GLOBAL ENTROPY STABILITY FOR A CLASS OF UNLIMITED HIGH-ORDER SCHEMES FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

LUDOVIC MARTAUD*, MEHDI BADSI*, CHRISTOPHE BERTHON*, ARNAUD DURAN†, AND KHALED SALEH†

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Abstract. We design high-order schemes to approximate the weak solutions of hyperbolic systems of conservation laws. These schemes are based on high order correction of the standard HLL flux. They are proved to satisfy a global entropy stability property under an appropriate CFL condition. These schemes do not involve limitation techniques and thus relevantly preserve the order of accuracy. Numerical experiments illustrate the accuracy and the stability of the proposed schemes.

1. Introduction.

The present work concerns the numerical approximation of the weak solutions of systems made of $d \geq 1$ conservation laws in one space dimension given by

$$\partial_t w + \partial_x f(w) = 0, \quad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

The unknown state vector $w(x,t)$ is assumed to belong to Ω a non-empty convex open subset of $\mathbb{R}^d$. Here, $f : \Omega \to \mathbb{R}^d$ is a given smooth flux function. It is assumed to be such that the $d \times d$ Jacobian matrix $\nabla f(w)$ is diagonalizable in $\mathbb{R}$ so that the system (1.1) is a hyperbolic system of conservation laws. We consider the Cauchy problem for (1.1), that is we prescribe an initial data at time $t = 0$ as follows:

$$w(x, t = 0) = w_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $w_0 : \mathbb{R} \to \Omega$ is a given measurable function. According to [34, 35, 42] (see also [18, 19, 36, 47]), it is well-known that the solutions of (1.1)-(1.2) may develop, in a finite time, discontinuities and that the weak solutions are in general non unique. In order to rule out non-admissible weak solutions, the system (1.1) must be endowed with entropy inequalities. In this regard, we assume the existence of both a strictly convex function $\eta \in C^2(\Omega, \mathbb{R})$, called entropy function, and an entropy flux function $G \in C^2(\Omega, \mathbb{R})$ such that

$$\nabla \eta(w)^T \nabla f(w) = \nabla G(w)^T, \quad \forall w \in \Omega. \tag{1.3}$$

We then note that smooth solutions of (1.1) satisfy the following additional conservation law

$$\partial_t \eta(w) + \partial_x G(w) = 0,$$

while weak solutions, containing discontinuities, verify an entropy inequality (for instance, see [34, 35, 42]) given by

$$\partial_t \eta(w) + \partial_x G(w) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0, +\infty)). \tag{1.4}$$

*Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, Université de Nantes, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France.

†Institut Camille Jordan, CNRS UMR 5208, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France.

A weak solution of (1.1) is called an entropy satisfying solution if and only if the entropy inequality (1.4) holds for any pair entropy-entropy flux (η, G). Integrating in space the entropy inequality (1.4) results in a global entropy stability inequality,

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(w(x,t))\,dx \le 0.$$

As a consequence, provided $\int_{\mathbb{R}} \eta(w_0(x))\,dx$ is finite, we have for all $t > 0$

$$\int_{\mathbb{R}} \eta(w(x,t))\,dx \le \int_{\mathbb{R}} \eta(w_0(x))\,dx. \tag{1.5}$$

The inequality (1.5) is a global entropy stability inequality. Within the specific context of scalar conservation laws, we may use $\eta(w) = w^2/2$ or for symmetric system of conservation laws $\eta(w) = \frac{1}{2}\sum_{j=1}^{d} w_j^2$ so that (1.5) reformulates as follows for all $t > 0$

$$\|w(t,.)\|_{L^2} \le \|w_0\|_{L^2},$$

which expresses the decrease of the L^2-norm satisfied by the solution. For general hyperbolic systems of conservation laws (1.1), the global entropy decreasing property (1.5) is reminiscent of a L^2 weighted type stability since the strict convexity of the entropy function η yields that the hyperbolic system (1.1) is symmetrizable.

As for the numerical approximation, we approximate the weak solutions of (1.1), at time t^n, by the following piecewise constant function

$$w_\Delta(x,t^n) = w_i^n \quad \text{ if } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \tag{1.6}$$

where $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ define the sequence of the mesh nodes. The quantities w_i^n are approximations of the average of the solution over the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ as follows,

$$w_i^n \backsimeq \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)\,dx$$

where $w(x,t^n)$ naturally belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$. For the sake of simplicity, we consider a uniform mesh made of constant size mesh cells $\Delta x > 0$. As a consequence, we have $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$ for all $i \in \mathbb{Z}$. In addition, we introduce the time increment $\Delta t > 0$ so that $t^{n+1} = t^n + \Delta t$. Over the past fifty years, numerous strategies have been proposed to evolve in time the approximation (1.6) and to define suitable updated states $(w_i^{n+1})_{i\in\mathbb{Z}}$ (for instance, see [11, 18, 19, 26, 36, 41, 47] and references therein). In the present work, we use conservative finite volume schemes so that the updated state reads

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\right), \tag{1.7}$$

where $\mathcal{F}_{i+\frac{1}{2}} \in \mathbb{R}^d$ is a numerical flux function. According to [19, 26, 47], provided $w_i^n = w$ for all $i \in \mathbb{Z}$, where w denotes here an arbitrary constant state, if we get

$$\mathcal{F}_{i+\frac{1}{2}} = f(w), \quad \forall w \in \Omega, \tag{1.8}$$

then the scheme (1.7) is known to be first-order consistent and in conservative form. As a consequence, we may expect from the famous Theorem by Lax and Wendroff [25], some convergence results. Namely, if the sequence $(w_i^n)_{i\in\mathbb{Z},n\in\mathbb{N}}$ converges in a sense to be defined then the limit function is a weak solution of (1.1). However, the obtained limit solution is not necessarily entropy satisfying and non-admissible discontinuous waves may appear (for instance, see [9,31]). To correct such unphysical solutions, one asks the approximate solution to satisfy, in addition, discrete entropy inequalities in the form

$$\eta(w_i^{n+1}) \leq \eta(w_i^n) - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{i+\frac{1}{2}} - \mathcal{G}_{i-\frac{1}{2}}\right), \tag{1.9}$$

where $\mathcal{G}_{i+\frac{1}{2}} \in \mathbb{R}^d$ denotes a numerical entropy flux function, which must be consistent; namely $\mathcal{G}_{i+\frac{1}{2}} = G(w)$ as long as $w_i^n = w$ for all $i \in \mathbb{Z}$ where $G(w)$ is the entropy flux function given by (1.3). From (1.9), we immediately recover the numerical counterpart to the global entropy stability condition (1.5) so that

$$\sum_{i\in\mathbb{Z}} \eta(w_i^{n+1})\Delta x \leq \sum_{i\in\mathbb{Z}} \eta(w_i^n)\Delta x. \tag{1.10}$$

The design of numerical schemes able to provide discrete entropy inequalities (1.9) and thus able to satisfy the global entropy stability (1.10) turns out to be very challenging. Among the few first-order approaches able to exhibit such estimates, we refer to the exact Godunov scheme [20,26], the kinetic schemes [4,32], the HLL scheme [25, 26], the HLLC scheme [48], some relaxation schemes such as Suliciu relaxation approaches [3,5,10] or the numerical strategy introduced by Tadmor [45,46]. Staggered schemes introduce an appropriate framework with respect to entropy stability, as illustrated in [27] in the implicit case for instance. In the same formalism, fully explicit results were proposed in [16, 17, 28] for the Euler equations, and in [15] for the shallow water equations. In [8, 29, 30], the global estimation (1.10) is established to justify the stability of the derived schemes. Let us however underline that, from a general viewpoint, time discretization is an important technical obstacle and stability is often considered in the semi-discrete setting [1, 7, 45]. Unfortunately, such semi-discrete entropy inequalities are known not to be sufficient to rule out non-admissible discontinuities in the converged solutions.

As far as high-order numerical approximations are concerned, the situation turns out to be drastically distinct. We may quote Bouchut [5] page 54, "*It is extremely difficult to obtain second-order schemes that verify an entropy inequality*". Several works devoted to high-order schemes attempted to exhibit discrete entropy inequalities (1.9). For instance, in [46], semi-discrete entropy estimates associated with (1.9), are established. In [2,6] (see also [39,40]), fully discrete entropy estimates are introduced as follows:

$$\eta(w_i^{n+1}) \leq \bar{\eta}_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{i+\frac{1}{2}} - \mathcal{G}_{i-\frac{1}{2}}\right), \tag{1.11}$$

for a suitable entropy average $\bar{\eta}_i^n$. Such discrete entropy inequality strategies are not fully relevant since Lax-Wendroff Theorem [33] cannot be successfully applied. It has been illustrated in [13] where the authors proved that, in the convergence limit, the expected entropy inequality (1.4) is satisfied up to a positive measure. In addition, in [13], numerical experiments exhibited the capture of non-admissible shock solutions for MUSCL schemes which satisfy (1.11).

Hopefully, recent formal developments, proposed in [12, 24, 38], may indicate that a discrete entropy global stability (1.10) is reachable. The key ingredient in their strategy consists in a suitable control of the high-order diffusion term in the numerical fluxes to get the required global numerical entropy stability (1.10). Thus, the aim of the present work is the design of high-order schemes to approximate the weak solutions of (1.1) which satisfy the global entropy stability condition (1.10). Although it is a global stability criterion, a local stability of the approximate solution may be observed numerically [8, 29, 30].

The paper is organized as follows. In the next section, we introduce a class of high-order schemes. This class is derived from the well-known HLL scheme [26] complemented with suitable higher-order corrections obtained by ensuring the high order consistency of the numerical flux function with the physical flux function. For the sake of conciseness in the paper, we derive second-, third- and fourth-order space accurate schemes to approximate the weak solutions of (1.1). The reader will be easily convinced by the possibility of high-order accurate extensions. In Section 3, we establish (1.10). The proof relies on the design of a relevant CFL-like condition to restrict the time step, and the use of the large enough dissipation granted by the first order viscosity of the HLL scheme to control (likely anti-dissipative) high order corrective terms. In the last section, several numerical experiments are carried out to illustrate both the stability and the accuracy of the proposed schemes.

2. Unlimited high-order HLL schemes. We derive high-order space accurate schemes. The starting point is the original first-order HLL scheme [26] that reads as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) - \mathcal{F}_\lambda^{O1}(w_{i-1}^n, w_i^n)\right), \tag{2.1}$$

where the numerical flux function is given by

$$\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) = \frac{1}{2}\left(f(w_i^n) + f(w_{i+1}^n)\right) - \frac{\lambda}{2}\left(w_{i+1}^n - w_i^n\right). \tag{2.2}$$

Here, $\lambda > 0$ stands for the numerical viscosity coefficient. Under the following CFL conditions:

$$\frac{\Delta t}{\Delta x}\lambda \leq \frac{1}{2} \quad \text{with} \quad \lambda \geq \max_{i\in\mathbb{Z}}\left(|\mu(w_i^n)|\right),$$

where $\mu(w)$ denotes the spectral radius of $\nabla f(w)$, the scheme (2.1) is known to be entropy preserving (see [26]). As a consequence, there exists a numerical entropy flux function $\mathcal{G}_\lambda^{O1}(w_i^n, w_{i+1}^n)$, consistent with the entropy flux function $G(w)$, such that for all $i \in \mathbb{Z}$ we have

$$\eta(w_i^{n+1}) \leq \eta(w_i^n) - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_\lambda^{O1}(w_i^n, w_{i+1}^n) - \mathcal{G}_\lambda^{O1}(w_{i-1}^n, w_i^n)\right), \tag{2.3}$$

for all entropy pairs (η, G).

Equipped with this first-order scheme, we are in position to increase the order of accuracy in space. Before doing so, we first recall the following result that characterizes the accuracy of finite volume schemes (for instance, see [5] Proposition 2.26 for the proof).

LEMMA 2.1. *Consider a numerical scheme of the form*

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(w_{i-\nu}^n, \cdots, w_{i+\nu+1}^n) - \mathcal{F}(w_{i-\nu-1}^n, \cdots, w_{i+\nu}^n)\right),$$

where $\nu \geq 0$ is an integer. The scheme is k^{th}-order of space accuracy if, for a fixed $x_{i+\frac{1}{2}}$, we have

$$\mathcal{F}(u_{i-\nu}, \cdots, u_{i+\nu+1}) = f(u(x_{i+\frac{1}{2}})) + \mathcal{O}(\Delta x^k),$$

where, for a given smooth function $u(x)$, we have set

$$u_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x)\, dx. \tag{2.4}$$

Thanks to this result, we easily notice that the numerical flux function (2.2) is first-order. More precisely, with (2.4), a standard Taylor expansion in a neighborhood of $x_{i+\frac{1}{2}}$ gives

$$\begin{aligned} \mathcal{F}_\lambda^{O1}(u_i, u_{i+1}) = {} & f\left(u(x_{i+\frac{1}{2}})\right) - \frac{\lambda \Delta x}{2} \partial_x u(x_{i+\frac{1}{2}}) \\ & + \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(u(x_{i+\frac{1}{2}})\right) + \frac{1}{3} \nabla f\left(u(x_{i+\frac{1}{2}})\right) \partial_{xx} u(x_{i+\frac{1}{2}}) \right) \\ & - \frac{\lambda \Delta x^3}{24} \partial_{xxx} u(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^4). \end{aligned} \tag{2.5}$$

The main idea is then to define a high-order correction of the numerical flux function $\mathcal{F}_\lambda^{O1}$ that is based on the Taylor expansion (2.5). We therefore consider numerical flux functions of the form

$$\mathcal{F}_\lambda^{Ok}(w_{i-\nu}^n, \cdots, w_{i+\nu+1}^n) = \mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) + \frac{1}{2}\left(\alpha_i^{Ok} + \alpha_{i+1}^{Ok}\right).$$

where the superscript Ok refers to the term "k^{th}-order" and will take values in the set $\{O2, O3, O4\}$ according to the space order accuracy of the scheme. From the Taylor expansion (2.5), we observe that the following consistency, in a neighborhood of a fixed $x_{i+\frac{1}{2}}$, must be satisfied by the correction α_i^{Ok} according to the selected order of accuracy:

$$\alpha_i^{O2} = \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^2), \tag{2.6}$$

$$\begin{aligned} \alpha_i^{O3} = {} & \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) \\ & - \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(w(x_{i+\frac{1}{2}}, t^n)\right) + \frac{1}{3} \nabla f\left(w(x_{i+\frac{1}{2}}, t^n)\right) \partial_{xx} w(x_{i+\frac{1}{2}}, t^n) \right) + \mathcal{O}(\Delta x^3), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \alpha_i^{O4} = {} & \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) \\ & - \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(w(x_{i+\frac{1}{2}}, t^n)\right) + \frac{1}{3} \nabla f\left(w(x_{i+\frac{1}{2}}, t^n)\right) \partial_{xx} w(x_{i+\frac{1}{2}}, t^n) \right) \\ & + \frac{\lambda \Delta x^3}{24} \partial_{xxx} w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^4), \end{aligned} \tag{2.8}$$

where respectively α_i^{O2} is the second-order correction, α_i^{O3} the third-order correction and α_i^{O4} the fourth-order correction. We thereby stress that the high order numerical flux function $\mathcal{F}_\lambda^{Ok}$ contains both approximation of the term $-\frac{\lambda\Delta x}{2}\partial_x w$ which inherits from the HLL flux function $\mathcal{F}_\lambda^{O1}$ and approximation of the same term but with the opposite sign $+\frac{\lambda\Delta x}{2}\partial_x w$ which inherits from the corrective term α_i^{Ok}. At the continuous level the sum of these two terms is equal to zero. However at the discrete level, since these two terms are not discretized within the same stencil, they do not generally compensate. The difference controls the numerical viscosity of the scheme and thus its stability.

We now give the definition of the corrective terms α_i^{Ok}. For the sake of clarity in the forthcoming notations, we set

$$\delta_{i+\frac{1}{2}} = w_{i+1}^n - w_i^n. \tag{2.9}$$

Concerning the second-order correction, we propose

$$\alpha_i^{O2} = \frac{\lambda}{2}\Delta x \overline{\partial_x w_i}^{O2}, \tag{2.10}$$

where

$$\Delta x \overline{\partial_x w_i}^{O2} = \Theta_i^{O2}\delta_{i+\frac{1}{2}} + \left(I - \Theta_i^{O2}\right)\delta_{i-\frac{1}{2}}. \tag{2.11}$$

Here, I is the $d\times d$ identity matrix while Θ_i^{O2} is a free $d\times d$ diagonal matrix parameter to be defined. This matrix parameter will play a central role to establish the required global entropy stability and it will be defined later on. We mention that other discretizations of the term $\Delta x\overline{\partial_x w_i}^{O2}$ are likely possible and extensions with more general matrices Θ_i^{O2} could be considered as well. We emphasize that the second-order consistency statement (2.6) is immediately satisfied provided the diagonal matrices Θ_i^{O2} remains bounded as Δx tends to 0.

Next, concerning the third-order correction, we choose

$$\alpha_i^{O3} = \frac{\lambda}{2}\Delta x\overline{\partial_x w_i}^{O3} - \frac{\Delta x^2}{8}\overline{\partial_{xx} f(w)_i} - \frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx} w_i}, \tag{2.12}$$

where

$$\begin{aligned}
\Delta x\overline{\partial_x w_i}^{O3} &= \frac{1}{3}\Theta_i^{O3}\left(\delta_{i+\frac{3}{2}} + \delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}}\right) + \frac{1}{3}\left(I - \Theta_i^{O3}\right)\left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} + \delta_{i-\frac{3}{2}}\right) \\
&\quad + \frac{1}{2}\left(I - 2\Theta_i^{O3}\right)\left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}\right), && (2.13) \\
\Delta x^2\overline{\partial_{xx} f(w)_i} &= f(w_{i+1}^n) - 2f(w_i^n) + f(w_{i-1}^n), && (2.14) \\
\Delta x^2\overline{\nabla f(w)\partial_{xx} w_i} &= \nabla f(w_i^n)\left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}\right). && (2.15)
\end{aligned}$$

We end up the definition of the scheme with the fourth-order correction that is given by

$$\alpha_i^{O4} = \frac{\lambda}{2}\Delta x\overline{\partial_x w_i}^{O4} - \frac{\Delta x^2}{8}\overline{\partial_{xx} f(w)_i} - \frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx} w_i} + \lambda\frac{\Delta x^3}{24}\overline{\partial_{xxx} w_i}^{O4}, \tag{2.16}$$

where we have set

$$\begin{aligned}\Delta x\overline{\partial_x w_i}^{O4} &= \Theta_i^{O4}\Delta x\overline{\partial_x w}_{i+\frac12}^{O4} + (I-\Theta_i^{O4})\Delta x\overline{\partial_x w}_{i-\frac12}^{O4}\\ &\quad+\frac14\left(-\Theta_i^{O4}\Delta x^2\overline{\partial_{xx}w}_{i+1}^{O4} + (I-2\Theta_i^{O4})\Delta x^2\overline{\partial_{xx}w_i}^{O4} + (I-\Theta_i^{O4})\Delta x^2\overline{\partial_{xx}w}_{i-1}^{O4}\right)\\ &\quad+\frac{\Delta x^3}{8}\overline{\partial_{xxx}w_i}^{O4},\end{aligned} \tag{2.17}$$

$$\Delta x\overline{\partial_x w}_{i+\frac12}^{O4} = \frac{1}{24}\left(-\delta_{i+\frac32} + 26\,\delta_{i+\frac12} - \delta_{i-\frac12}\right), \tag{2.18}$$

$$\Delta x^2\overline{\partial_{xx}w_i}^{O4} = \delta_{i+\frac12} - \delta_{i-\frac12}, \tag{2.19}$$

$$\Delta x^3\overline{\partial_{xxx}w_i}^{O4} = (\delta_{i+\frac32} - \delta_{i+\frac12}) - (\delta_{i+\frac12} - \delta_{i-\frac12}). \tag{2.20}$$

Note that the terms $\Delta x\overline{\partial_x w_i}^{O2}, \Delta x\overline{\partial_x w_i}^{O3}, \Delta x\overline{\partial_x w_i}^{O4}$ are consistent with $\Delta x\partial_x w(x_{i+\frac12})$ but with different order of consistency according to the consistency relations (2.6)-(2.8). This is why different formulas are proposed.

Equipped with the correction terms α_i^{O2}, α_i^{O3} and α_i^{O4}, we are now able to give the high-order scheme of interest as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac12}^{Ok} - \mathcal{F}_{i-\frac12}^{Ok}\right), \tag{2.21}$$

where we have set

$$\mathcal{F}_{i+\frac12}^{Ok} = \mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) + \frac12\left(\alpha_i^{Ok} + \alpha_{i+1}^{Ok}\right), \tag{2.22}$$

with

$$\begin{aligned}\alpha_i^{Ok} &= \alpha_i^{O2} && \text{for the second-order scheme,}\\ \alpha_i^{Ok} &= \alpha_i^{O3} && \text{for the third-order scheme,}\\ \alpha_i^{Ok} &= \alpha_i^{O4} && \text{for the fourth-order scheme.}\end{aligned}$$

We complete this section by establishing the order of accuracy of the schemes.

Proposition 2.2. *Let be given $u(x)$ a smooth function and define u_i by (2.4). Let the sequence of matrices $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$ be bounded as $\Delta x\to 0$. For a fixed $x_{i+\frac12}$ and $k\in\{2,3,4\}$ we have*

$$\mathcal{F}_\lambda^{O1}(u_i, u_{i+1}) + \frac12\left(\alpha_i^{Ok} + \alpha_{i+1}^{Ok}\right) = f(u(x_{i+\frac12})) + \mathcal{O}(\Delta x^k).$$

As a consequence, the high-order scheme (2.21) is space second-, third- or fourth-order according to the selected order of accuracy.

Proof. A direct Taylor expansion and the application of Lemma 2.1 achieve the proof. □

To conclude this section, we highlight that the high-order schemes do not involve limitations techniques in contrast with other usual approaches (MUSCL technique [49] or ENO/WENO schemes [37,43,44] or DG schemes [14,29,44], for instance). We do not need limitations in the high-order correction terms to establish global entropy stability.

3. Global entropy stability. In this section we establish the global entropy stability (1.10) satisfied by the high-order scheme (2.21). In order to deal simultaneously with second-, third- and fourth-order of space accuracy, the high-order correction α_i^{Ok} is reformulated as follows:

$$\begin{aligned}\alpha_i^{Ok} =& \frac{\lambda}{2}\Delta x\overline{\partial_x w_i}^{Ok} - \varepsilon^{O3}\frac{\Delta x^2}{8}\overline{\partial_{xx} f(w)_i} - \varepsilon^{O3}\frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx} w_i} \\ &+ \lambda\varepsilon^{O4}\frac{\Delta x^3}{24}\overline{\partial_{xxx} w_i}^{O4},\end{aligned}$$

where

$$\varepsilon^{O3} = \begin{cases} 1 & \text{for third- and fourth-order,} \\ 0 & \text{otherwise,}\end{cases} \qquad \varepsilon^{O4} = \begin{cases} 1 & \text{for fourth-order,} \\ 0 & \text{otherwise,}\end{cases}$$

where $\Delta x\overline{\partial_x w_i}^{Ok}$ is given by definitions (2.11), (2.13) and (2.17) according to the selected order of accuracy $Ok \in \{O2, O3, O4\}$.

For the forthcoming developments, it is convenient to condense the high-order scheme (2.21)-(2.22) in the form

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}, \tag{3.1}$$

where

$$\mathcal{R}_i^{Ok} = -\frac{1}{2}\left(f(w_{i+1}^n) - f(w_{i-1}^n)\right) + \frac{\lambda}{2}\left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}\right) - \frac{1}{2}\left(\alpha_{i+1}^{Ok} - \alpha_{i-1}^{Ok}\right). \tag{3.2}$$

We now state our main result.

THEOREM 3.1. *Consider $\eta \in C^2(\Omega, \mathbb{R})$ a strictly convex entropy. Let the approximation at time t^n, $w_\Delta(\cdot, t^n)$ given by (1.6) being a non zero function in $L^2(\mathbb{R})$ and such that $\int_{\mathbb{R}} \eta(w_\Delta(x, t^n))dx$ is finite. We assume the following:*

a) There exists a compact set $K \subset \Omega$ such that $w_\Delta(x, t^n) \in K$ for every $x \in \mathbb{R}$.

b) The sequence of bounded (as $\Delta x \to 0$) diagonal matrices $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$, defined according to the selected order of accuracy, satisfies for all $i \in \mathbb{Z}$ the following condition

$$\begin{aligned}&\sum_{i\in\mathbb{Z}} \left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac{1}{2}} \\ &- \frac{1}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx} w_i}^{O4}\right) > 0,\end{aligned} \tag{3.3}$$

where $\Delta x\overline{\partial_x w_i}^{Ok}$ and $\Delta x^3\overline{\partial_{xxx} w_i}^{O4}$ linearly depend on Θ_i^{Ok}.

Let μ^{Ok} be a positive bounded (as $\Delta x \to 0$) constant such that

$$\begin{aligned}&\mu^{Ok}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac{1}{2}} \\ &\le \sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac{1}{2}} \\ &\quad - \frac{1}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx} w_i}^{O4}\right).\end{aligned} \tag{3.4}$$

Then there exists positive constants, denoted $C_{\eta,n}$, $C^{Ok}_{\eta,f,n}$ independent from λ and $\frac{\Delta t}{\Delta x}$ and positive constants $r_n(\lambda)$, $C^{Ok}_n(\lambda)$ that depend on $\lambda > 0$ but not on $\frac{\Delta t}{\Delta x}$ such that if $\lambda > 0$ large enough and $\frac{\Delta t}{\Delta x} > 0$ small enough verify both

$$\lambda\mu^{Ok} \geq 2\max\left(\max_{i\in\mathbb{Z}}|\mu(w_i^n)|, 4\varepsilon^{O4}\frac{C^{Ok}_{\eta,f,n}}{C_{\eta,n}}\right), \tag{3.5}$$

$$\frac{\Delta t}{\Delta x} < \min\left(\frac{1}{\lambda\mu^{Ok}}, \frac{\frac{\lambda\mu^{Ok}}{8}C_{\eta,n} - \varepsilon^{O3}C^{Ok}_{\eta,f,n}}{C^{Ok}_n(\lambda)}, \frac{\operatorname{dist}(K,\partial\Omega)}{r_n(\lambda)}\right), \tag{3.6}$$

where $\mu(w)$ stands for the spectral radius of $\nabla f(w)$ and $\operatorname{dist}(K,\partial\Omega) > 0$ is the distance from the compact K to the boundary $\partial\Omega$, then updated approximation given by the high-order scheme (2.21) verifies $w_\Delta(\cdot,t^{n+1}) \subset \Omega$ and one has the global entropy stability inequality (1.10), that is

$$\int_{\mathbb{R}}\eta(w_\Delta(x,t^{n+1}))dx \leq \int_{\mathbb{R}}\eta(w_\Delta(x,t^n))dx. \tag{3.7}$$

Before going any further in the establishment of the main result, let us comment on the technical assumptions :

- The assumption a) has to be understood as an L^∞ bound on the solution. It is however much stronger since we require that the solution belongs to a compact subset of Ω where Ω is an open set. It is used for several purposes, namely; to obtain a lower bound for the smallest eigenvalue of the Hessian of the entropy $\nabla^2\eta$, to get a L^∞ bound on the physical flux f and last but not the least to obtain the robustness of the scheme for $\frac{\Delta t}{\Delta x} > 0$ small enough. Note that in the case of scalar conservation laws or symmetric system of conservation laws this assumption could easily be removed since the entropy function to be considered $\eta(w) = \frac{1}{2}|w|^2$ is a strongly convex function and the admissible set is $\Omega = \mathbb{R}^d$. In the case of Euler equations with a perfect gas, the admissible set is $\Omega = \{w = (\rho, \rho u, E) \in \mathbb{R}^3 \ : \ \rho > 0, E - \rho u^2/2 > 0\}$. Our assumption therefore implies that the density ρ and the pressure $p = (\gamma-1)(E - \rho u^2/2)$, where $\gamma \in (1,3)$ stands for the adiabatic constant, are strictly away from the vacuum and bounded from above. It is a somehow standard assumption (see [42]).
- The CFL condition $\frac{\Delta t}{\Delta x}r_n(\lambda) < \operatorname{dist}(K,\partial\Omega)$ with $\operatorname{dist}(K,\partial\Omega) > 0$ is used to prove the robustness of the scheme, namely; $w_\Delta(\cdot,t^{n+1}) \subset \Omega$. This CFL condition can be quite restrictive. However in practice we always consider datum that are far away from the border $\partial\Omega$. We mention that it is difficult to prove robustness for high order scheme under a less restrictive condition, except in the case where limitation techniques are used (for instance, see [2,40]).
- The assumption b) about the definition of the matrix parameters Θ_i^{Ok} is used to control the dissipation of the global entropy. Once again this assumption is easily satisfied in the case of scalar conservation laws or symmetric systems with the quadratic entropy $\eta(w) = \frac{1}{2}|w|^2$. For this specific entropy, and the second order in space scheme O2, the inequality (3.3) reformulates as follows:

$$\sum_{i\in\mathbb{Z}}\delta_{i+\frac{1}{2}}\cdot\delta_{i+\frac{1}{2}} - \frac{1}{2}\sum_{i\in\mathbb{Z}}\left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}}\right)\cdot\left(\Theta_i^{O2}\delta_{i+\frac{1}{2}} + (I - \Theta_i^{O2})\delta_{i-\frac{1}{2}}\right) > 0,$$

which after a translation of indices is equal to

$$\frac{1}{4}\sum_{i\in\mathbb{Z}}\left(\delta_{i+\frac{1}{2}}-\delta_{i-\frac{1}{2}}\right)^2-\frac{1}{2}\sum_{i\in\mathbb{Z}}\Theta_i^{O2}\left(\delta_{i+\frac{1}{2}}+\delta_{i-\frac{1}{2}}\right)\cdot\left(\delta_{i+\frac{1}{2}}-\delta_{i-\frac{1}{2}}\right)>0.$$

For instance the following choice:

$$\forall i\in\mathbb{Z},\quad \Theta_i^{O2}=\operatorname{diag}_{1\le j\le d}\left(-\operatorname{sign}\left((\delta_{i+\frac{1}{2}})_j^2-(\delta_{i-\frac{1}{2}})_j^2\right)\right)$$

yields the desired inequality. Moreover, since it is bounded as $\Delta x\to 0$, it preserves the order of consistency of the scheme. Another possible choice is:

$$\forall i\in\mathbb{Z},\quad \Theta_i^{O2}=0\in\mathcal{M}_d(\mathbb{R}).$$

This choice also gives the desired inequality since for a non trivial solution $w_\Delta(\cdot,t^n)\in L^2(\mathbb{R})$ the sum $\frac{1}{4}\sum_{i\in\mathbb{Z}}\left(\delta_{i+\frac{1}{2}}-\delta_{i-\frac{1}{2}}\right)^2$ is positive.

- For an arbitrary entropy η, we propose a systematic way to design a sequence of matrices $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$ such that the inequality (3.3) of assumption b) holds (see Proposition (A.1) in the appendix). We did not manage to prove that the proposed sequence of matrices stays bounded as $\Delta x\to 0$, we however observed it numerically.

3.1. Dissipation estimates. To prove our main result 3.1, we shall need several technical lemmata that arise in the study of the quantity $\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}$. This formal quantity with an appropriate CFL condition controls the dissipation rate of the global entropy of the scheme (2.21) and results from the following expansion of the scheme (2.21),

$$\begin{aligned}\eta(w_i^{n+1})=&\eta(w_i^n)+\frac{\Delta t}{\Delta x}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}\\&+\left(\frac{\Delta t}{\Delta x}\right)^2\int_0^1(1-s)\nabla^2\eta\left(w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds.\end{aligned}$$

The quantity $\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}$ thus must necessarily be negative for (3.7) to hold. Actually, we shall be more precise and prove a quantitative estimate that essentially shows that the global entropy dissipation rate of the the high order scheme (2.21) can be controlled by the global dissipation rate granted by the first order HLL scheme. In this respect, we begin with the following lemma. It gives an estimate on how the first order part of the scheme (2.21) dissipates the global entropy.

Lemma 3.2. *For any sequence $(v_i)_{i\in\mathbb{Z}}\in l^2(\mathbb{Z})$ with values in Ω such that $\sum_{i\in\mathbb{Z}}\eta(v_i)\Delta x$ is finite, for all $\Lambda\ge\max_{i\in\mathbb{Z}}|\mu(v_i)|$ where $\mu(v)$ denotes the sequence of the eigenvalues of $\nabla f(v)$, under the CFL condition*

$$\frac{\Delta t}{\Delta x}\Lambda\le\frac{1}{2},\tag{3.8}$$

we have

$$\begin{aligned}-\frac{1}{2}\sum_{i\in\mathbb{Z}}\nabla\eta(v_i)\cdot(f(v_{i+1})-f(v_{i-1}))-\frac{\Lambda}{2}\sum_{i\in\mathbb{Z}}(\nabla\eta(v_{i+1})-\nabla\eta(v_i))\cdot(v_{i+1}-v_i)\\ \le-\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(v_i+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\Lambda}\right)\mathcal{R}_i^{O1,\Lambda}\cdot\mathcal{R}_i^{O1,\Lambda}\,ds,\end{aligned}$$

where

$$\mathcal{R}_i^{O1,\Lambda} = -\frac{1}{2}\left(f(v_{i+1}) - f(v_{i-1})\right) + \frac{\Lambda}{2}(v_{i+1} - v_i - (v_i - v_{i-1})). \tag{3.9}$$

Proof. Let $(v_i)_{i\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be an arbitrary sequence with values in Ω. Consider the updated sequence $(\tilde{v}_i)_{i\in\mathbb{Z}}$ determined by the first order HLL scheme (2.1) with a numerical viscosity Λ and a CFL condition given by (3.8). That is

$$\tilde{v}_i = v_i - \frac{\Delta t}{2\Delta x}\left(f(v_{i+1}) - f(v_{i-1})\right) + \frac{\Lambda\Delta t}{2\Delta x}(v_{i+1} - v_i - (v_i - v_{i-1})). \tag{3.10}$$

Since $\Lambda \geq \max_{i\in\mathbb{Z}} |\mu(v_i)|$, it is known that the first order HLL scheme verifies $\tilde{v}_i \in \Omega$ (because Ω is convex) and is entropy preserving (see [26]). As a consequence, we get

$$\eta(\tilde{v}_i) \leq \eta(v_i) - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_\Lambda^{O1}(v_i, v_{i+1}) - \mathcal{G}_\Lambda^{O1}(v_{i-1}, v_i)\right),$$

where $\mathcal{G}_\Lambda^{O1}$ is the numerical entropy flux function. We then obtain

$$\sum_{i\in\mathbb{Z}} \eta(\tilde{v}_i) - \sum_{i\in\mathbb{Z}} \eta(v_i) \leq 0. \tag{3.11}$$

Besides, since $\tilde{v}_i \in \Omega$, using (3.10) we have

$$\begin{aligned}\eta(\tilde{v}_i) =& \eta(v_i) + \frac{\Delta t}{\Delta x}\nabla\eta(v_i)\cdot\mathcal{R}_i^{O1,\Lambda}\\ &+ \left(\frac{\Delta t}{\Delta x}\right)^2 \int_0^1 (1-s)\nabla^2\eta\left(v_i + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\Lambda}\right)\mathcal{R}_i^{O1,\Lambda}\cdot\mathcal{R}_i^{O1,\Lambda}\,ds,\end{aligned}$$

where $\mathcal{R}_i^{O1,\Lambda}$ is given by (3.9). Considering (3.11), we necessarily have

$$\sum_{i\in\mathbb{Z}} \nabla\eta(v_i)\cdot\mathcal{R}_i^{O1,\Lambda} \leq -\frac{\Delta t}{\Delta x}\int_0^1 (1-s)\sum_{i\in\mathbb{Z}} \nabla^2\eta\left(v_i + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\Lambda}\right)\mathcal{R}_i^{O1,\Lambda}\cdot\mathcal{R}_i^{O1,\Lambda}\,ds.$$

Eventually remark that

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \nabla\eta(v_i)\cdot\mathcal{R}_i^{O1,\Lambda} &= -\frac{1}{2}\sum_{i\in\mathbb{Z}} \nabla\eta(v_i)\cdot\left(f(v_{i+1}) - f(v_{i-1})\right)\\ &\qquad + \frac{\Lambda}{2}\sum_{i\in\mathbb{Z}} \nabla\eta(v_i)\cdot(v_{i+1} - v_i - (v_i - v_{i-1})),\\ &= -\frac{1}{2}\sum_{i\in\mathbb{Z}} \nabla\eta(v_i)\cdot\left(f(v_{i+1}) - f(v_{i-1})\right)\\ &\qquad - \frac{\Lambda}{2}\sum_{i\in\mathbb{Z}} (\nabla\eta(v_{i+1}) - \nabla\eta(v_i))\cdot(v_{i+1} - v_i),\end{aligned}$$

which yields the desired inequality and achieves the proof. □

Our second lemma states that the upper bound in (3.4) which controls the numerical viscosity $\mu^{Ok} > 0$ is bounded as $\Delta x \to 0$ and thus the scheme does not need to be infinitely viscous.

LEMMA 3.3. *Let the approximation at time t^n, $w_\Delta(\cdot,t^n)$ given by* (1.6) *being a non zero function in $L^2(\mathbb{R})$ and such that it verifies the assumption a) of Theorem 3.1. Let $(\Theta_i)_{i\in\mathbb{Z}}$ a sequence of bounded as ($\Delta x \to 0$) matrices that verifies (3.3). Then the upper bound in (3.4), is bounded as $\Delta x \to 0$.*

Proof. First, since $\eta \in C^2(\Omega,\mathbb{R})$ is a strictly convex function and $w_\Delta(\cdot,t^n)$ belongs to a compact set $K\subset\Omega$ there exists a constant $\alpha_{\eta,n}>0$ such that we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12} &= \int_0^1\sum_{i\in\mathbb{Z}}\nabla^2\eta(w_i^n+s\delta_{i+\frac12})\delta_{i+\frac12}\cdot\delta_{i+\frac12}\,ds,\\ &\geq \alpha_{\eta,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.\end{aligned}$$

Since $w_\Delta(\cdot,t^n)$ is in $L^2(\mathbb{R})$, the sum $\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}>0$ is convergent. Let us set

$$\begin{aligned}\mathcal{S} = &\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}\\ &-\frac12\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w_i}^{Ok}+\frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx}w_i}^{O4}\right)\end{aligned}$$

which is positive since the sequence of matrices $(\Theta_i)_{i\in\mathbb{Z}}$ satisfies (3.3). Since η is smooth and $w_\Delta(\cdot,t^n)$ lives in a compact set $K\subset\Omega$ then there also exists a positive constant $\tilde\beta_{\eta,n}$ such that

$$\left|\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}\right| \leq \tilde\beta_{\eta,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

Moreover, since $\Delta x\overline{\partial_x w_i}^{Ok}$ and $\Delta x^3\overline{\partial_{xxx}w_i}^{O4}$ are linear functions with respect to $(\delta_{i+\nu+\frac12})_{-2\leq\nu\leq+1}$ and with bounded coefficients, there exists positive constant $\bar\beta_{\eta,n}$ such that

$$\begin{aligned}&\left|\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w_i}^{Ok}+\frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx}w_i}^{O4}\right)\right|\\ &\qquad\leq \bar\beta_{\eta,n}\sum_{i\in\mathbb{Z}}\sum_{\nu=-2}^{1}\left(\delta_{i+\frac12}+\delta_{i-\frac12}\right)\cdot\delta_{i+\nu+\frac12},\\ &\qquad\leq 8\bar\beta_{\eta,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.\end{aligned}$$

It results in the existence of a positive constant $\beta_{\eta,n}>0$ such that

$$\mathcal{S}\leq\beta_{\eta,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

As a consequence, we get

$$0<\frac{\mathcal{S}}{\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}}\leq\frac{\beta_{\eta,n}}{\alpha_{\eta,n}},$$

and thus, from (3.4), μ^{Ok} remains bounded as $\Delta x\to 0$. The proof of Lemma 3.3 is completed. ∎

Our last lemma is the cornerstone of this work. It is an estimate on how the high order global entropy dissipation rate can be controlled by the first order dissipation rate.

LEMMA 3.4. *Let the approximation at time t^n, $w_\Delta(\cdot,t^n)$ given by (1.6) being a non zero function in $L^2(\mathbb{R})$. Let the assumption a) and b) of Theorem 3.1 hold. Let $\mu^{Ok}>0$ be large enough to satisfy (3.4). Let D^{Ok} be the high order dissipation rate given by*

$$D^{Ok} = \frac{\lambda}{2}\sum_{i\in\mathbb{Z}} \nabla\eta(w_i^n)\cdot(\delta_{i+\frac12}-\delta_{i-\frac12}) - \frac{1}{2}\sum_{i\in\mathbb{Z}} \nabla\eta(w_i^n)\cdot\left(\alpha_{i+1}^{Ok}-\alpha_{i-1}^{Ok}\right). \tag{3.12}$$

Then there exists a positive constant $C^{Ok}_{\eta,f,n}$ which does not depend on λ such that

$$D^{Ok} \le -\frac{\lambda\mu^{Ok}}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12} + \varepsilon^{O3}C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12},$$

Proof. Let us begin with a bound on the second term $|\frac12\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\left(\alpha_{i+1}^{Ok}-\alpha_{i-1}^{Ok}\right)|$. The approximation at time t^n, $w_\Delta(\cdot,t^n)$ is in $L^2(\mathbb{R})$ and $w_\Delta(\cdot,t^n)\subset K$ where K is a compact set of Ω. Besides, since the functions f and η are smooth there exists a positive constant $C^{Ok}_{\eta,f,n}$, which does not depend on λ such that

$$\begin{aligned}
&\Bigg|\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\Bigg(\frac{\Delta x^2}{8}\overline{\partial_{xx}f(w)}_{i+1}+\frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx}w}_{i+1}\\
&\qquad\qquad -\frac{\Delta x^2}{8}\overline{\partial_{xx}f(w)}_{i-1}-\frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx}w}_{i-1}\Bigg)\Bigg|\\
&=\Bigg|\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\frac{\Delta x^2}{8}\overline{\partial_{xx}f(w)}_i+\frac{\Delta x^2}{24}\overline{\nabla f(w)\partial_{xx}w}_i\right)\Bigg|,\\
&\le C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.
\end{aligned}$$

Expanding D^{Ok}, we obtain

$$\begin{aligned}
D^{Ok} &\le \frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot(\delta_{i+\frac12}-\delta_{i-\frac12}) - \frac{\lambda}{4}\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot(\Delta x\overline{\partial_x w}^{Ok}_{i+1}-\Delta x\overline{\partial_x w}^{Ok}_{i-1})\\
&\quad -\varepsilon^{O4}\frac{\lambda}{48}\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)(\Delta x^3\overline{\partial_{xxx}w}^{O4}_{i+1}-\Delta x^3\overline{\partial_{xxx}w}^{O4}_{i-1}))\\
&\quad +\varepsilon^{O3}C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12},\\
&\le -\frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}\\
&\quad +\frac{\lambda}{4}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w}^{Ok}_i+\frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx}w}^{O4}_i\right)\\
&\quad +\varepsilon^{O3}C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.
\end{aligned}$$

Multiplying the inequality (3.4) by $-\lambda/2$ one has the bound

$$\begin{aligned}
&-\frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}\\
&\quad+\frac{\lambda}{4}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\Delta x\overline{\partial_x w_i}^{Ok}+\frac{\varepsilon^{O4}}{12}\Delta x^3\overline{\partial_{xxx}w_i}^{O4}\right)\\
&\le -\frac{\lambda\mu^{Ok}}{2}\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12}
\end{aligned}$$

gathering all the terms together yields the desired inequality. ∎

3.2. Proof of the main result. Let $\lambda>0$ to be fixed later. We first prove the robustness of the scheme (3.1). Namely, there exists a compact set $K'\subset\Omega$ such that for all $s\in[0,1]$ and $i\in\mathbb{Z}$, $w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\in K'$ for some small enough $\frac{\Delta t}{\Delta x}>0$. We argue as follows: since $w_\Delta(\cdot,t^n)$ is assumed to belong to a compact set $K\subset\Omega$, by a standard continuity argument, one can find a positive constant $r_n(\lambda)$ that depends on λ but not on $\frac{\Delta t}{\Delta x}$ such that for all $i\in\mathbb{Z}$ $|\mathcal{R}_i^{Ok}|\le r_n(\lambda)$. Consequently, one has the following embedding

$$\{w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\ :\ s\in[0,1],i\in\mathbb{Z}\}\subset K+\frac{\Delta t}{\Delta x}\mathcal{B}(r_n(\lambda)):=K'$$

where $\mathcal{B}(r_n(\lambda))$ is the ball in $\mathbb{R}^d$ of radius $r_n(\lambda)$. For any $\frac{\Delta t}{\Delta x}>0$, the set K' is a compact subset of $\mathbb{R}^d$. Since K is a compact subset of Ω and Ω is an open set, then $\mathrm{dist}(K,\partial\Omega)>0$. Provided $0<\frac{\Delta t}{\Delta x}r_n(\lambda)<\mathrm{dist}(K,\partial\Omega)$, one has $K'=K+\frac{\Delta t}{\Delta x}\mathcal{B}(r_n(\lambda))\subset\Omega$ which proves the robustness.

We now prove the global entropy stability. Since $\eta\in C^2(\Omega;\mathbb{R})$ is a smooth enough function and the updated approximation $w_\Delta(\cdot,t^{n+1})$ given by (3.1) belongs to Ω, one has using a Taylor expansion,

$$\begin{aligned}
\eta(w_i^{n+1})=&\eta(w_i^n)+\frac{\Delta t}{\Delta x}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}\\
&+\left(\frac{\Delta t}{\Delta x}\right)^2\int_0^1(1-s)\nabla^2\eta\left(w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds.
\end{aligned}$$

We have to prove the following inequality

$$\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}+\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}ds\le 0. \tag{3.13}$$

We decompose the first term as follows

$$\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok}=-\frac12\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\left(f(w_{i+1}^n)-f(w_{i-1}^n)\right)+D^{Ok},$$

where $\mathcal{R}_i^{Ok}$ is given by (3.2) and D^{Ok} is given by (3.12). Using Lemma 3.4, the second term of the right hand side of the above equality can be from bounded above so that we have,

$$\sum_{i\in\mathbb{Z}} \nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok} \le -\frac{1}{2}\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\big(f(w_{i+1}^n)-f(w_{i-1}^n)\big)$$
$$-\frac{\lambda\mu^{Ok}}{2}\sum_{i\in\mathbb{Z}}\big(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\big)\,\delta_{i+\frac12} + \varepsilon^{O3}C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

Using Lemma 3.2 with the sequence $(v_i = w_i^n)_{i\in\mathbb{Z}}$ and with the numerical viscosity $\Lambda = \lambda\mu^{Ok}/2$ and the CFL condition (3.8), one can bound the first term of the right hand side of the above inequality as follows,

$$-\frac{1}{2}\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\big(f(w_{i+1}^n)-f(w_{i-1}^n)\big) - \frac{\lambda\mu^{Ok}}{4}\sum_{i\in\mathbb{Z}}\big(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\big)\,\delta_{i+\frac12}$$
$$\le -\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\,ds,$$

where $\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}$ is given by (3.9). As a consequence, gathering all the terms together we glean

$$\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok} \le -\frac{\lambda\mu^{Ok}}{4}\sum_{i\in\mathbb{Z}}\big(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\big)\,\delta_{i+\frac12}$$
$$-\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\,ds$$
$$+\varepsilon^{O3}C^{Ok}_{\eta,f,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

Moreover, since η is strictly convex and continuous and $w_\Delta(\cdot,t_n)$ is assumed to live in a compact set $K\subset\Omega$, there exists $C_{\eta,n}>0$ such that

$$\sum_{i\in\mathbb{Z}}\big(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\big)\cdot\delta_{i+\frac12} = \int_0^1\sum_{i\in\mathbb{Z}}\nabla^2\eta(w_i^n+s\delta_{i+\frac12})\delta_{i+\frac12}\cdot\delta_{i+\frac12}\,ds,$$
$$\ge C_{\eta,n}\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12},$$

so that we get the intermediate following inequality

$$\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok} \le \left(-\frac{\lambda\mu^{Ok}}{4}C_{\eta,n}+\varepsilon^{O3}C^{Ok}_{\eta,f,n}\right)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}$$
$$-\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\,ds. \tag{3.14}$$

Adding the term $\frac{\Delta t}{\Delta x}\int_0^1(1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n+s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}ds$ to the inequality

(3.14) results in

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok} + \frac{\Delta t}{\Delta x}\int_0^1 (1-s)\sum_{i\in\mathbb{Z}} \nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds \\
&\le \left(-\frac{\lambda\mu^{Ok}}{4}C_{\eta,n} + \varepsilon^{O3}C_{\eta,f,n}^{Ok}\right)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12} \\
&\quad - \frac{\Delta t}{\Delta x}\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\,ds \\
&\quad + \frac{\Delta t}{\Delta x}\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds.
\end{aligned}$$

To complete the proof, we have to provide an upper bound for the two last terms. For the first term, using once again the numerical viscosity $\Lambda = \lambda\mu^{Ok}/2$ and the CFL condition (3.8), one has for all $s\in[0,1]$ and $i\in\mathbb{Z}$, $w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\in\Omega$. Therefore by standard continuity argument, there exists a positive constant $C_1(\lambda)$ that depends on λ such that

$$\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}ds \le C_1(\lambda)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

We now deal with the second term. Using a continuity argument in the compact set $K'\subset\Omega$, there exists a positive constant $C_2(\lambda)$ that depends on λ such that

$$\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds \le C_2(\lambda)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.$$

We then get with a positive constant $C^{Ok}(\lambda) = C_1(\lambda)+C_2(\lambda)$

$$\begin{aligned}
&-\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\right)\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\cdot\mathcal{R}_i^{O1,\lambda\mu^{Ok}/2}\,ds \\
&+\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds \\
&\qquad\qquad \le C^{Ok}(\lambda)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.
\end{aligned}$$

It eventually yields

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\nabla\eta(w_i^n)\cdot\mathcal{R}_i^{Ok} + \frac{\Delta t}{\Delta x}\int_0^1 (1-s)\sum_{i\in\mathbb{Z}}\nabla^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathcal{R}_i^{Ok}\right)\mathcal{R}_i^{Ok}\cdot\mathcal{R}_i^{Ok}\,ds \\
&\qquad\qquad \le \left(-\frac{\lambda\mu^{Ok}}{4}C_\eta + \varepsilon^{O3}C_{\eta,f}^{Ok} + \frac{\Delta t}{\Delta x}C^{Ok}(\lambda)\right)\sum_{i\in\mathbb{Z}}\delta_{i+\frac12}\cdot\delta_{i+\frac12}.
\end{aligned}$$

To conclude the proof, it is enough to choose λ large enough to satisfy (3.5) and $\frac{\Delta t}{\Delta x}$ small enough to verify the additional CFL restriction (3.6) so that

$$-\frac{\lambda\mu^{Ok}}{4}C_\eta + \varepsilon^{O3}C_{\eta,f}^{Ok} + \frac{\Delta t}{\Delta x}C^{Ok}(\lambda)\le 0,$$

and the required inequality (3.13) is satisfied. The proof of Theorem 3.1 is thus achieved.

4. Numerical experiments. In this section, we provide several numerical examples that illustrate the accuracy and the stability of the proposed schemes. In order to be complete, some details in the scheme implementation must be given.

As far as the time order of accuracy is concerned, the scheme (2.21) is first-order in time. To increase the time accuracy, we use the well-known SSP Rung-Kutta methods introduced in [21–23]. Since this high-order time approach is based on convex combination of first-order time sub-steps, the global entropy stability result (1.10) is preserved.

Let us now explain how the parameters of the scheme are settled. Being given a strictly convex entropy η, we design the matrix parameter $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$ that satisfies the criteria (3.3). Then we have to choose an explicit definition of the numerical viscosity coefficient λ and the time step Δt. For a fixed $\Delta x > 0$, according to Theorem 3.1, there exists $\lambda > 0$ large enough and $\Delta t > 0$ small enough such that the inequality (3.13) is satisfied which implies the global entropy stability. From practical point of view λ and $\Delta t/\Delta x$ are chosen such that $\lambda = \max_i|\mu(w_i^n)|$ and $\frac{\lambda\Delta t}{\Delta x} \leq \frac{1}{2}$. We shall verify systematically at the numerical level that this choice ensures the decrease of the total entropy.

Equipped with this numerical parameters, we performed numerical simulations considering mainly the scalar Burgers equation and the Euler equations. For each case, we propose three different choice of the matrix parameter $\Theta_i^{Ok} \in \{\Theta_{a,i}^{Ok}, \Theta_{b,i}^{Ok}, \Theta_{c,i}^{Ok}\}$. We systematically measure the error in L^1, L^2 and L^∞ norms between the numerical solutions and an exact solution. Plots of the obtained numerical solutions and the total entropy are also given. A particular attention must be paid on the choice $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$ whose results are surprisingly good, notably because very few oscillations are observed in the discontinuities.

4.1. Burgers equation. The Burgers equations consists in taking $w \in \mathbb{R}$ and the flux function given by $f(w) = w^2/2$. We consider the entropy function $\eta(w) = w^2/2$ so that the global entropy stability (1.10) coincides with a L^2-decreasing property. We shall present several test with the following parameters:

$$\begin{aligned}
\Theta_{a,i}^{Ok} &= -\theta \,\mathrm{sign}\left(\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)(\mathcal{A}_i)\right), \\
\Theta_{b,i}^{Ok} &= \frac{\left(\delta_{i-1/2}^2 - \delta_{i+1/2}^2\right)\left(\delta_{i-1/2}^2 + \delta_{i+1/2}^2\right)}{\left(\delta_{i-1/2}^2 + \delta_{i+1/2}^2\right)^2 + \varepsilon}, \\
\Theta_{c,i}^{Ok} &= \frac{1}{2},
\end{aligned} \tag{4.1}$$

where we fix $\theta = -\min(0, S/D)$, with S given by (A.2) and D given by (A.5), and $\varepsilon = 10^{-12}$. Numerically, we verified that these choices of Θ_i^{Ok} satisfy the criteria (3.3).

4.1.1. Smooth solution. We take a smooth initial data $w_0(x) = 0.25+0.5\sin(\pi x)$ over a periodic domain $[-1, 1)$. With a final time small enough, here given by $t = 0.3$, the exact solution remains smooth so that the order of accuracy can be evaluated.

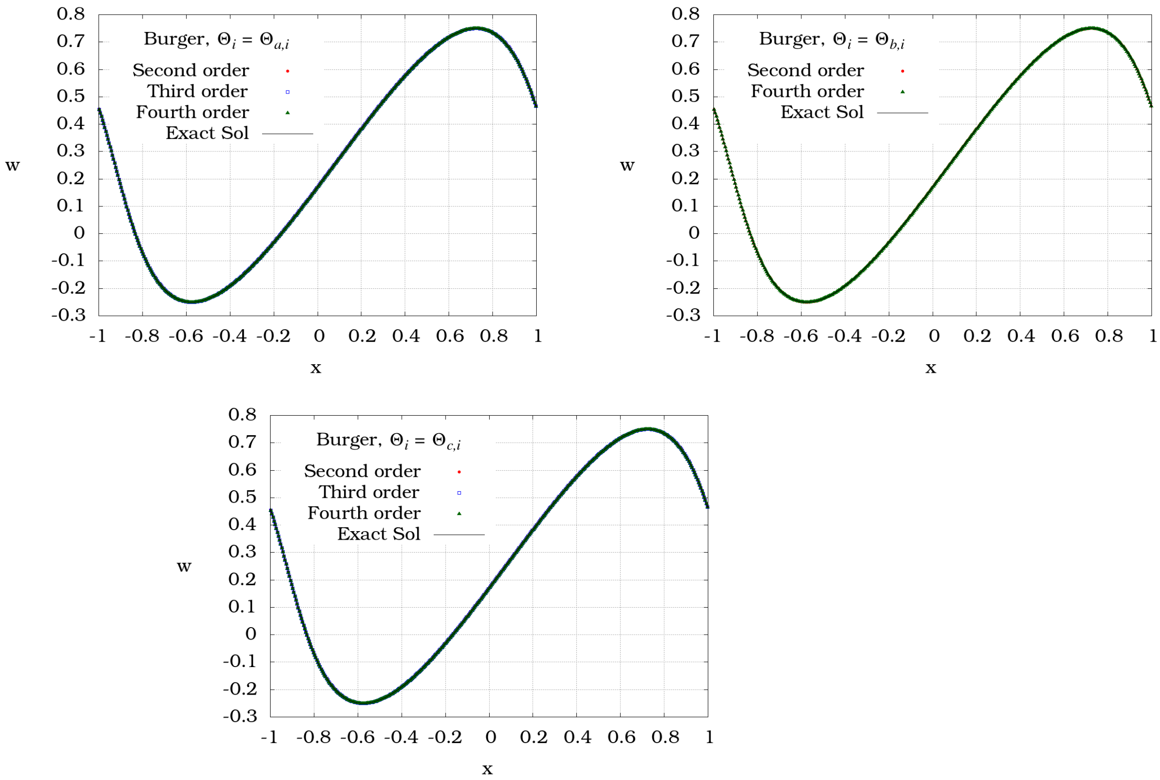

Table 4.1: Second-, third- and fourth-order accurate approximation of the smooth Burgers with a mesh made of 400 cells.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	5.8E-04	-	6.9E-04	-	8.2E-04	-
200	1.4E-04	2.0	1.7E-04	2.0	2.0E-04	2.0
400	3.6E-05	2.0	4.2E-05	2.0	5.0E-05	2.0
800	8.9E-06	2.0	1.0E-05	2.0	1.2E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	3.1E-06	2.0

	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	8.2E-05	-	1.3E-04	-	3.4E-05	-
200	1.0E-05	3.0	1.6E-05	3.0	2.4E-06	3.8
400	1.3E-06	3.0	2.0E-06	3.0	1.9E-07	3.7
800	1.6E-07	3.0	2.5E-07	3.0	1.6E-08	3.6
1600	2.0E-08	3.0	3.1E-08	3.0	1.4E-09	3.6

	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	6.1E-06	-	1.0E-05	-	1.2E-05	-
200	3.8E-07	4.0	6.4E-07	4.0	7.9E-07	4.0
400	2.4E-08	4.0	4.0E-08	4.0	4.9E-08	4.0
800	1.5E-09	4.0	2.5E-09	4.0	3.0E-09	4.0
1600	9.1E-11	4.0	1.5E-10	4.0	1.9E-10	4.0

Table 4.2: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	1.4E-03	-	1.6E-03	-	1.0E-03	-
200	2.5E-04	2.5	2.8E-04	2.5	9.9E-05	3.4
400	3.9E-05	2.7	4.3E-05	2.7	4.8E-05	1.1
800	8.9E-06	2.1	1.0E-05	2.1	1.2E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	3.1E-06	2.0
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-06	-	9.8E-06	-	1.2E-05	-
200	3.7E-07	4.0	6.3E-07	4.0	7.6E-07	3.9
400	2.3E-08	4.0	3.9E-08	4.0	4.8E-08	4.0
800	1.5E-09	4.0	2.5E-09	4.0	3.0E-09	4.0
1600	9.1E-11	4.0	1.5E-10	4.0	1.9E-10	4.0

Table 4.3: Errors and order evaluations for the second- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	4.5E-04	-	4.8E-04	-	4.5E-04	-
200	1.1E-04	2.0	1.2E-04	2.1	1.1E-04	2.1
400	2.7E-05	2.0	2.8E-05	2.0	2.7E-05	2.0
800	6.6E-06	2.0	6.9E-06	2.0	6.6E-06	2.0
1600	1.6E-06	2.0	1.7E-06	2.0	1.6E-06	2.0
	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	8.2E-05	-	1.3E-04	-	1.3E-05	-
200	1.0E-05	3.0	1.6E-05	3.0	1.0E-06	3.7
400	1.3E-06	3.0	2.0E-06	3.0	8.7E-08	3.6
800	1.6E-07	3.0	2.5E-07	3.0	7.6E-09	3.5
1600	2.0E-08	3.0	3.1E-08	3.0	6.8E-10	3.5

	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-06	-	5.9E-06	-	1.8E-06	-
200	2.0E-07	4.1	3.3E-07	4.1	1.1E-07	4.0
400	1.2E-08	4.1	1.9E-08	4.1	6.6E-09	4.1
800	7.2E-10	4.0	1.2E-09	4.1	4.0E-10	4.0
1600	4.5E-11	4.0	7.1E-11	4.0	2.2E-11	4.2

Table 4.4: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{Ok} = \Theta_{c,i}^{Ok}$.

The obtained numerical solutions are presented in Fig. 4.1. We notice the very good behavior of the approximations. This remark is completed by Table 4.2, 4.3, 4.4 where the evaluation of the order of accuracy is presented. Since the high-order scheme is unlimited, we get the expected order.

4.1.2. Discontinuous solution. We take a discontinuous initial data over the periodic domain $[-1, 1)$ defined by $w_0(x) = \begin{cases} 1 & \text{if} \quad -0.25 \leq x \leq 0.25, \\ 0 & \text{otherwise.} \end{cases}$
The exact solution is made of both rarefaction and shock waves. With a final time $t = 0.3$, the waves do not interact. In Table 4.5, 4.6, 4.7 we present the evaluated order of accuracy. The obtained approximations are presented in Fig. 4.1. We notice a remarkable behavior of the approximate solutions since very little spurious oscillations appear.

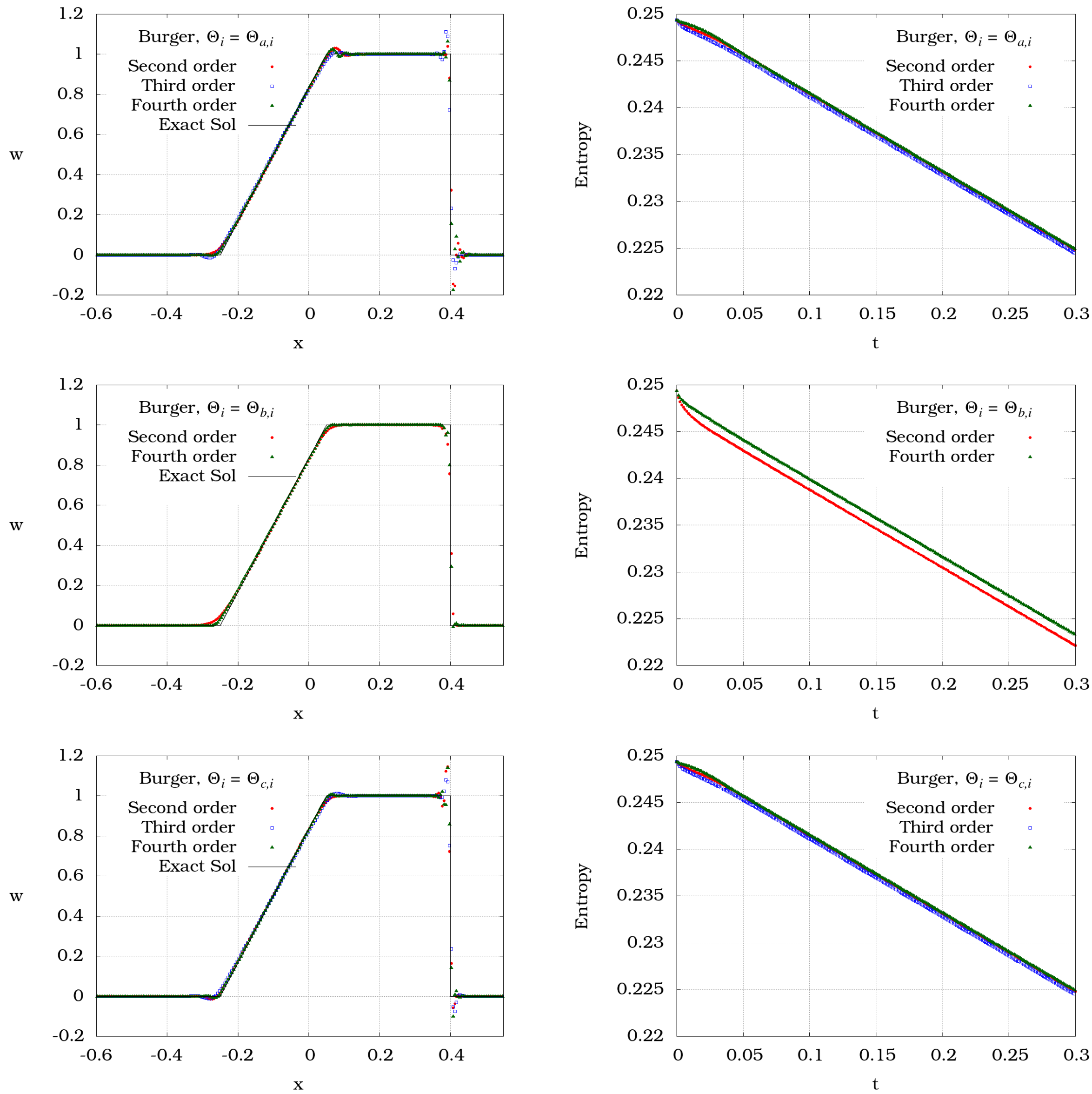

Fig. 4.1: Second-, third- and fourth-order accurate approximation of the Burgers solution made of rarefaction and shock waves with a mesh made of 400 cells.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	6.4E-02	-	8.0E-02	-
200	1.7E-02	1.0	4.3E-02	0.6	6.4E-02	0.3
400	8.4E-03	1.0	3.0E-02	0.5	5.0E-02	0.3
800	4.2E-03	1.0	2.1E-02	0.5	4.0E-02	0.3
1600	2.1E-03	1.0	1.5E-02	0.5	3.2E-02	0.3

	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.6E-02	-	6.3E-02	-	5.3E-02	-
200	1.7E-02	1.1	4.2E-02	0.6	4.2E-02	0.3
400	8.4E-03	1.0	2.9E-02	0.5	3.3E-02	0.3
800	4.2E-03	1.0	2.0E-02	0.5	2.6E-02	0.3
1600	2.1E-03	1.0	1.4E-02	0.5	2.1E-02	0.3
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	2.6E-02	-	4.5E-02	-	3.7E-02	-
200	1.2E-02	1.1	3.0E-02	0.6	2.9E-02	0.3
400	5.7E-03	1.0	2.1E-02	0.5	2.4E-02	0.3
800	2.8E-03	1.0	1.5E-02	0.5	1.9E-02	0.3
1600	1.4E-03	1.0	1.1E-02	0.5	1.5E-02	0.3

Table 4.5: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	7.0E-02	-	8.6E-02	-
200	1.8E-02	0.9	4.9E-02	0.5	6.8E-02	0.3
400	9.2E-03	1.0	3.3E-02	0.5	5.4E-02	0.3
800	4.6E-03	1.0	2.3E-02	0.5	4.3E-02	0.3
1600	2.3E-03	1.0	1.6E-02	0.5	3.4E-02	0.3
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	2.3E-02	-	5.5E-02	-	7.3E-02	-
200	1.3E-02	0.8	3.9E-02	0.5	5.6E-02	0.4
400	6.5E-03	1.0	2.7E-02	0.5	4.4E-02	0.3
800	3.3E-03	1.0	1.9E-02	0.5	3.5E-02	0.3
1600	1.6E-03	1.0	1.3E-02	0.5	2.8E-02	0.3

Table 4.6: Errors and order evaluations for the second- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.1E-02	-	5.8E-02	-	6.2E-02	-
200	1.4E-02	1.1	4.0E-02	0.6	5.0E-02	0.3
400	7.1E-03	1.0	2.8E-02	0.5	4.0E-02	0.3
800	3.5E-03	1.0	1.9E-02	0.5	3.1E-02	0.3
1600	1.8E-03	1.0	1.4E-02	0.5	2.5E-02	0.3

	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	5.9E-02	-	3.2E-02	-
200	1.6E-02	1.1	3.9E-02	0.6	2.6E-02	0.3
400	7.8E-03	1.0	2.7E-02	0.5	2.0E-02	0.3
800	3.9E-03	1.0	1.9E-02	0.5	1.6E-02	0.3
1600	1.9E-03	1.0	1.3E-02	0.5	1.3E-02	0.3

	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	2.3E-02	-	4.2E-02	-	3.1E-02	-
200	9.7E-03	1.2	2.8E-02	0.6	2.5E-02	0.3
400	4.8E-03	1.0	2.0E-02	0.5	2.0E-02	0.3
800	2.4E-03	1.0	1.4E-02	0.5	1.6E-02	0.3
1600	1.2E-03	1.0	9.7E-03	0.5	1.3E-02	0.3

Table 4.7: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for $\Theta_i^{Ok} = \Theta_{c,i}^{Ok}$.

4.2. Euler system. The second numerical experiment concerns the Euler system for a perfect diatomic gas where the unknown vector and the flux function are given as follows:

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)\, u \end{pmatrix}, \quad \text{where} \quad p = (\gamma - 1)\left(\rho E - \frac{\rho u^2}{2}\right).$$

We fix $\gamma = 1.4$ and we endow the system with the following entropy:

$$\eta(w) = -\rho \ln\left(\frac{p}{\rho^\gamma}\right). \tag{4.2}$$

We set the following matrix parameter Θ_i^{Ok} values for the Euler problem

$$\begin{aligned} \Theta_{a,i}^{Ok} &= -\theta \,\mathrm{diag}_{1\le j\le d}\left(\mathrm{sign}\left(\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)_j (\mathcal{A}_i)_j\right)\right), \\ \Theta_{b,i}^{Ok} &= \mathrm{diag}_{1\le j\le d}\left(\frac{\left(\left(\delta_{i-1/2}\right)_j^2 - \left(\delta_{i+1/2}\right)_j^2\right)\left(\left(\delta_{i-1/2}\right)_j^2 + \left(\delta_{i+1/2}\right)_j^2\right)}{\left(\left(\delta_{i-1/2}\right)_j^2 + \left(\delta_{i+1/2}\right)_j^2\right)^2 + \varepsilon}\right), \end{aligned} \tag{4.3}$$

where θ and ε are taken equal to $-\min(0, S/D)$ and 10^{-12} respectively. Once again, we perform two numerical simulations respectively concerned with a continuous solution, to relevantly evaluate the order of accuracy, and with a shock tube to illustrate the behavior of the approximate solution within shock waves and the absence of spurious oscillations.

4.2.1. Smooth solution. The initial data is given as follows over the periodic domain $[-1, 1)$: $\rho_0(x) = 1 + 0.5\sin^2(\pi x)$, $\quad u_0(x) = 0.5$, $\quad p_0(x) = 1$. For such an initial data the Euler equations reduces to a linear transport problem and the solution remains smooth for all time $t > 0$.

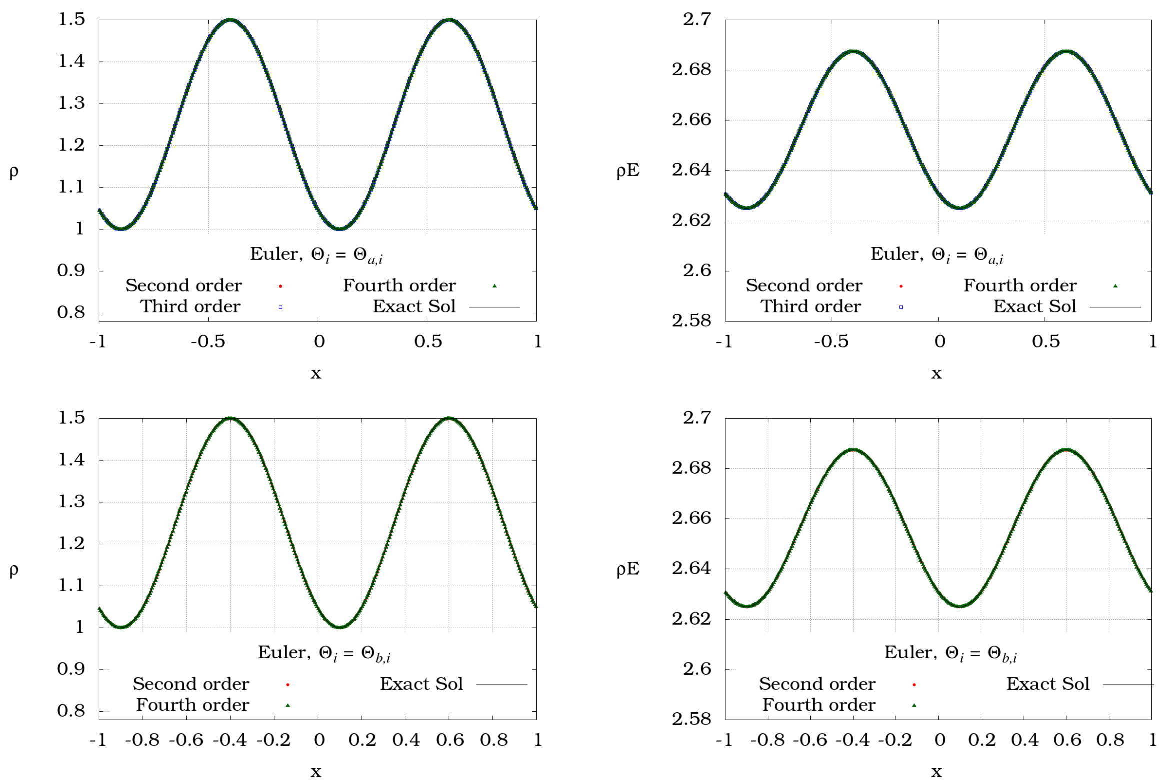

Fig. 4.2: Second-, third- and fourth-order accurate approximation of the smooth Euler solution and entropy with a mesh made of 400 cells.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0

	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	6.3E-04	-	3.4E-04	-	3.0E-04	-
200	7.9E-05	3.0	4.3E-05	3.0	3.8E-05	3.0
400	9.9E-06	3.0	5.4E-06	3.0	4.8E-06	3.0
800	1.2E-06	3.0	6.7E-07	3.0	6.0E-07	3.0
1600	1.5E-07	3.0	8.4E-08	3.0	7.5E-08	3.0

	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	2.0E-05	-	1.1E-05	-	9.6E-06	-
200	1.3E-06	4.0	6.8E-07	4.0	6.0E-07	4.0
400	7.8E-08	4.0	4.3E-08	4.0	3.8E-08	4.0
800	4.9E-09	4.0	2.7E-09	4.0	2.4E-09	4.0
1600	3.1E-10	4.0	1.7E-10	4.0	1.5E-10	4.0

Table 4.8: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the continuous Euler solution and for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	1.2E-02	-	7.6E-03	-	1.1E-02	-
200	2.4E-03	2.3	1.6E-03	2.2	2.9E-03	1.9
400	3.4E-04	2.8	2.0E-04	3.0	3.7E-04	3.0
800	6.0E-05	2.5	3.1E-05	2.7	2.6E-05	3.8
1600	1.4E-05	2.1	7.4E-06	2.1	6.5E-06	2.0
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	5.5E-05	-	3.9E-05	-	7.2E-05	-
200	3.4E-06	4.0	2.7E-06	3.9	6.4E-06	3.5
400	1.2E-07	4.9	6.8E-08	5.3	1.4E-07	5.5
800	5.3E-09	4.4	2.8E-09	4.6	2.4E-09	5.9
1600	3.1E-10	4.1	1.7E-10	4.1	1.5E-10	4.0

Table 4.9: Errors and order evaluations for the second- and fourth-order accurate schemes with the continuous Euler solution and for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

The obtained approximate solutions are displayed in Fig. 4.2. Once again, we notice a very good agreement of the approximate solution when compared to the exact one. This remark is emphasized with Tables 4.8 4.9 where we show that the expected order of accuracy are obtained even surprisingly a greater order for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

4.2.2. Shock tube solution. We perform a shock tube as described in over the domain $[0, 1]$ where the initial data is given by

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \qquad u_0(x) = 0, \qquad p_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.1 & \text{otherwise,} \end{cases}$$

The final time is 0.2. In order to impose periodic conditions on the boundaries, we work on the domain $[-1, 1]$ and we set a symmetric tube shock problem on $[-1, 0]$.

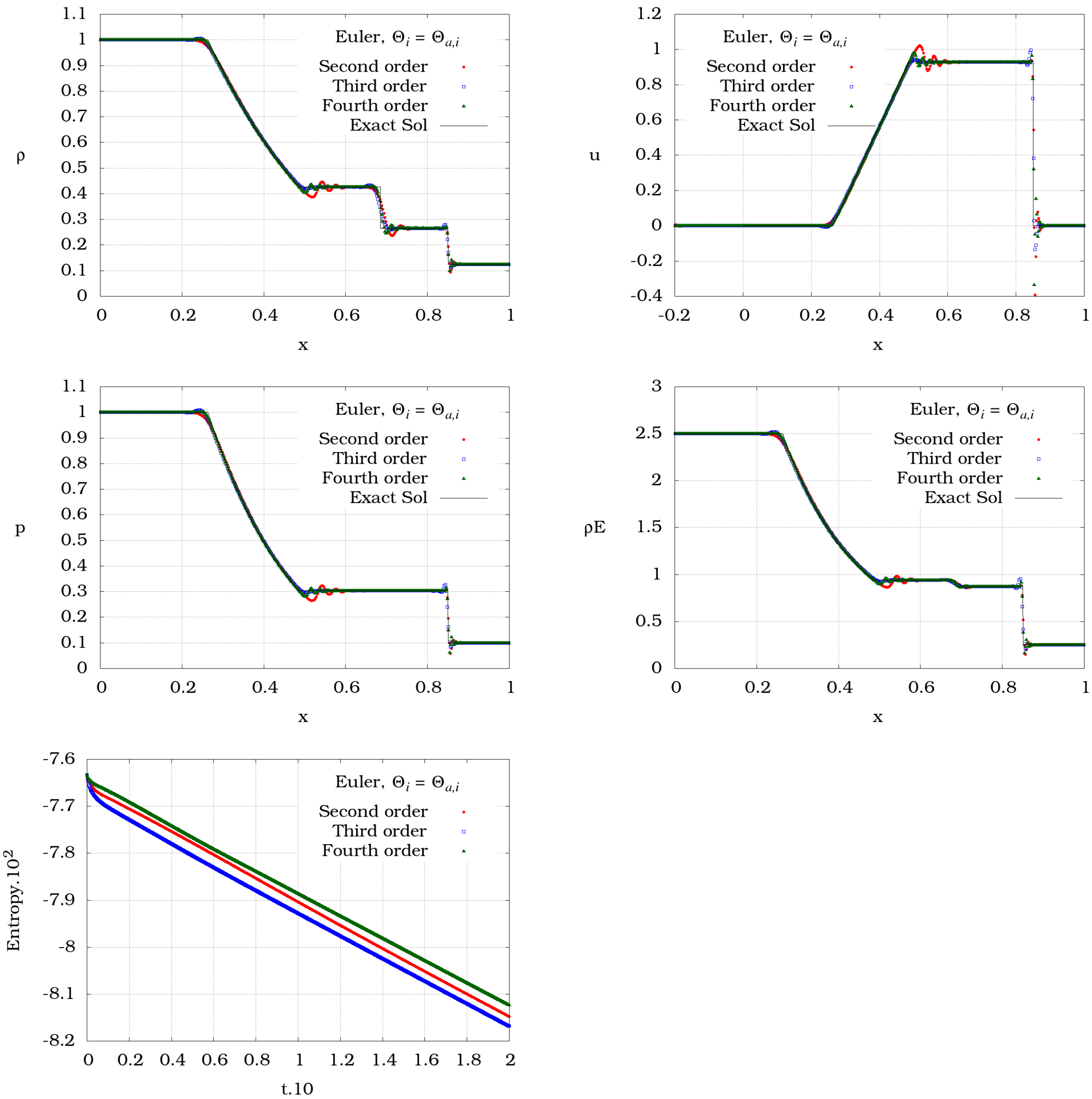

Fig. 4.3: Second-, third- and fourth-order accurate approximation of the shock tube Euler solution and entropy with a mesh made of 400 cells for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

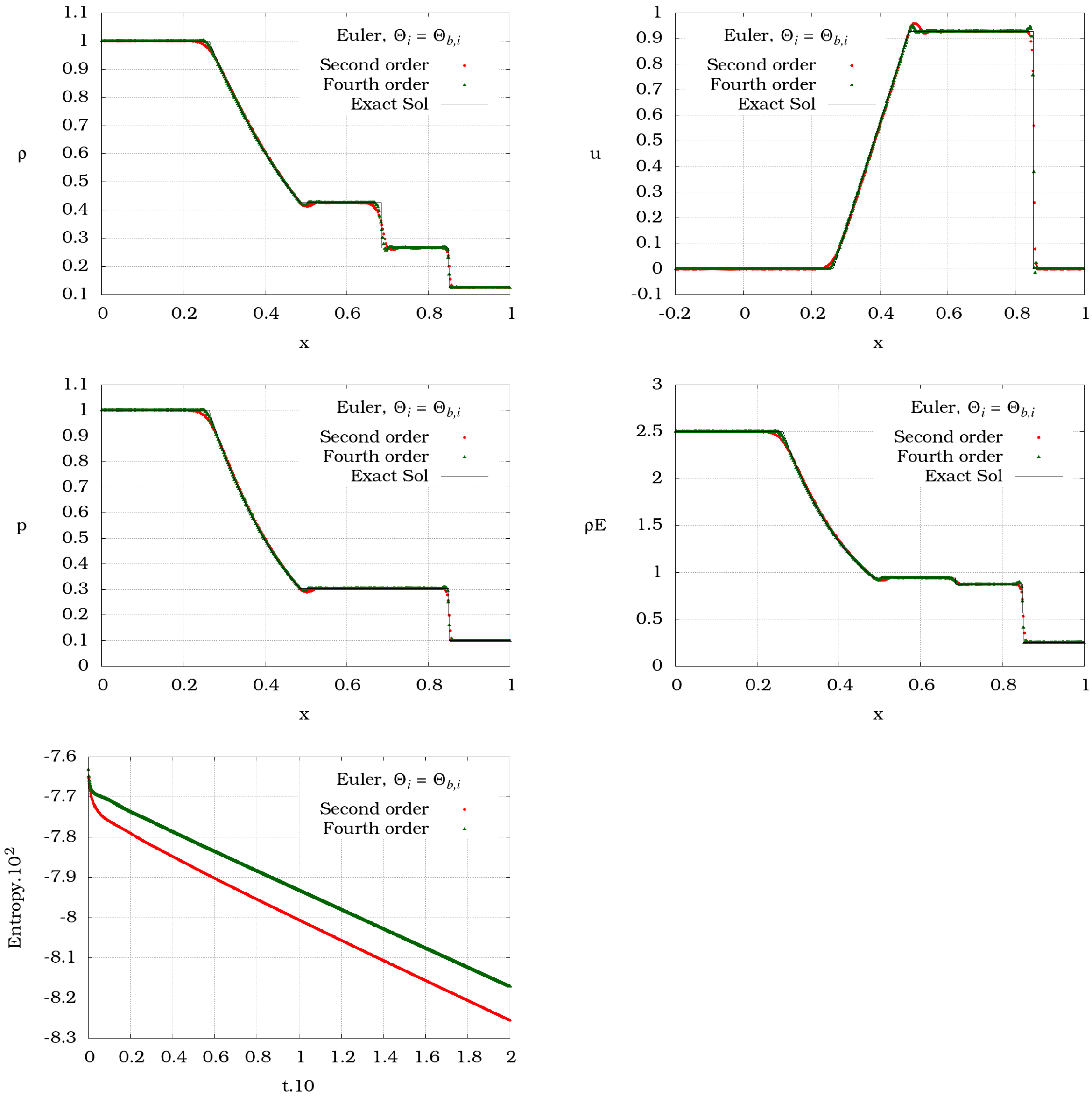

Fig. 4.4: Second-, third- and fourth-order accurate approximation of the shock tube Euler solution and entropy with a mesh made of 400 cells for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	7.2E-02	-	7.1E-02	-	2.7E-01	-
200	4.0E-02	0.8	4.5E-02	0.7	2.4E-01	0.1
400	2.2E-02	0.9	2.9E-02	0.6	2.4E-01	0.0
800	1.2E-02	0.9	1.9E-02	0.6	1.9E-01	0.4
1600	6.4E-03	0.9	1.3E-02	0.6	1.7E-01	0.1
	Third-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	4.5E-02	-	5.0E-02	-	2.0E-01	-
200	2.3E-02	0.9	3.1E-02	0.7	1.9E-01	0.1
400	1.2E-02	1.0	2.0E-02	0.7	1.7E-01	0.1
800	5.9E-03	1.0	1.2E-02	0.7	1.3E-01	0.4
1600	3.2E-03	0.9	8.5E-03	0.5	1.0E-01	0.4
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	3.8E-02	-	1.4E-01	-
200	1.9E-02	0.9	2.4E-02	0.6	1.2E-01	0.3
400	9.9E-03	0.9	1.6E-02	0.6	1.2E-01	0.0
800	4.9E-03	1.0	9.5E-03	0.7	8.1E-02	0.6
1600	2.5E-03	1.0	6.5E-03	0.6	8.5E-02	0.1

Table 4.12: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the shock tube Euler solution and for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

	Second-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-02	-	6.3E-02	-	2.5E-01	-
200	3.2E-02	0.9	4.0E-02	0.7	2.3E-01	0.1
400	1.7E-02	0.9	2.6E-02	0.6	2.4E-01	0.0
800	8.7E-03	1.0	1.6E-02	0.7	1.7E-01	0.5
1600	4.5E-03	0.9	1.1E-02	0.5	2.2E-01	0.4
	Fourth-order scheme errors					
cells	L^1	order	L^2	order	L^∞	order
100	3.2E-02	-	3.9E-02	-	1.7E-01	-
200	1.6E-02	1.0	2.5E-02	0.7	1.6E-01	0.1
400	8.0E-03	1.0	1.5E-02	0.7	1.4E-01	0.2
800	3.8E-03	1.1	8.8E-03	0.8	1.1E-01	0.4
1600	2.0E-03	0.9	6.0E-03	0.5	8.9E-02	0.3

Table 4.13: Errors and order evaluations for the second- and fourth-order accurate schemes with the shock tube Euler solution and for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

The obtained approximate solutions are displayed Fig. 4.3 4.4. Once again, we remark only little spurious oscillations. In Table 4.12 4.13, we detail the evaluated orders of accuracy.

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Appendix. Existence of the matrix parameter $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$.

PROPOSITION A.1. *Consider $\eta \in C^2(\Omega,\mathbb{R})$ a strictly convex entropy. Let the approximation at time t^n, $w_\Delta(\cdot,t^n)$ given by (1.6) being a non zero function in $L^2(\mathbb{R})$. Assume there exists a compact set $K\subset\Omega$ such that $w_\Delta(x,t^n)\in K$ for every $x\in\mathbb{R}$. Then there exists a sequence of bounded (as $\Delta x\to 0$) diagonal matrices $(\Theta_i^{Ok})_{i\in\mathbb{Z}}$ such that the inequality (3.3) is satisfied.*

Proof. According to the selected order of accuracy, we remark that both $\Delta x\overline{\partial_x w_i}^{Ok}$ and $\Delta x^3\overline{\partial_{xxx} w_i}^{O4}$ are affine functions with respect to Θ_i^{Ok} so that we may write

$$\left(\Delta x\overline{\partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{24}\Delta x^3\overline{\partial_{xxx} w_i}^{O4}\right) = \Theta_i^{Ok}\mathcal{A}_i + \mathcal{B}_i,$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ are vectors of size d that come with a linear dependency on $(\delta_{i+\nu+\frac12})_{-2\le\nu\le 1}$ (with respect of $(w_i^n)_{i\in\mathbb{Z}}$). The condition (3.3) then reformulates as follows

$$\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12} - \frac12\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\Theta_i^{Ok}\mathcal{A}_i+\mathcal{B}_i\right) > 0$$

where the above sums are convergent since $w_\Delta(\cdot,t^n)$ belongs to a compact set $K\subset\Omega$ and $w_\Delta(\cdot,t^n)$ belongs to $L^2(\mathbb{R})$. Since the matrices Θ_i^{Ok} are diagonal, it equivalently reformulates

$$S - \frac12\sum_{i\in\mathbb{Z}}\Theta_i^{Ok}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\mathcal{A}_i > 0, \tag{A.1}$$

where we have set

$$S = \sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_i^n)\right)\cdot\delta_{i+\frac12} - \frac12\sum_{i\in\mathbb{Z}}\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\mathcal{B}_i. \tag{A.2}$$

We now choose $\Theta_i^{Ok}\in\mathcal{M}_d(\mathbb{R})$ under the form

$$\Theta_i^{Ok} = -\theta\,\mathrm{diag}_{1\le j\le d}\left(\mathrm{sign}\left(\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)_j(\mathcal{A}_i)_j\right)\right), \tag{A.3}$$

with $\theta>0$ a free constant to be fixed. Using such a formula for Θ_i^{Ok}, the inequality (A.1) now reformulates as follows:

$$S+\theta D > 0, \tag{A.4}$$

where D is positive number (because $w_\Delta(\cdot,t^n)$ is non constant) given by

$$D = \frac12\sum_{i\in\mathbb{Z}}\sum_{j=1}^d\left|\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)_j(\mathcal{A}_i)_j\right| > 0. \tag{A.5}$$

Since $D>0$, it is therefore sufficient to choose $\theta>0$ such that $\theta>\frac{S^-}{D}$ where $S^-=-\min(S,0)\ge 0$ is the negative part of S. $\square$