{\mathscr{S},\mathscr{C}} : (A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)}) &\rightsquigarrow (A^{(0)}, (M^{(\infty)})^{-1} M^{(0)}, A^{(\infty)}) \\ &\quad = (A^{(0)}, M, A^{(\infty)}) \\ (F, S^{(0)}, S^{(\infty)}) &\rightsquigarrow (S^{(0)}, S^{(\infty)}) \end{aligned}$$

are equivalences of categories from $\mathscr{S}$ to $\mathscr{E}$ and from $\mathscr{S}$ to $\mathscr{C}$ respectively. In particular, $\mathscr{E}$ and $\mathscr{C}$ are equivalent.

Concretely, consider an object A of $\mathscr{E}$ of size n. Using shearing transformations, one can prove that there exists $\widetilde{R}^{(0)} \in \mathrm{GL}_n(\mathbb{C}(z))$ such that $\widetilde{A} := (\sigma_q \widetilde{R}^{(0)})^{-1} A \widetilde{R}^{(0)}$ belongs to $\mathrm{GL}_n(\mathbb{C}\{z\})$ and satisfies $\mathrm{Sp}(\widetilde{A}(0)) \cap q^{\mathbb{Z}^*} \mathrm{Sp}(\widetilde{A}(0)) = \emptyset$. Moreover, one can prove that there exists $\widetilde{M}^{(0)} \in I_n + z\,\mathrm{M}_n(\mathbb{C}\{z\})$ such that $(\sigma_q \widetilde{M}^{(0)})^{-1} \widetilde{A} \widetilde{M}^{(0)} = \widetilde{A}(0)$. Then it is easily seen that $M^{(0)} := \widetilde{R}^{(0)} \widetilde{M}^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ satisfies $(\sigma_q M^{(0)})^{-1} A M^{(0)} = \widetilde{A}(0)$. One can use a similar method at ∞ in order to construct some $M^{(\infty)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^* \cup \{\infty\}))$ satisfying similar properties. Then $(A^{(0)}, M, A^{(\infty)}) := (\widetilde{A}(0), (M^{(\infty)})^{-1} M^{(0)}, \widetilde{A}(\infty))$ is an object of $\mathscr{C}$ corresponding to A in the above equivalence.

4. The categories $\mathscr{E}_c$ and $\mathscr{C}_c$, local isomorphy and rigidity

4.1. The categories $\mathscr{E}_c$ and $\mathscr{C}_c$.—

***Definition 8* (Completely regular singular connection data)**
We denote by $\mathscr{C}_c$ the full subcategory of $\mathscr{C}$ made of the objects $(A^{(0)}, M, A^{(\infty)})$ such that M has at most simple poles on $\mathbb{C}^*$. The objects of $\mathscr{C}_c$ are called completely regular singular connection data.

Proposition 9. — *The category $\mathscr{C}_c$ is closed under isomorphism in $\mathscr{C}$ i.e. any object of $\mathscr{C}$ isomorphic to an object of $\mathscr{C}_c$ is actually an object of $\mathscr{C}_c$.*

Proof. — Let $(A^{(0)}, M, A^{(\infty)})$ be an object of $\mathscr{C}$ isomorphic to some object $(B^{(0)}, N, B^{(\infty)})$ of $\mathscr{C}_c$. So there exist $S^{(0)}, S^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ such that $M = (S^{(\infty)})^{-1} N S^{(0)}$. Since, for all $s \in \mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{C}[z, z^{-1}]) \subset \mathrm{GL}_n(\mathbb{C}\{z - s\})$, the previous equality shows that M has at most simple poles on $\mathbb{C}^*$ and hence $(A^{(0)}, M, A^{(\infty)})$ is an object of $\mathscr{C}_c$. □

***Definition 10* (Completely regular singular q-difference systems)**
We denote by $\mathscr{E}_c$ the full subcategory of $\mathscr{E}$ corresponding to $\mathscr{C}_c$ in the Riemann-Hilbert correspondence (Theorem 7). The objects of $\mathscr{E}_c$ are called completely regular singular q-difference systems.

Explicitly, an object A of $\mathscr{E}$ is actually an object of $\mathscr{E}_c$ if there exists an object S of $\mathscr{S}$ with the property that A is isomorphic to $\mathcal{F}_{\mathscr{S},\mathscr{E}}(S)$ and that $\mathcal{F}_{\mathscr{S},\mathscr{C}}(S)$ is an object of $\mathscr{C}_c$.

Proposition 11. — *The category $\mathscr{E}_c$ is closed under isomorphism in $\mathscr{E}$ i.e. any object of $\mathscr{E}$ isomorphic to an object of $\mathscr{E}_c$ is actually an object of $\mathscr{E}_c$.*

Proof. — Immediate from the definition of $\mathscr{E}_c$. □

4.2. Local isomorphy and rigidity for $\mathscr{C}_c$. —

***Definition 12* (Local isomorphy for $\mathscr{C}_c$)**. — *Let $C = (A^{(0)}, M, A^{(\infty)}), C' = (B^{(0)}, N, B^{(\infty)})$ be objects of $\mathscr{C}_c$ of size n.*

We say that C and C' are locally isomorphic if the following properties hold :

(i) $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ *such that*

$$\begin{cases} (\sigma_q S_0^{(0)}) A^{(0)} = B^{(0)} S_0^{(0)} \\ (\sigma_q S_\infty^{(\infty)}) A^{(\infty)} = B^{(\infty)} S_\infty^{(\infty)}; \end{cases}$$

(ii) $\forall u \in \mathbb{C}^*$, $\exists S_u^{(0)}, S_u^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ *such that*

$$\begin{cases} (\sigma_q S_u^{(0)}) A^{(0)} = B^{(0)} S_u^{(0)} \\ (\sigma_q S_u^{(\infty)}) A^{(\infty)} = B^{(\infty)} S_u^{(\infty)} \\ S_u^{(\infty)}(u)(\mathrm{Res}_u M) = (\mathrm{Res}_u N) S_u^{(0)}(u). \end{cases}$$

We say that C and C' are weakly locally isomorphic if the following properties hold :

(i') $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z,z^{-1}])$ *such that*

$$\begin{cases}(\sigma_q S_0^{(0)})A^{(0)} = B^{(0)}S_0^{(0)}\\(\sigma_q S_\infty^{(\infty)})A^{(\infty)} = B^{(\infty)}S_\infty^{(\infty)};\end{cases}$$

(ii') $\forall u \in \mathbb{C}^*$, $\exists S_u^{(0)}, S_u^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z,z^{-1}])$ *such that*

$$S_u^{(\infty)}(u)(\mathrm{Res}_u M) = (\mathrm{Res}_u N)S_u^{(0)}(u);$$

i.e. $\mathrm{rk}\,\mathrm{Res}_u M = \mathrm{rk}\,\mathrm{Res}_u N$.

The proof of the following proposition is easy and left to the reader.

Proposition 13. — *For any objects C, C' of $\mathscr{C}_c$, we have*

C isom. to $C' \Rightarrow C$ locally isom. to $C' \Rightarrow C$ weakly locally isom. to C'.

Moreover, "being locally isomorphic" and "being weakly locally isomorphic" are equivalence relations.

We have the following reformulation of Definition 12 :

Proposition 14. — *Let $C = (A^{(0)}, M, A^{(\infty)}), C' = (B^{(0)}, N, B^{(\infty)})$ be objects of $\mathscr{C}_c$ of size n.*

Then C and C' are locally isomorphic if and only if the following properties hold :

(α) $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z,z^{-1}])$ *such that*

$$\begin{cases}(\sigma_q S_0^{(0)})A^{(0)} = B^{(0)}S_0^{(0)}\\(\sigma_q S_\infty^{(\infty)})A^{(\infty)} = B^{(\infty)}S_\infty^{(\infty)};\end{cases}$$

(β) M and N have the same set of poles on $\mathbb{C}^*$; we let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ for the action by multiplication by $q^{\mathbb{Z}}$;

(γ) $\forall i \in \{1, ..., m\}$, $\exists S_i^{(0)}, S_i^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z,z^{-1}])$ *such that*

$$\begin{cases}(\sigma_q S_i^{(0)})A^{(0)} = B^{(0)}S_i^{(0)}\\(\sigma_q S_i^{(\infty)})A^{(\infty)} = B^{(\infty)}S_i^{(\infty)}\\S_i^{(\infty)}(s_i)(\mathrm{Res}_{s_i} M) = (\mathrm{Res}_{s_i} N)S_i^{(0)}(s_i).\end{cases}$$

Moreover, C and C' are weakly locally isomorphic if and only if the following properties hold :

(α') $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z,z^{-1}])$ *such that*

$$\begin{cases}(\sigma_q S_0^{(0)})A^{(0)} = B^{(0)}S_0^{(0)}\\(\sigma_q S_\infty^{(\infty)})A^{(\infty)} = B^{(\infty)}S_\infty^{(\infty)};\end{cases}$$

(β') M and N have the same set of poles on $\mathbb{C}^*$; we let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ for the action by multiplication by $q^{\mathbb{Z}}$;

(γ') $\forall i \in \{1, ..., m\}$, $\mathrm{rk}\,\mathrm{Res}_{s_i} M = \mathrm{rk}\,\mathrm{Res}_{s_i} N$.

Proof. — It is clear that if C and C' are locally isomorphic then properties (α) to (γ) hold. Let us prove the converse implication. So, we assume that C and C' satisfy properties (α) to (γ). Let us consider $u \in \mathbb{C}^*$. In virtue of property (β), u is either [not a pole of M and not a pole of N] or [a pole of M and a pole of N]. In the fist case, we have $\mathrm{Res}_u M = 0 = \mathrm{Res}_u N$ and hence, if we set $S_u^{(0)} = S_0^{(0)}$ and $S_u^{(\infty)} = S_\infty^{(\infty)}$ (given by property (α)), we have

$$\begin{cases} (\sigma_q S_u^{(0)})A^{(0)} = B^{(0)} S_u^{(0)} \\ (\sigma_q S_u^{(\infty)})A^{(\infty)} = B^{(\infty)} S_u^{(\infty)} \\ S_u^{(\infty)}(u)(\mathrm{Res}_u M) = (\mathrm{Res}_u N) S_u^{(0)}(u). \end{cases}$$

Let us now consider the case that u is a pole of M. There exist $k \in \mathbb{Z}$ and $i \in \{1, ..., m\}$ such that $u = q^k s_i$. The relation $(\sigma_q M)A^{(0)} = A^{(\infty)} M$ entails that $(\sigma_q^k M)(A^{(0)})^k = (A^{(\infty)})^k M$ and hence $\mathrm{Res}_u M = q^k (A^{(\infty)})^k (\mathrm{Res}_{s_i} M)(A^{(0)})^{-k}$. Similarly, $\mathrm{Res}_u N = q^k (B^{(\infty)})^k (\mathrm{Res}_{s_i} N)(B^{(0)})^{-k}$. We set $S_u^{(0)} = S_i^{(0)}$ and $S_u^{(\infty)} = S_i^{(\infty)}$ (given by property (γ)). We have

$$\begin{cases} (\sigma_q S_u^{(0)})A^{(0)} = B^{(0)} S_u^{(0)} \\ (\sigma_q S_u^{(\infty)})A^{(\infty)} = B^{(\infty)} S_u^{(\infty)} \\ S_u^{(\infty)}(u)(\mathrm{Res}_u M) = S_i^{(\infty)}(q^k s_i) q^k (A^{(\infty)})^k (\mathrm{Res}_{s_i} M)(A^{(0)})^{-k} \\ = q^k (B^{(\infty)})^k S_i^{(\infty)}(s_i)(\mathrm{Res}_{s_i} M)(A^{(0)})^{-k} = q^k (B^{(\infty)})^k (\mathrm{Res}_{s_i} N) S_i^{(0)}(s_i)(A^{(0)})^{-k} \\ = q^k (B^{(\infty)})^k (\mathrm{Res}_{s_i} N)(B^{(0)})^{-k} S_i^{(0)}(q^k s_i) = (\mathrm{Res}_u N) S_u^{(0)}(u). \end{cases}$$

It is now clear that C and C' are isomorphic.

The case of weak local isomorphy is left to the reader. □

The following result will be useful (it will allow us to work in $\mathrm{GL}_n(\mathbb{C})$ rather than in $\mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$).

Proposition 15. — *Let $C = (A^{(0)}, M, A^{(\infty)}), C' = (B^{(0)}, N, B^{(\infty)})$ be normalized objects of $\mathscr{C}_c$ of size n.*

Then C and C' are locally isomorphic if and only if the following properties hold :

(α) $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\begin{cases} S_0^{(0)} A^{(0)} = B^{(0)} S_0^{(0)} \\ S_\infty^{(\infty)} A^{(\infty)} = B^{(\infty)} S_\infty^{(\infty)}; \end{cases}$$

(β) M and N have the same set of poles on $\mathbb{C}^*$; we let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ for the action by multiplication by $q^{\mathbb{Z}}$;

(γ) $\forall i \in \{1, ..., m\}$, $\exists S_i^{(0)}, S_i^{(\infty)} \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\begin{cases} S_i^{(0)} A^{(0)} = B^{(0)} S_i^{(0)} \\ S_i^{(\infty)} A^{(\infty)} = B^{(\infty)} S_i^{(\infty)} \\ S_i^{(\infty)} (\mathrm{Res}_{s_i} M) = (\mathrm{Res}_{s_i} N) S_i^{(0)}. \end{cases}$$

Moreover, C and C' are weakly isomorphic if and only if the following properties hold :

(α') $\exists S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\begin{cases} S_0^{(0)} A^{(0)} = B^{(0)} S_0^{(0)} \\ S_\infty^{(\infty)} A^{(\infty)} = B^{(\infty)} S_\infty^{(\infty)}; \end{cases}$$

(β') M and N have the same set of poles on $\mathbb{C}^*$; we let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ for the action by multiplication by $q^{\mathbb{Z}}$;

(γ') $\forall i \in \{1, ..., m\}$,
$\mathrm{rk}\,\mathrm{Res}_{s_i} M = \mathrm{rk}\,\mathrm{Res}_{s_i} N$.

Proof. — Direct consequence of Proposition 2. □

***Definition 16* (Rigidity for $\mathscr{C}_c$).** — *We say that an object C of $\mathscr{C}_c$ is rigid (resp. strongly rigid) if any object of $\mathscr{C}_c$ locally isomorphic (resp. weakly locally isomorphic) to C is actually isomorphic to C.*

The proof of the following result is left to the reader.

Proposition 17. — *For any objects C of $\mathscr{C}_c$, we have :*

$$C \text{ strongly rigid} \Rightarrow C \text{ rigid.}$$

The following result will be used later.

Proposition 18. — *An object $C = (A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}_c$ is rigid (resp. strongly rigid) if and only if any object of $\mathscr{C}_c$ of the form $(A^{(0)}, N, A^{(\infty)})$ locally isomorphic (resp. weakly locally isomorphic) to C is actually isomorphic to C.*

Proof. — The "only if" part of the proposition is obvious. We now prove the "if" part of the proposition. Let $(B^{(0)}, N, B^{(\infty)})$ be an object of $\mathscr{C}_c$ locally isomorphic (resp. weakly locally isomorphic) to C. Then there exists $S_0^{(0)}, S_\infty^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ such that $(\sigma_q S_0^{(0)}) A^{(0)} = B^{(0)} S_0^{(0)}$ and $(\sigma_q S_\infty^{(\infty)}) A^{(\infty)} = B^{(\infty)} S_\infty^{(\infty)}$. It follows that $(B^{(0)}, N, B^{(\infty)})$ is isomorphic to $(A^{(0)}, (S_\infty^{(\infty)})^{-1} N S_0^{(0)}, A^{(\infty)})$. So $(A^{(0)}, (S_\infty^{(\infty)})^{-1} N S_0^{(0)}, A^{(\infty)})$ is locally isomorphic (resp. weakly isomorphic) to C and hence isomorphic to C. Therefore $(B^{(0)}, N, B^{(\infty)})$ is isomorphic to C. $\square$

4.3. Local isomorphy and rigidity for $\mathscr{E}_c$. —

***Definition 19* (Local isomorphy for $\mathscr{E}_c$).** — *Let A, B be objects of $\mathscr{E}_c$ and let C_A, C_B be corresponding connection data in the Riemann-Hilbert correspondence. We say that A and B are locally isomorphic (resp. weakly locally isomorphic) if C_A and C_B are locally isomorphic (resp. weakly locally isomorphic).*

A priori, the above definition depends on the choice of C_A and C_B. Consider C'_A and C'_B alternative corresponding connection data. Then C_A and C'_A are isomorphic and C_B and C'_B are isomorphic. Using Proposition 13, we get that C_A and C_B are locally isomorphic (resp. weakly locally isomorphic) if and only if the same property holds for C'_A and C'_B. Hence the above definition is not ambiguous.

***Definition 20* (Rigidity for $\mathscr{E}_c$).** — *We say that an object A of $\mathscr{E}_c$ is rigid (resp. strongly rigid) if any object of $\mathscr{E}_c$ locally isomorphic (resp. weakly locally isomorphic) to A is actually isomorphic to A.*

So, if C_A is a connection data corresponding to A then A is rigid if and only if C_A is rigid.

5. Rigidity under the hypothesis $q^{\mathbb{Z}} \operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}} \operatorname{Sp}(A^{(\infty)}) = \emptyset$

5.1. Hypotheses and notations. — The following hypotheses and notations will be maintained in the whole section 5.

We let $A^{(0)}, A^{(\infty)}$ be elements of $\mathrm{GL}_n(\mathbb{C})$ such that

$$q^{\mathbb{Z}} \operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}} \operatorname{Sp}(A^{(\infty)}) = \emptyset.$$

We consider $\underline{s} = (s_1, ..., s_m) \in (\mathbb{C}^*)^m$ such that, for all $(i,j) \in \{1, ..., m\}^2$,

$$i \neq j \Rightarrow s_i \not\equiv s_j \text{ mod. } q^{\mathbb{Z}}.$$

We denote by $\mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}}$ the $\mathbb{C}$-vector space made of the matrices $M \in \mathrm{M}_n(\mathcal{M}(\mathbb{C}^*))$ such that :

- M is analytic on $\mathbb{C}^* \setminus \cup_{i=1}^m q^{\mathbb{Z}} s_i$;
- M has at most simple poles on $\mathbb{C}^*$;
- $(\sigma_q M) A^{(0)} = A^{(\infty)} M$.

We set

$$\mathscr{C}_{\underline{s};A^{(0)},A^{(\infty)}} = \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \cap \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^*)).$$

Note that $\mathscr{C}_{\underline{s};A^{(0)},A^{(\infty)}}$ is nothing but the set made of the matrices $M \in \mathrm{M}_n(\mathcal{M}(\mathbb{C}^*))$ analytic on $\mathbb{C}^* \setminus \cup_{i=1}^m q^{\mathbb{Z}} s_i$ such that $(A^{(0)}, M, A^{(\infty)})$ is an object of $\mathscr{C}_c$.

We consider

$$\mathscr{R}_{\underline{s};A^{(0)},A^{(\infty)}} = \mathrm{Im}(\mathrm{Res}_{\underline{s}} : \mathscr{C}_{\underline{s};A^{(0)},A^{(\infty)}} \to \mathrm{M}_n(\mathbb{C})^m)$$

and

$$\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}} = \mathrm{Im}(\mathrm{Res}_{\underline{s}} : \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \to \mathrm{M}_n(\mathbb{C})^m)$$

where $\mathrm{Res}_{\underline{s}}$ is defined by

$$\mathrm{Res}_{\underline{s}} M = (\mathrm{Res}_{s_1} M, ..., \mathrm{Res}_{s_m} M).$$

It is clear that the set of poles of any $M \in \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}}$ is invariant by the natural action of $q^{\mathbb{Z}}$ on $\mathbb{C}^*$. For this reason, we will only consider the poles of M in some fundamental domain of $\mathbb{C}^*$ for the action of $q^{\mathbb{Z}}$.

5.2. Theta functions. — For any $X, Y \in \mathrm{GL}_n(\mathbb{C})$, for any $U \in \mathrm{M}_n(\mathbb{C})$, for any $s \in \mathbb{C}^*$, we consider the analytic function $\Theta_{X,Y;U;s} : \mathbb{C}^* \to \mathrm{M}_n(\mathbb{C})$ defined by

$$\Theta_{X,Y;U;s}(z) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{(k-1)k}{2}} Y^{-k} U X^k (z/s)^k.$$

We have

$$\sigma_q \Theta_{X,Y;U;s} = -(z/s)^{-1} Y \Theta_{X,Y;U;s} X^{-1}. \tag{1}$$

In the special case $n = X = Y = U = s = 1$, we get an usual Jacobi theta function $\theta := \Theta_{1,1;1;1}$:

$$\theta(z) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{(k-1)k}{2}} z^k.$$

We recall the so-called Jacobi triple product formula

$$\theta(z) = (q;q)_\infty \, (z;q)_\infty \, (q/z;q)_\infty$$

where we have used the usual notation for the q-Pochhammer symbol :

$$(a;q)_\infty = \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$

For any $s \in \mathbb{C}^*$, we will also use the short-hand notation $\theta_s := \Theta_{1,1;1;s}$ (so $\theta_s(z) = \theta(z/s)$). The function θ_s is analytic on $\mathbb{C}^*$, its set of zeros is $q^{\mathbb{Z}} s$ and its zeros are simple (these two last statements follow from the triple product formula). Moreover, we have

$$\sigma_q \theta_s = -(z/s)^{-1}\theta_s. \tag{2}$$

The reason why we consider the above functions lies in the fact that, for all $U \in \mathrm{M}_n(\mathbb{C})$, for all $i \in \{1, ..., m\}$,

$$\frac{\Theta_{A^{(0)},A^{(\infty)};U;s_i}}{\theta_{s_i}} \in \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}}$$

(this follows immediately from (1) and (2)).

5.3. Structure of sets of residues. — In this section, we study $\mathscr{R}_{\underline{s};A^{(0)},A^{(\infty)}}$ and $\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}}$.

Proposition 21. — *The following properties hold :*

i) the map

$$\begin{array}{rcl} \Psi := \Psi_{A^{(0)},A^{(\infty)};\underline{s}} : \mathrm{M}_n(\mathbb{C})^m & \to & \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \\ \underline{U} = (U_1, ..., U_m) & \mapsto & \displaystyle\sum_{i=1}^{m} \frac{\Theta_{A^{(0)},A^{(\infty)};U_i;s_i}}{\theta_{q;s_i}} \end{array}$$

is a $\mathbb{C}$-linear isomorphism;

ii) the map

$$\mathrm{Res}_{\underline{s}} \circ \Psi : \mathrm{M}_n(\mathbb{C})^m \to \mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}}$$

is a $\mathbb{C}$-linear isomorphism; in particular $\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}} = \mathrm{M}_n(\mathbb{C})^m$.

Proposition 22. — *$\mathscr{R}_{\underline{s};A^{(0)},A^{(\infty)}}$ a Zariski-dense open subset of $\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}} = \mathrm{M}_n(\mathbb{C})^m$.*

Before proceeding with the proofs of these propositions, we state and prove a series of lemmas.

Lemma 23. — *The $\mathbb{C}$-linear morphism $\mathrm{Res}_{\underline{s}} : \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \to \mathrm{M}_n(\mathbb{C})^m$ is injective.*

Proof. — Let us consider $M \in \mathrm{Ker}(\mathrm{Res}_{\underline{s}} : \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \to \mathrm{M}_n(\mathbb{C})^m)$. Then $M : \mathbb{C}^* \to \mathrm{M}_n(\mathbb{C})$ is analytic and such that $(\sigma_q M)A^{(0)} = A^{(\infty)}M$. Hence, denoting by $M = \sum_{j\in\mathbb{Z}} M_j z^j$ the Taylor expansion of M on $\mathbb{C}^*$, we have, for all $j \in \mathbb{Z}$, $q^j M_j A^{(0)} = A^{(\infty)} M_j$. Lemma 3 ensures that, for all $j \in \mathbb{Z}$, $M_j = 0$ and hence $M = 0$. □

Lemma 24. — *For any diagonal matrices $X = \mathrm{diag}(x_1, ..., x_n), Y = \mathrm{diag}(y_1, ..., y_n) \in \mathrm{GL}_n(\mathbb{C})$, for any $U = (u_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{C})$, for any $s \in \mathbb{C}^*$, $\Theta_{X,Y;U;s}(z) = (u_{i,j}\theta_s(x_i y_j^{-1} z))_{1\leq i,j\leq n}$.*

Proof. — Indeed, for all $z \in \mathbb{C}^*$, we have :

$$\begin{aligned}\Theta_{X,Y;U;s}(z) &= \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{(k-1)k}{2}} Y^{-k} U X^k (z/s)^k\\ &= \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{(k-1)k}{2}} (y_i^{-k} u_{i,j} x_j^k)_{1\leq i,j\leq n} (z/s)^k\\ &= (u_{i,j}\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{(k-1)k}{2}} y_i^{-k} x_j^k (z/s)^k)_{1\leq i,j\leq n}\\ &= (u_{i,j}\theta_s(y_i^{-1}x_j z))_{1\leq i,j\leq n}.\end{aligned}$$

□

We set

$$\mathcal{N} = \{(n_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{C}) \mid \forall (i,j) \in [[1,n]]^2, (j \neq i+1 \Rightarrow n_{i,j} = 0)\\ \text{and } \forall i \in [[1, n-1]], n_{i,i+1} \in \{0,1\}\}.$$

We denote by $\odot$ the termwise multiplication of matrices i.e. for all $U = (u_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{C})$, for all $V = (v_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{C})$,

$$U \odot V = (u_{i,j}v_{i,j})_{1\leq i,j\leq n}.$$

Lemma 25. — *Let X, Y be elements of $\mathcal{N}$ and let $K = (K(k,l))_{0\leq k,l\leq n}$ be a family of elements of $\mathrm{M}_n(\mathbb{C})$ such that the entries of $K(0,0)$ are non zero. Then, the map*

$$\begin{aligned}\Phi_K : \mathrm{M}_n(\mathbb{C}) &\to \mathrm{M}_n(\mathbb{C})\\ U &\mapsto \sum_{0\leq k,l\leq n} Y^k (K(k,l)\odot U) X^l\end{aligned}$$

is a $\mathbb{C}$-linear automorphism.

Proof. — It is sufficient to prove that $\operatorname{Ker}\Phi_K = \{0\}$. Let us consider $U = (u_{i,j})_{1\leq i,j\leq n} \in \operatorname{Ker}\Phi_K$. In what follows, the symbols $*$ denote some complex numbers. Note that :

$$Y^k(K(k,l)\odot U)X^l = \begin{pmatrix} 0 & \cdots & 0 & *u_{k+1,1} & \cdots & *u_{k+1,n-l}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & *u_{n,1} & \cdots & *u_{n,n-l}\\ 0 & \cdots & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

So the fact that $U=(u_{i,j})_{1\leq i,j\leq n}\in \operatorname{Ker}\Phi_K$ means that the n^2 entries of U satisfy a system of n^2 linear equations of the form

$$\left(K(0,0)_{i,j}u_{i,j}+\sum_{\substack{i\leq k\leq n\\ 1\leq l\leq j\\ (k,l)\neq(i,j)}} *u_{k,l}\right)_{1\leq i,j\leq n}=0.$$

Considering the first columns in this equality, we get

$$K(0,0)_{n,1}u_{n,1}=K(0,0)_{n-1,1}u_{n-1,1}+*u_{n,1}=K(0,0)_{n-2,1}u_{n-2,1}+*u_{n-1,1}+*u_{n,1}=\cdots$$
$$\cdots=K(0,0)_{1,1}u_{1,1}+*u_{2,1}+\cdots+*u_{n-1,1}+*u_{n,1}=0.$$

So $u_{n,1}=u_{n-1,1}=\cdots=u_{1,1}=0$. Similarly, considering the last rows, we get $u_{n,1}=u_{n,2}=\cdots=u_{n,n}=0$. So the entries of the first column and of the last row of U are zero and its remaining $(n-1)^2$ entries satisfy a system of $(n-1)^2$ linear equations of the form

$$\left(K(0,0)_{i,j}u_{i,j}+\sum_{\substack{i\leq k\leq n-1\\ 2\leq l\leq j\\ (k,l)\neq(i,j)}} *u_{k,l}\right)_{1\leq i\leq n-1,2\leq j\leq n}=0.$$

The result follows clearly by induction. $\square$

Lemma 26. — *For all $s\in\mathbb{C}^*$, the map*

$$\begin{array}{rcl}\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\\ U & \mapsto & \operatorname{Res}_s \dfrac{\Theta_{A^{(0)},A^{(\infty)};U;s}}{\theta_s}=(\operatorname{Res}_s 1/\theta_s)\Theta_{A^{(0)},A^{(\infty)};U;s}(s)\end{array}$$

is a $\mathbb{C}$-linear automorphism.

Proof. — By Dunford-Jordan decomposition, we can clearly assume that $A^{(0)}=D^{(0)}+N^{(0)}$ for some diagonal matrix $D^{(0)}\in\mathrm{GL}_n(\mathbb{C})$ and for some $N^{(0)}\in\mathcal{N}$ commuting with $D^{(0)}$ and that a similar decomposition $(A^{(\infty)})^{-1}=D^{(\infty)}+N^{(\infty)}$ holds. It is easily seen that there exists a family $(K(k,l))_{0\leq k,l\leq n,(k,l)\neq(0,0)}$ of elements of $\mathrm{M}_n(\mathbb{C})$ such that, for all $U\in\mathrm{M}_n(\mathbb{C})$:

$$\Theta_{A^{(0)},A^{(\infty)};U;s}(s)=\Theta_{D^{(0)},(D^{(\infty)})^{-1};U;s}(s)+\sum_{\substack{0\leq k,l\leq n\\ (k,l)\neq(0,0)}}(N^{(\infty)})^k(K(k,l)\odot U)(N^{(0)})^l.$$

Since $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)}) = \emptyset$, Lemma 24 implies that there exists $K(0,0) \in \mathrm{M}_n(\mathbb{C})$ with non zero entries such that, for all $U \in \mathrm{M}_n(\mathbb{C})$, $\Theta_{D^{(0)},D^{(\infty)};U;s}(s) = K(0,0) \odot U$. The result follows from Lemma 25. □

We now prove Proposition 21 and Proposition 22.

Proof of Proposition 21. — The fact that $\operatorname{Res}_{\underline{s}} \circ \Psi$ is an isomorphism is an immediate consequence of Lemma 26 because

$$\operatorname{Res}_{\underline{s}} \circ \Psi(\underline{U}) = \left(\operatorname{Res}_{s_1} \frac{\Theta_{A^{(0)},A^{(\infty)};U_1;s_1}}{\theta_{s_1}}, ..., \operatorname{Res}_{s_m} \frac{\Theta_{A^{(0)},A^{(\infty)};U_m;s_m}}{\theta_{s_m}}\right).$$

But, in virtue of Lemma 23, $\operatorname{Res}_{\underline{s}} : \mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \to \mathrm{M}_n(\mathbb{C})^m$ is injective. So Ψ is an isomorphism. □

Proof of Proposition 22. — Note that $\Psi^{-1}(\mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \setminus \mathscr{C}_{\underline{s};A^{(0)},A^{(\infty)}})$ is a Zariski-closed subset of $\mathrm{M}_n(\mathbb{C})^m$ because it coincides with the inverse image of 0 by the map

$$\begin{array}{rcl} \mathrm{M}_n(\mathbb{C})^m & \to & \mathcal{O}(\mathbb{C}^*) \\ \underline{U} = (U_1, ..., U_m) & \mapsto & \det\left(\sum_{i=1}^{m} \Theta_{A^{(0)},A^{(\infty)};U_i;s_i} \prod_{j \in \{1,...,m\} \setminus \{i\}} \theta_{s_j}\right) \end{array}$$

which is a polynomial with coefficients in $\mathcal{O}(\mathbb{C}^*)$ in the entries of $\underline{U}$ (and hence, by taking the Taylor expansions at 0 of the coefficients, we get that $\Psi^{-1}(\mathscr{C}'_{\underline{s};A^{(0)},A^{(\infty)}} \setminus \mathscr{C}_{\underline{s};A^{(0)},A^{(\infty)}})$ is the zero locus of polynomials with complex coefficients in the entries of $\underline{U}$). So its image $\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}} \setminus \mathscr{R}_{\underline{s};A^{(0)},A^{(\infty)}}$ by the $\mathbb{C}$-linear isomorphism $\operatorname{Res}_{\underline{s}} \circ \Psi$ is Zariski-closed in $\mathscr{R}'_{\underline{s};A^{(0)},A^{(\infty)}}$. The result follows from the fact that $\mathscr{R}_{\underline{s};A^{(0)},A^{(\infty)}}$ is non empty. □

5.4. Numerical characterization of rigidity. — For all $A \in \mathrm{GL}_n(\mathbb{C})$, we denote by $\mathrm{Z}(A)$ the centralizer of A in $\mathrm{GL}_n(\mathbb{C})$:

$$\mathrm{Z}(A) = \{X \in \mathrm{GL}_n(\mathbb{C}) \mid XA = AX\}.$$

For all $R \in \mathrm{M}_n(\mathbb{C})$, we consider the complex linear algebraic group

$$G(R) = \{(X, Y) \in \mathrm{Z}(A^{(0)}) \times \mathrm{Z}(A^{(\infty)}) \mid YR = RX\}.$$

For all $A \in \mathrm{M}_n(\mathbb{C})$, we denote by $\mathfrak{z}(A)$ the centralizer of A in $\mathrm{M}_n(\mathbb{C})$:

$$\mathfrak{z}(A) = \{X \in \mathrm{M}_n(\mathbb{C}) \mid XA = AX\}.$$

For all $R \in \mathrm{M}_n(\mathbb{C})$, we consider the complex Lie algebra

$$\mathfrak{g}(R) = \{(X, Y) \in \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) \mid YR = RX\}.$$

We have $\dim \mathrm{Z}(A) = \dim \mathfrak{z}(A)$ and $\dim G(R) = \dim \mathfrak{g}(R)$.

We will need the following result.

Proposition 27. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be an irreducible object of $\mathscr{C}_c$ of size n. Let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$ and set, for all $i \in \{1, ..., m\}$, $R_i = \operatorname{Res}_{s_i} M$.*

Then $\cap_{i=1}^{m} \mathfrak{g}(R_i) = \mathbb{C}(I_n, I_n)$.

Consider moreover an object $C' = (A^{(0)}, M', A^{(\infty)})$ of $\mathscr{C}_c$ of size n such that the set of poles of M' is included in $\cup_{i \in \{1,...,m\}} q^{\mathbb{Z}} s_i$ and set, for all $i \in \{1, ..., m\}$, $R'_i = \operatorname{Res}_{s_i} M'$. If $X \in \mathrm{Z}(A^{(0)})$ and $Y \in \mathrm{Z}(A^{(\infty)})$ are such that, for all $i \in \{1, ..., m\}$, $Y R_i = R'_i X$ then either $(X, Y) = (0, 0)$ or $((X, Y) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and C is isomorphic to C').

Proof. — We will use the map $\Psi := \Psi_{A^{(0)}, A^{(\infty)}; \underline{s}}$ defined in Proposition 21. Proposition 21 ensures that there exists $\underline{U} \in \mathrm{M}_n(\mathbb{C})^m$ such that $\Psi(\underline{U}) = M$. Let us consider $(X, Y) \in \cap_{i=1}^{m} \mathfrak{g}(R_i)$. We have $Y \operatorname{Res}_{\underline{s}} \circ \Psi(\underline{U}) - \operatorname{Res}_{\underline{s}} \circ \Psi(\underline{U}) X = 0$. But, using the fact that X commutes with $A^{(0)}$ and that Y commutes with $A^{(\infty)}$, it is easily seen that $Y \operatorname{Res}_{\underline{s}} \circ \Psi(\underline{U}) - \operatorname{Res}_{\underline{s}} \circ \Psi(\underline{U}) X = \operatorname{Res}_{\underline{s}} \circ \Psi(Y\underline{U} - \underline{U}X)$. So $\operatorname{Res}_{\underline{s}} \circ \Psi(Y\underline{U} - \underline{U}X) = 0$. Proposition 21 ensures that $Y\underline{U} - \underline{U}X = 0$ and hence that $YM - MX = Y\Psi(\underline{U}) - \Psi(\underline{U})X = \Psi(Y\underline{U} - \underline{U}X) = 0$. Hence (X, Y) is an endomorphism of C. Schur's Lemma ensures that $X = Y \in \mathbb{C} I_n$.

The proof of the second assertion is similar. □

Theorem 28. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}_c$ of size n such that $q^{\mathbb{Z}} \operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}} \operatorname{Sp}(A^{(\infty)}) = \emptyset$. Let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$ and set, for all $i \in \{1, ..., m\}$, $\operatorname{Res}_{s_i} M = R_i$. Then :*

i) $\sum_{i=1}^{m} \dim G(R_i) \leq (m-1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1$;

ii) C is rigid if and only if $\sum_{i=1}^{m} \dim G(R_i) = (m-1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1$.

Proof. — We first prove i). For any $i \in \{1, ..., m\}$, we consider the $\mathbb{C}$-linear map

$$\begin{array}{rcl} \varphi_i : \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) & \to & \mathrm{M}_n(\mathbb{C}) \\ (X, Y) & \mapsto & YR_i - R_i X \end{array}$$

whose kernel is $\mathfrak{g}(R_i)$ and hence whose rank is $\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)}) - \dim \mathfrak{g}(R_i)$. We also consider the $\mathbb{C}$-linear map

$$\begin{array}{rcl} \varphi = (\varphi_1, ..., \varphi_m) : \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) & \to & \mathrm{M}_n(\mathbb{C})^m \\ (X, Y) & \mapsto & (YR_1 - R_1 X, ..., YR_m - R_m X) \end{array}$$

whose kernel is, in virtue of Proposition 27, $\mathbb{C}(I_n, I_n)$ and hence whose rank is $\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)}) - 1$. The inequality $\operatorname{rk} \varphi \leq \sum_{i=1}^{m} \operatorname{rk} \varphi_i$ entails that

$$\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)}) - 1 \leq \sum_{i=1}^{m} (\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)}) - \dim \mathfrak{g}(R_i))$$

and hence that

$$\sum_{i=1}^{m} \dim \mathfrak{g}(R_i) \leq (m-1)(\dim \mathfrak{z}(A^{(0)}) + \dim \mathfrak{z}(A^{(\infty)})) + 1$$

as expected.

Let us now prove ii). In virtue of i), we must prove that C is rigid if and only if

$$\sum_{i=1}^{m} \dim G(R_i) \geq (m-1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1.$$

We first assume that C is rigid. Let us consider the complex affine algebraic variety

$$V = \mathrm{Z}(A^{(0)}) \times \left(\prod_{i=1}^{m} (\mathrm{Z}(A^{(0)}) \times \mathrm{Z}(A^{(\infty)})) \right) \times \mathrm{Z}(A^{(\infty)}).$$

Proposition 22 ensures that

$$\begin{aligned} U = \{ (S_0^{(0)}, (S_1^{(0)}, S_1^{(\infty)}), ..., (S_m^{(0)}, S_m^{(\infty)}), S_\infty^{(\infty)}) \in V \mid \\ (S_0^{(0)} A^{(0)} (S_0^{(0)})^{-1}, S_1^{(\infty)} R_1 (S_1^{(0)})^{-1}, ... \\ ..., S_m^{(\infty)} R_m (S_m^{(0)})^{-1}, S_\infty^{(\infty)} A^{(\infty)} (S_\infty^{(\infty)})^{-1}) \in \mathscr{R}_{\underline{s}; A^{(0)}, A^{(\infty)}} \} \end{aligned}$$

is a Zariski-dense open subset of V.

Let $(S_0^{(0)}, (S_1^{(0)}, S_1^{(\infty)}), ..., (S_m^{(0)}, S_m^{(\infty)}), S_\infty^{(\infty)})$ be an element of U and consider $C' = (A^{(0)}, N, A^{(\infty)})$ where N is the unique element of $\mathscr{C}_{\underline{s}; A^{(0)}, A^{(\infty)}}$ such that, for all $i \in \{1, ..., m\}$, $\operatorname{Res}_{s_i} N = S_i^{(\infty)} R_i (S_i^{(0)})^{-1}$. Then C and C' are clearly locally isomorphic and hence isomorphic because C is rigid. Let $(S^{(0)}, S^{(\infty)}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ be an isomorphism from C to C' (the fact that this isomorphism has coefficients in $\mathbb{C}$ follows from Proposition 6). We have

$$\begin{cases} S^{(0)} A^{(0)} = A^{(0)} S^{(0)} \\ S^{(\infty)} A^{(\infty)} = A^{(\infty)} S^{(\infty)} \\ S^{(\infty)} M = N S^{(0)} \end{cases}$$

so (the third assertion is obtained by taking residues)

$$\begin{cases} S^{(0)} \in \mathrm{Z}(A^{(0)}) \\ S^{(\infty)} \in \mathrm{Z}(A^{(\infty)}) \\ \forall i \in \{1,...,m\}, S^{(\infty)}R_i = S_i^{(\infty)}R_i(S_i^{(0)})^{-1}S^{(0)} \\ \qquad\qquad \text{i.e. } ((S^{(0)})^{-1}S_i^{(0)}, (S^{(\infty)})^{-1}S_i^{(\infty)}) \in G(R_i). \end{cases}$$

Hence, setting $T^{(0)} = (S^{(0)})^{-1} \in \mathrm{Z}(A^{(0)})$, $T^{(\infty)} = (S^{(\infty)})^{-1} \in \mathrm{Z}(A^{(\infty)})$, $T_0^{(0)} = I_n \in \mathrm{Z}(A^{(0)})$, $T_\infty^{(\infty)} = I_n \in \mathrm{Z}(A^{(\infty)})$, $(T_i^{(0)}, T_i^{(\infty)}) = ((S^{(0)})^{-1}S_i^{(0)}, (S^{(\infty)})^{-1}S_i^{(\infty)}) \in G(R_i)$, we have :

$$\begin{cases} S^{(0)} = (T^{(0)})^{-1}T_0^{(0)} \\ S^{(\infty)} = (T^{(\infty)})^{-1}T_\infty^{(\infty)} \\ \forall i \in \{1,...,m\}, (S_i^{(0)}, S_i^{(\infty)}) = ((T^{(0)})^{-1}T_i^{(0)}, (T^{(\infty)})^{-1}T_i^{(\infty)}). \end{cases} \tag{3}$$

This has the following consequence in terms of action of groups. We denote by K the complex algebraic group which is the quotient of

$$\mathrm{Z}(A^{(0)}) \times \mathrm{Z}(A^{(0)}) \times \left(\prod_{i=1}^{m} G(R_i)\right) \times \mathrm{Z}(A^{(\infty)}) \times \mathrm{Z}(A^{(\infty)})$$

by the central subgroup

$$\mathbb{C}^*(I_n, I_n, (I_n, I_n), ..., (I_n, I_n), I_n, I_n).$$

We define a right action of K on $\mathrm{M}_n(\mathbb{C})^{2(m+1)}$ by letting the class $k \in K$ of

$$(T^{(0)}, T_0^{(0)}, (T_1^{(0)}, T_1^{(\infty)}), ..., (T_m^{(0)}, T_m^{(\infty)}), T_\infty^{(\infty)}, T^{(\infty)})$$

act on

$$S = (S_0^{(0)}, (S_1^{(0)}, S_1^{(\infty)}), ..., (S_m^{(0)}, S_m^{(\infty)}), S_\infty^{(\infty)}) \in \mathrm{M}_n(\mathbb{C})^{2(m+1)}$$

as follows :

$$\begin{aligned} Sk = ((T^{(0)})^{-1}S_0^{(0)}T_0^{(0)}, ((T^{(0)})^{-1}S_1^{(0)}T_1^{(0)}, (T^{(\infty)})^{-1}S_1^{(\infty)}T_1^{(\infty)}), ... \\ ..., ((T^{(0)})^{-1}S_m^{(0)}T_m^{(0)}, (T^{(\infty)})^{-1}S_m^{(\infty)}T_m^{(\infty)}), (T^{(\infty)})^{-1}S_\infty^{(\infty)}T_\infty^{(\infty)}). \end{aligned}$$

But (3) shows that U is contained in the orbit of $(I_n, (I_n, I_n), ..., (I_n, I_n), I_n)$ under the action of K. So $\dim K \geq \dim U$ i.e.

$$\begin{aligned} 2\dim \mathrm{Z}(A^{(0)}) + \sum_{i=1}^{m} \dim G(R_i) + 2\dim \mathrm{Z}(A^{(\infty)}) - 1 \\ \geq (m+1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) \end{aligned}$$

whence the result.

Assume conversely that

$$\sum_{i=1}^{m} \dim G(R_i) = (m-1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1$$

and let us prove that C is rigid. Let us consider an object $C' = (A^{(0)}, M', A^{(\infty)})$ of $\mathscr{C}$ locally isomorphic to C and set, for all $i \in \{1, ..., m\}$, $\mathrm{Res}_{s_i} M' = R'_i$. So, for all $i \in \{1, ..., m\}$, there exists $(S_i^{(0)}, S_i^{(\infty)}) \in \mathrm{Z}(A^{(0)}) \times \mathrm{Z}(A^{(\infty)})$ such that $R'_i = (S_i^{(\infty)})^{-1} R_i S_i^{(0)}$. For all $i \in \{1, ..., m\}$, we introduce the $\mathbb{C}$-linear morphism

$$\begin{array}{rcl} \psi_i : \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) & \rightarrow & \mathrm{M}_n(\mathbb{C}) \\ (X, Y) & \mapsto & Y R_i - R'_i X. \end{array}$$

Note that

$$\mathrm{Ker}\, \psi_i = \{(X,Y) \in \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) \mid (S_i^{(\infty)} Y) R_i - R_i (S_i^{(0)} X) = 0\}$$

and hence

$$\begin{array}{rcl} \dim \mathrm{Ker}\, \psi_i & = & \dim \{(X,Y) \in \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) \mid Y R_i - R_i X = 0\} \\ & = & \dim \mathfrak{g}(R_i). \end{array}$$

So

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{m} \dim \mathrm{Ker}\, \psi_i = \sum_{i=1}^{m} \dim \mathfrak{g}(R_i) & = & (m-1)(\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1 \\ & > & (m-1) \dim(\mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(0)})) \end{array}$$

hence $\cap_{i=1}^{m} \mathrm{Ker}\, \psi_i \neq \{0\}$ (here, we use the elementary fact that if E is a finite dimensional vector space and if $F_1, ..., F_m$ are subspaces of E such that $\sum_{i=1}^{m} \dim F_i > (m-1) \dim E$ then $\cap_{i=1}^{n} F_i \neq \{0\}$). Let us consider a non zero $(S^{(0)}, S^{(\infty)}) \in \cap_{i=1}^{m} \mathrm{Ker}\, \psi_i$. So, for all $i \in \{1, ..., m\}$, $S^{(\infty)} R_i = R'_i S^{(0)}$. Proposition 27 ensures that $(S^{(0)}, S^{(\infty)})$ belongs to $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and that C and C' are isomorphic. This concludes the proof in virtue of Proposition 18. □

5.5. Numerical characterization of strong rigidity. — For all $R \in \mathrm{M}_n(\mathbb{C})$, we consider the complex linear algebraic group

$$H(R) = \{(X,Y) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \mid YR = RX\}.$$

For all $R \in \mathrm{M}_n(\mathbb{C})$, we consider the Lie algebra

$$\mathfrak{h}(R) = \{(X,Y) \in \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \mid YR = RX\}.$$

We have $\dim H(R) = \dim \mathfrak{h}(R)$.

Theorem 29. — *Let $C=(A^{(0)},M,A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}_c$ of size n such that $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)})\cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)})=\emptyset$. Let $s_1,...,s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$ and set, for all $i\in\{1,...,m\}$, $\operatorname{Res}_{s_i} M=R_i$. Then :*

i) $\sum_{i=1}^m \dim H(R_i)\leq 2mn^2-(\dim \mathrm{Z}(A^{(0)})+\dim \mathrm{Z}(A^{(\infty)}))+1$;

ii) C is strongly rigid if and only if $\sum_{i=1}^m \dim H(R_i)=2mn^2-(\dim \mathrm{Z}(A^{(0)})+\dim \mathrm{Z}(A^{(\infty)}))+1$.

Proof. — The proof is similar to the proof of Theorem 28. For this reason, we just explain how to modify this proof.

Considering, for all $i\in\{1,...,m\}$, the $\mathbb{C}$-linear morphism

$$\begin{array}{rcl}\varphi_i:\mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\\ (X,Y) & \mapsto & YR_i-R_iX\end{array}$$

and the $\mathbb{C}$-linear morphisms

$$\begin{array}{rcl}\varphi_0:\mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\\ (X,Y) & \mapsto & XA^{(0)}-A^{(0)}X\end{array}$$

and

$$\begin{array}{rcl}\varphi_\infty:\mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\\ (X,Y) & \mapsto & YA^{(\infty)}-A^{(\infty)}Y\end{array}$$

and setting $\varphi=(\varphi_0,\varphi_1,...,\varphi_m,\varphi_\infty)$ the proof of i) is similar to the proof of the assertion i) of Theorem 28.

Considering the affine algebraic variety

$$V=\mathrm{Z}(A^{(0)})\times\left(\prod_{i=1}^m(\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C}))\right)\times\mathrm{Z}(A^{(\infty)})$$

and the complex algebraic group K which is the quotient of

$$\mathrm{Z}(A^{(0)})\times\mathrm{Z}(A^{(0)})\times\left(\prod_{i=1}^m H(R_i)\right)\times\mathrm{Z}(A^{(\infty)})\times\mathrm{Z}(A^{(\infty)})$$

by the central subgroup $\mathbb{C}^*(I_n,I_n,(I_n,I_n),...,(I_n,I_n),I_n,I_n)$, the proof of the "only if" part of ii) is similar to the proof of the "only if" part of ii) of Theorem 28.

Considering, for all $i\in\{1,...,m\}$, the $\mathbb{C}$-linear morphism

$$\begin{array}{rcl}\psi_i:\mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\\ (X,Y) & \mapsto & YR_i-R_i'X\end{array}$$

and the $\mathbb{C}$-linear morphism

$$\begin{array}{rcl}\psi_{0,\infty}:\mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C}) & \to & \mathrm{M}_n(\mathbb{C})\times\mathrm{M}_n(\mathbb{C})\\ (X,Y) & \mapsto & (XA^{(0)}-A^{(0)}X,YA^{(\infty)}-A^{(\infty)}Y),\end{array}$$

the proof of the proof of the "if" part of ii) is similar to the proof of the "if" part of ii) of Theorem 28. □

5.6. Calculation of $\dim H(R)$ and $\dim \mathrm{Z}(A)$. —

Lemma 30. — *Let $R \in \mathrm{M}_n(\mathbb{C})$ whose rank is denoted by r. We have* $\dim H(R) = r^2 + 2n^2 - 2nr$.

Proof. — It is clearly sufficient to treat the case that $R = diag(1, ..., 1, 0, ...0)$. In this case an elementary calculation shows that

$$H(R) = \left\{ \left(\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \begin{pmatrix} A & F \\ 0 & G \end{pmatrix} \right) \;\middle|\; A \in \mathrm{GL}_r(\mathbb{C}), C \in \mathrm{GL}_{n-r}(\mathbb{C}), \right.$$
$$\left. G \in \mathrm{GL}_{n-r}(\mathbb{C}), B \in \mathrm{M}_{n-r,r}(\mathbb{C}), F \in \mathrm{M}_{r,n-r}(\mathbb{C}) \right\}.$$

So $\dim H(R) = r^2 + 2(n-r)^2 + 2r(n-r) = r^2 + 2n^2 - 2nr$. □

The following result is classical.

Lemma 31. — *Let us consider $A \in \mathrm{GL}_n(\mathbb{C})$. Let $P_1, ..., P_r \in \mathbb{C}[X]$ be the invariant factors of A (i.e. $P_1, ..., P_r \in \mathbb{C}[X]$ are monic polynomials of degree ≥ 1 such that $P_1 | \cdots | P_r$ and such that A is conjugate to* $\mathrm{diag}(C_{P_1}, ..., C_{P_r})$ *where C_{P_i} is the companion matrix for P_i). Then* $\dim \mathrm{Z}(A) = \sum_{m=1}^{r} (2r - 2m + 1) \deg P_m$.

Lemma 32. — *Let us consider $A \in \mathrm{GL}_n(\mathbb{C})$. Then* $\dim \mathrm{Z}(A) \geq n$ *and the equality holds if and only if A is conjugate to some companion matrix. In particular, if* $\dim \mathrm{Z}(A) = n$ *then the eigenspaces of A have dimension* 1.

Proof. — Indeed, we have $\dim \mathrm{Z}(A) = \sum_{m=1}^{r} (2r - 2m + 1) \deg P_m = \sum_{m=1}^{r} (2r - 2m) \deg P_m + \sum_{m=1}^{r} \deg P_m = (\sum_{m=1}^{r} (2r - 2m) \deg P_m) + n$. Therefore, $\dim \mathrm{Z}(A) \geq n$ and the equality holds if and only if $\sum_{m=1}^{r} (2r - 2m) \deg P_m = 0$ if and only if $r = 1$. □

6. Generalized q-hypergeometric equations

6.1. Generalized q-hypergeometric objects of $\mathscr{E}$ and $\mathscr{C}$. —

We denote by $\mathbb{C}(z)\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1} \rangle$ the non commutative algebra of non commutative polynomials with coefficients in $\mathbb{C}(z)$ satisfying to the relation $\boldsymbol{\sigma}_q z = qz\boldsymbol{\sigma}_q$. The generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ with parameters $\underline{a} = (a_1, ..., a_n) \in (\mathbb{C}^*)^n$, $\underline{b} = (b_1, ..., b_n) \in (\mathbb{C}^*)^n$ and $\lambda \in \mathbb{C}^*$ is the regular singular q-difference operator given by :

$$\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) = \prod_{j=1}^{n} (\frac{b_j}{q} \boldsymbol{\sigma}_q - 1) - z\lambda \prod_{i=1}^{n} (a_i \boldsymbol{\sigma}_q - 1) \in \mathbb{C}(z)\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1} \rangle. \tag{4}$$

We denote by $f_0, ..., f_n \in \mathbb{C}[z]$ be the coefficients of the generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$:

$$\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) = f_0 \boldsymbol{\sigma}_q^n + f_1 \boldsymbol{\sigma}_q^{n-1} + \cdots + f_n.$$

Note that $f_0, f_1, ..., f_n$ are degree one polynomials with complex coefficients and that $f_0 = \prod_{j=1}^n \frac{b_j}{q} - z\lambda \prod_{i=1}^n a_i$ and $f_n = (-1)^n(1 - \lambda z)$.

The generalized q-hypergeometric system with parameters $\underline{a}$, $\underline{b}$ and λ is the object A of $\mathscr{E}$ given by :

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{f_n}{f_0} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{f_{n-1}}{f_0} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{f_{n-2}}{f_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{f_1}{f_0} \end{pmatrix}^{-1} \in \mathrm{GL}_n(\mathbb{C}(z)). \tag{5}$$

Definition 33. — *An object of $\mathscr{E}$ is q-hypergeometric with parameters $\underline{a}$, $\underline{b}$ and λ if it is isomorphic to (5).*

An object of $\mathscr{C}$ is q-hypergeometric with parameters $\underline{a}$, $\underline{b}$ and λ if it corresponds to the generalized q-hypergeometric system with parameters $\underline{a}$, $\underline{b}$ and λ in the Riemann-Hilbert correspondence.

We know (Proposition 7 and Proposition 8 in [**12**]) that a generalized q-hypergeometric object with parameters $\underline{a}$, $\underline{b}$ and λ is irreducible if and only if, for all $i, j \in \{1, ..., n\}$, we have $a_i/b_j \notin q^{\mathbb{Z}}$.

6.2. Rigidity and characterizations of the generalized q-hypergeometric equations. —

Lemma 34. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be an object of $\mathscr{C}$ of size n. If M is analytic on $\mathbb{C}^*$ then :*
- *M belongs to $\mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$;*
- *C is isomorphic to $(A^{(\infty)}, I_n, A^{(\infty)})$ and, hence, is reducible.*

Proof. — Let $M(z) = \sum_{k=-\infty}^{+\infty} M_k z^k$ be the Taylor expansion of M on $\mathbb{C}^*$. Since $(\sigma_q M) A^{(0)} = A^{(\infty)} M$, we have, for all $k \in \mathbb{Z}$, $q^k M_k A^{(0)} = A^{(\infty)} M_k$. Lemma 3 implies that, for $|k|$ large enough, we have $M_k = 0$ and hence $M \in \mathrm{M}_n(\mathbb{C}[z, z^{-1}])$. So $M^{-1} \in M_n(\mathbb{C}(\{z\}))$ and arguing as above we get $M^{-1} \in \mathrm{M}_n(\mathbb{C}[z, z^{-1}])$. Whence the first part of the lemma. It is immediate that (M, I_n) is an isomorphism in $\mathscr{C}$ form $(A^{(0)}, M, A^{(\infty)})$ to $(A^{(\infty)}, I_n, A^{(\infty)})$. $\square$

***Definition 35* (Property $H(\underline{a}, \underline{b}, z_0)$)**. — *Let us consider $\underline{a} = (a_1, ..., a_n) \in (\mathbb{C}^*)^n$, $\underline{b} = (b_1, ..., b_n) \in (\mathbb{C}^*)^n$ and $z_0 \in \mathbb{C}^*$. We say that an object $C = (A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}$ satisfies the condition $H(\underline{a}, \underline{b}, z_0)$ if :*

1) the poles of M on $\mathbb{C}^*$ are simple;
2) the set of poles of M is a subset of $q^{\mathbb{Z}}z_0$;
3) $\operatorname{rk}\operatorname{Res}_{z_0} M = 1$;
4) $\underline{a} = (a_1, ..., a_n)$ and $\underline{b} = (b_1, ..., b_n)$ are the lists of eigenvalues (repeated with algebraic multiplicity) of $A^{(\infty)}$ and $qA^{(0)}$ respectively.

Proposition 36. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ and $C' = (B^{(0)}, N, B^{(\infty)})$ be isomorphic normalized irreducible objects of $\mathscr{C}$ of size n. Then C satisfies $H(\underline{a}, \underline{b}, z_0)$ if and only if C' satisfies $H(\underline{a}, \underline{b}, z_0)$.*

Proof. — Let $(S^{(0)}, S^{(\infty)})$ be isomorphism from C to C'. Proposition 6 ensures that $(S^{(0)}, S^{(\infty)}) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. So, we have :

$$\begin{cases} S^{(0)}A^{(0)} = B^{(0)}S^{(0)} \\ S^{(\infty)}A^{(\infty)} = B^{(\infty)}S^{(\infty)} \\ S^{(\infty)}M = NS^{(0)}. \end{cases}$$

The result is now clear. □

Theorem 37 **(Rigidity and "monodromic" characterization of the q-hypergeometrics)**

Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}$ such that $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)}) = \emptyset$.

If C satisfies $H(\underline{a}, \underline{b}, z_0)$ then :
- the eigenspaces of $A^{(0)}$ and $A^{(\infty)}$ are one dimensional;
- C is strongly rigid.

We let $\underline{a} = (a_1, ..., a_n)$ and $\underline{b} = (b_1, ..., b_n)$ be the lists of eigenvalues (repeated with algebraic multiplicity) of $A^{(\infty)}$ and $qA^{(0)}$ respectively. The following properties are equivalent :
i) C satisfies $H(\underline{a}, \underline{b}, z_0)$;
ii) C is q-hypergeometric with parameters $\underline{a}$, $\underline{b}$ and $\lambda = (\prod_{j=1}^n \frac{b_j}{q})(z_0 \prod_{i=1}^n a_i)^{-1}$.

Proof. — Let us consider a normalized irreducible object $C = (A^{(0)}, M, A^{(\infty)})$ of $\mathscr{C}$ such that $q^{\mathbb{Z}}\operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}}\operatorname{Sp}(A^{(\infty)}) = \emptyset$ satisfying $H(\underline{a}, \underline{b}, z_0)$. Lemma 34 ensures that M has a least one pole on $\mathbb{C}^*$ and, hence, that the set of poles of M on $\mathbb{C}^*$ is $q^{\mathbb{Z}}z_0$. Using Lemma 30, we see that Theorem 29 i) can be rewritten as follows

$$(6) \qquad 2n^2 - 2n + 1 \leq 2n^2 - (\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1.$$

So

$$\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)}) \leq 2n.$$

Since $\dim \mathrm{Z}(A^{(0)}) \geq n$ and $\dim \mathrm{Z}(A^{(\infty)}) \geq n$ (Lemma 32), we get $\dim \mathrm{Z}(A^{(0)}) = \dim \mathrm{Z}(A^{(\infty)}) = n$ and hence the eigenspaces of $A^{(0)}$ and $A^{(\infty)}$ have dimension 1 (Lemma 32), whence i).

Moreover, we obtain that the inequality (6) is actually an equality. Therefore, Theorem 29 ii) ensures that C is strongly rigid.

We shall now prove ii) $\Rightarrow$ i). Since, by definition, two q-hypergeometric objects with same parameters are isomorphic, Proposition 36 shows that it is sufficient to prove this implication for a specific q-hypergeometric object with parameters $\underline{a}$, $\underline{b}$ and λ. We keep the notations of section 6.1 for the generalized q-hypergeometric system with parameters $\underline{a}$, $\underline{b}$ and λ. The eigenvalues of $A(0)$ and $A(\infty)$ (repeated with algebraic multiplicity) are respectively $\underline{b}/q$ and $\underline{a}$ and hence the system is non resonant (terminology of [**14**], section 1.2.2). There exists (see section 1.2.2 of [**14**])

$$(M^{(0)}, M^{(\infty)}) \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C})) \times \mathrm{GL}_n(\mathcal{M}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0\}))$$

such that $(\sigma_q M^{(0)})A(0) = AM^{(0)}$ and $(\sigma_q M^{(\infty)})A(\infty) = AM^{(\infty)}$. Hence

$$C = (A^{(0)}, M, A^{(\infty)}) := (A(0), (M^{(\infty)})^{-1}M^{(0)}, A(\infty))$$

is a normalized irreducible q-hypergeometric object of $\mathscr{C}$ with parameters $\underline{a}$, $\underline{b}$ and λ. We will prove the result for this specific q-hypergeometric object.

We have, for all $k \in \mathbb{N}^*$:

$$M = (A^{(\infty)})^k (\sigma_q^{-k} M^{(\infty)})^{-1} (\sigma_q^{-k} A)^{-1} \cdots (\sigma_q^{-1} A)^{-1} A^{-1} \cdots \cdots (\sigma_q^{k-1} A)^{-1} (\sigma_q^{k} M^{(0)}) (A^{(0)})^k. \tag{7}$$

Let us consider $s \in \mathbb{C}^* \setminus q^{\mathbb{Z}} z_0$. For $k \in \mathbb{N}^*$ large enough, $\sigma_q^k M^{(0)} \in \mathrm{GL}_n(\mathbb{C}\{z - s\})$ and $\sigma_q^{-k} M^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}\{z-s\})$. Moreover, for any $k \in \mathbb{Z}$, $\sigma_q^k A^{-1} \in \mathrm{M}_n(\mathbb{C}\{z-s\})$. Hence, (7) shows that $M \in \mathrm{M}_n(\mathbb{C}\{z-s\})$. So the set of poles of M on $\mathbb{C}^*$ is included in $q^{\mathbb{Z}} z_0$.

For $k \in \mathbb{N}^*$ large enough, $\sigma_q^k M^{(0)} \in \mathrm{GL}_n(\mathbb{C}\{z-z_0\})$ and $\sigma_q^{-k} M^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}\{z-z_0\})$. Moreover $A^{-1} = \frac{R}{z-z_0}$ mod. $\mathrm{M}_n(\mathbb{C}\{z-z_0\})$ for some $R \in \mathrm{M}_n(\mathbb{C})$ with rank at most one and, for any $k \in \mathbb{Z}^*$, $\sigma_q^k A^{-1} \in \mathrm{M}_n(\mathbb{C}\{z-z_0\})$. Therefore, there exists $R_1 \in \mathrm{M}_n(\mathbb{C})$ with rank at most one such that :

$$M = \frac{R_1}{z - z_0} \text{ mod. } \mathrm{M}_n(\mathbb{C}\{z-z_0\}). \tag{8}$$

We claim that $R_1 \neq 0$. Indeed, if $R_1 = 0$ then M would be analytic near z_0 and, hence, it would be analytic near $q^{\mathbb{Z}} z_0$ (because of the functional equation $(\sigma_q M)A^{(0)} = A^{(\infty)}M$). Therefore, M would be analytic on $\mathbb{C}^*$ and, hence, C would be reducible in virtue of Lemma 34 : contradiction. So z_0 is a simple pole of M and $\operatorname{rk} \operatorname{Res}_{z_0} M = \operatorname{rk} R_1 = 1$. Once again, the functional equation $(\sigma_q M)A^{(0)} = A^{(\infty)}M$ implies that M has simple poles at any point of $q^{\mathbb{Z}} z_0$.

Hence C satisfies $H(\underline{a}, \underline{b}, z_0)$ as expected.

It remains to prove i) $\Rightarrow$ ii). Let $C' = (B^{(0)}, N, B^{(\infty)})$ be a normalized irreducible q-hypergeometric object of $\mathscr{C}$ with parameters $\underline{a}$, $\underline{b}$ and λ such that $\underline{a} = (a_1, ..., a_n)$ and $\underline{b} = (b_1, ..., b_n)$ are the lists of eigenvalues (repeated with algebraic multiplicity) of $B^{(\infty)}$ and $qB^{(0)}$ (we have seen during the proof of ii) $\Rightarrow$ i) that such an object exists). Then, the first assertion of the present theorem ensures that $A^{(0)}$ and $A^{(\infty)}$ are conjugated to $B^{(0)}$ and $B^{(\infty)}$ respectively. Moreover, the set of poles of M and that of N are equal to $q^{\mathbb{Z}} z_0$ and $\operatorname{rk} \operatorname{Res}_{z_0} M = \operatorname{rk} \operatorname{Res}_{z_0} N (= 1)$. Therefore, C and C' are weakly locally isomorphic and, hence, isomorphic because C is strongly rigid in virtue of the first part of this Proposition. $\square$

Proposition 38. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible object of $\mathscr{C}_c$ such that $q^{\mathbb{Z}} \operatorname{Sp}(A^{(0)}) \cap q^{\mathbb{Z}} \operatorname{Sp}(A^{(\infty)}) = \emptyset$. If both $A^{(0)}$ and $A^{(\infty)}$ have n distinct eigenvalues then the following properties are equivalent :*

- C is strongly rigid;

- C is q-hypergeometric.

Proof. — Let $s_1, ..., s_m$ be the poles of M on some fundamental domain of $\mathbb{C}^*$ with respect to the action by multiplication of $q^{\mathbb{Z}}$. We set, for all $i \in \{1, ..., m\}$, $\operatorname{Res}_{s_i} M = R_i$. In virtue of Theorem 29, C is strongly rigid if and only if

$$\sum_{i=1}^{m} \dim H(R_i) = 2mn^2 - (\dim \mathrm{Z}(A^{(0)}) + \dim \mathrm{Z}(A^{(\infty)})) + 1.$$

Using Lemma 30 and the fact that $\dim \mathrm{Z}(A^{(0)}) = \dim \mathrm{Z}(A^{(\infty)}) = n$, we get that this equality is equivalent to

$$\sum_{i=1}^{m} \operatorname{rk} R_i (\operatorname{rk} R_i - 2n) = 1 - 2n.$$

This equality holds if and only if $m = 1$ and $\operatorname{rk} R_1 = 1$. The result follows from Theorem 37. $\square$

6.3. Description of the q-hypergeometric residues. — For all $\underline{x} = (x_1, ..., x_\kappa) \in \mathbb{C}^\kappa$, we set :

$$\xi(\underline{x}) = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_2 \\ 0 & \cdots & & 0 & x_1 \end{pmatrix} \in \mathrm{M}_\kappa(\mathbb{C}).$$

For all $\underline{x_1} \in \mathbb{C}^{\kappa_1}$,..., $\underline{x_r} \in \mathbb{C}^{\kappa_r}$, we set

$$\xi(\underline{x_1};...;\underline{x_r}) = \mathrm{diag}(\xi(\underline{x_1}),...,\xi(\underline{x_r})) \in \mathrm{M}_{\kappa_1+\cdots+\kappa_r}(\mathbb{C}).$$

For all $\kappa_1,...,\kappa_r \in \mathbb{N}^*$, we set

$$\Xi(\kappa_1,...,\kappa_r) = \{\xi(\underline{x_1};...;\underline{x_r}) \mid \underline{x_1} \in \mathbb{C}^* \times \mathbb{C}^{\kappa_1-1},... \\ ..., \underline{x_k} \in \mathbb{C}^* \times \mathbb{C}^{\kappa_k-1}\} \subset \mathrm{GL}_{\kappa_1+\cdots+\kappa_r}(\mathbb{C}).$$

For all $\kappa_1,...,\kappa_r \in \mathbb{N}^*$ and $\tau_1,...,\tau_s \in \mathbb{N}^*$ such that $\kappa_1+\cdots+\kappa_r = \tau_1+\cdots+\tau_s = n$, we set

$$\mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s) = \{R = (r_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(\mathbb{C}) \mid \mathrm{rk}\, R = 1 \\ \text{and } \forall (k,l) \in [[1,r]] \times [[1,s]], r_{\kappa_1+\cdots+\kappa_k,\tau_1+\cdots+\tau_l} \neq 0\}. \tag{9}$$

Lemma 39. — *Let $R \in \mathrm{M}_n(\mathbb{C})$ be a rank one matrix. Let us consider $\kappa_1,...,\kappa_r \in \mathbb{N}^*$ and $\tau_1,...,\tau_s \in \mathbb{N}^*$ such that $\kappa_1+\cdots+\kappa_r = \tau_1+\cdots+\tau_s = n$. We consider the map*

$$\begin{array}{rcl}\eta : \Xi(\kappa_1,...,\kappa_r) \times \Xi(\tau_1,...,\tau_s) & \rightarrow & \mathrm{M}_n(\mathbb{C}) \\ (X,Y) & \mapsto & XR^tY.\end{array}$$

The following conditions are equivalent :
(i) $R \in \mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$;
(ii) $\eta^{-1}(R) \subset \mathbb{C}^*I_n \times \mathbb{C}^*I_n$.
Moreover, if these conditions hold then $\mathrm{Im}(\eta) = \mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$.

Proof. — Let $U = (\underline{u_1},...,\underline{u_r}) \in \mathbb{C}^{\kappa_1} \times \cdots \times \mathbb{C}^{\kappa_r}$ and $V = (\underline{v_1},...,\underline{v_s}) \in \mathbb{C}^{\tau_1} \times \cdots \times \mathbb{C}^{\tau_r}$ be such that $R = {}^tUV$. Then, for any $X = \xi(\underline{x_1};...;\underline{x_r}) \in \Xi(\kappa_1,...,\kappa_r)$ and $Y = \xi(\underline{y_1};..;\underline{y_{\tau_s}}) \in \Xi(\tau_1,...,\tau_s)$, we have

$$\eta(X,Y) = (X^tU)^t(Y^tV)$$

and we have

$$X^tU = {}^t(a_{1,1},...,a_{1,\kappa_1},...,a_{r,1},...,a_{r,\kappa_r}) \text{ where } a_{i,k} = x_{i,1}u_{i,k}+\cdots+x_{i,\kappa_i-k+1}u_{i,\kappa_i}$$

and

$$Y^tV = {}^t(b_{1,1},...,b_{1,\kappa_1},...,b_{s,1},...,b_{s,\tau_s}) \text{ where } b_{j,l} = y_{j,1}v_{j,l}+\cdots+y_{j,\tau_j-l+1}v_{j,\tau_j}.$$

Assume that (i) does not hold i.e. that $R \notin \mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$. Either there exists $i \in [[1,r]]$ such that $u_{i,\kappa_i} = 0$ or there exists $j \in [[1,s]]$ such that $v_{j,\tau_j} = 0$. Suppose for instance that $u_{i,\kappa_i} = 0$ for some $i \in [[1,r]]$ (the other case is similar). Then $\eta(X,Y)=R$ with $Y=I_n\in\Xi(\tau_1,...,\tau_s)$ and, if $\kappa_i\geq 2$,

$$X=\xi(\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_1-1}};...;\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_{i-1}-1}};\underbrace{(1,0,...,0,1)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_i-1}};\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_{i+1}-1}};...$$
$$...;\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_n-1}})\in\Xi(\kappa_1,...,\kappa_r)$$

and, if $\kappa_i=1$,

$$X=\xi(\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_1-1}};...;\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_{i-1}-1}};\underbrace{(2)}_{\in\mathbb{C}^*};\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_{i+1}-1}};...$$
$$...;\underbrace{(1,0,...,0)}_{\in\mathbb{C}^*\times\mathbb{C}^{\kappa_n-1}})\in\Xi(\kappa_1,...,\kappa_r).$$

So condition (ii) does not hold.

Assume that (i) holds i.e. that $R\in\mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$. So, for all $i\in[[1,r]]$, $u_{i,\kappa_i}\neq 0$ and, for all $j\in[[1,s]]$, $v_{j,\tau_j}\neq 0$. Let us consider $X=\xi(\underline{x_1};\cdots;\underline{x_r})\in\Xi(\kappa_1,...,\kappa_r)$ and $Y=\xi(\underline{y_1};..;\underline{y_s})\in\Xi(\tau_1,...,\tau_s)$ such that $\eta(X,Y)=R$. This equality ensures that

$$\begin{array}{rcl} a_{i,\kappa_i}b_{j,1} & = & u_{i,\kappa_i}v_{j,1}\\ \vdots & \vdots & \vdots\\ a_{i,\kappa_i}b_{j,\tau_j-1} & = & u_{i,\kappa_i}v_{j,\tau_j-1}\\ a_{i,\kappa_i}b_{j,\tau_j} & = & u_{i,\kappa_i}v_{j,\tau_j}\end{array}$$

i.e.

$$\begin{array}{rcl} x_{i,1}u_{i,\kappa_i}(y_{j,1}v_{j,1}+\cdots+y_{j,\tau_j}v_{j,\tau_j}) & = & u_{i,\kappa_i}v_{j,1}\\ \vdots & \vdots & \vdots\\ x_{i,1}u_{i,\kappa_i}(y_{j,1}v_{j,\tau_j-1}+y_{j,2}v_{j,\tau_j}) & = & u_{i,\kappa_i}v_{j,\tau_j-1}\\ x_{i,1}u_{i,\kappa_i}y_{j,1}v_{j,\tau_j} & = & u_{i,\kappa_i}v_{j,\tau_j}.\end{array}$$

Since u_{i,κ_i}, v_{j,τ_j} and $x_{i,1}$ are non zero, we see that $x_{i,1}y_{j,1}=1$ and $y_{j,2}=\cdots=y_{j,\tau_j}=0$. Similarly, $x_{j,2}=\cdots=x_{j,\tau_j}=0$. Note that the equalities, for all $(i,j)\in[[1,r]]\times[[1,s]]$, $x_{i,1}y_{j,1}=1$ ensure that $x_{1,1}=\cdots=x_{r,1}$ and $y_{1,1}=\cdots=y_{s,1}$. Hence X and Y belong to $\mathbb{C}^*I_n$ as expected and condition (ii) holds.

Therefore, we have proved that (i) and (ii) are equivalent.

Assume that (i) holds and hence that, for all $i \in [[1,r]]$, $u_{i,\kappa_i} \neq 0$ and, for all $j \in [[1,s]]$, $v_{j,\kappa_j} \neq 0$. Then it is clear that both

$$\begin{array}{rcl} \Xi(\kappa_1,...,\kappa_r) & \to & (\mathbb{C}^{\kappa_1-1}\times\mathbb{C}^*)\times\cdots\times(\mathbb{C}^{\kappa_r-1}\times\mathbb{C}^*) \\ X & \mapsto & {}^t(X{}^tU) \end{array}$$

and

$$\begin{array}{rcl} \Xi(\tau_1,...,\tau_s) & \to & (\mathbb{C}^{\tau_1-1}\times\mathbb{C}^*)\times\cdots\times(\mathbb{C}^{\tau_s-1}\times\mathbb{C}^*) \\ Y \mapsto {}^t(Y{}^tV) & & \end{array}$$

are surjective. Therefore $\mathrm{Im}(\psi_R) = R(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$. □

Theorem 40. — *Let $C = (A^{(0)}, M, A^{(\infty)})$ be a normalized irreducible q-hypergeometric object of $\mathscr{C}$ with parameters $\underline{a}$, $\underline{b}$ and λ. We set $z_0 = (\prod_{j=1}^n \frac{b_j}{q})(\lambda\prod_{i=1}^n a_i)^{-1}$. Then*

$$\begin{array}{l} \{\mathrm{Res}_{z_0} N \mid (A^{(0)}, N, A^{(\infty)})\ q\text{-hypergeometric with parameters } \underline{a},\ \underline{b} \text{ and } \lambda\} \\ = Q^{-1}\mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)P \end{array}$$

where :
- *κ_1,...,κ_r are the algebraic multiplicities of the eigenvalues of $A^{(\infty)}$;*
- *τ_1,...,τ_s are the algebraic multiplicities of the eigenvalues of $A^{(0)}$;*
- *$P, Q \in \mathrm{GL}_n(\mathbb{C})$ are such that $PA^{(0)}P^{-1} \in {}^t\Xi(\tau_1,...,\tau_s)$ and $QA^{(\infty)}Q^{-1} \in \Xi(\kappa_1,...,\kappa_r)$.*

Proof. — Using the fact that $(P,Q) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ is an isomorphism from C to $(PA^{(0)}P^{-1}, QMP^{-1}, QA^{(\infty)}Q^{-1})$, it is clearly sufficient to consider the case $P = Q = I_n$.

Let N be such that $(A^{(0)}, N, A^{(\infty)})$ is q-hypergeometric with parameters $\underline{a}$, $\underline{b}$ and λ. We set $R = \mathrm{Res}_{z_0} N$. Theorem 28 i) ensures that $\dim G(R) = \dim \mathfrak{g}(R) \leq 1$ therefore $\{(X,Y) \in \mathfrak{z}(A^{(0)}) \times \mathfrak{z}(A^{(\infty)}) \mid YR = RX\} = \mathbb{C}(I_n, I_n)$. Let us consider $(X,Y) \in \Xi(\kappa_1,...,\kappa_r) \times \Xi(\tau_1,...,\tau_s)$ such that $XR{}^tY = R$ i.e. $XR = R({}^tY)^{-1}$. Since $X \in \Xi(\kappa_1,...,\kappa_r) \subset \mathfrak{z}(A^{(\infty)})$ and $({}^tY)^{-1} \in {}^t\Xi(\tau_1,...,\tau_s) \subset \mathfrak{z}(A^{(0)})$, we get that $(({}^tY)^{-1}, X) \in G(R)$ and, hence, X and Y belong to $\mathbb{C}^*I_n$. Lemma 39 ensures that $R \in \mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)$. So we have proved the inclusion

(10)
$$\begin{array}{r} \{\mathrm{Res}_{z_0} N \mid (A^{(0)}, N, A^{(\infty)})\ q\text{-hypergeometric with parameters } \underline{a},\ \underline{b} \text{ and } \lambda\} \\ \subset Q^{-1}\mathscr{R}_{q-hyp}(\kappa_1,...,\kappa_r;\tau_1,...,\tau_s)P \end{array}$$

Moreover, for all $(X,Y) \in \Xi(\kappa_1,...,\kappa_r) \times \Xi(\tau_1,...,\tau_s)$, $(A^{(0)}, N, A^{(\infty)}) := (A^{(0)}, XM{}^tY, A^{(\infty)})$ is q-hypergeometric with parameters $\underline{a}$, $\underline{b}$ and λ (because $(({}^tY)^{-1}, X)$ is is an isomorphism between $(A^{(0)}, M, A^{(\infty)})$ and $(A^{(0)}, N, A^{(\infty)})$).

We obviously have $\mathrm{Res}_{z_0} N = X(\mathrm{Res}_{z_0} M)^t Y$. Since $\mathrm{Res}_{z_0} M$ belongs to $\mathscr{R}_{q-hyp}(\kappa_1, ..., \kappa_r; \tau_1, ..., \tau_s)$ (in virtue of the inclusion (10)), the fact that the inclusion (10) is actually an equality is a direct consequence of the last assertion of Lemma 39. □

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Julien Roques, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 St Martin d'Hères • *E-mail :* `Julien.Roques@ujf-grenoble.fr`