NORMAL FORMS OF FUNCTIONS IN NEIGHBOURHOODS OF DEGENERATE CRITICAL POINTS

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An analysis of the normal forms to which functions can be reduced in neighbourhoods of degenerate critical points shows that many of them are quasihomogeneous or semiquasihomogeneous.

A semiquasihomogeneous function is a sum of a quasihomogeneous (or weighted homogeneous) polynomial with an isolated critical point and summands of a higher degree of quasihomogeneity. The normal form to which a semiquasihomogeneous function can be reduced is described in terms of the local ring of the gradient mapping given by the quasihomogeneous part of the function. The number of parameters in this normal form is called the inner modality of the quasihomogeneous part.

A classification is given of all quasihomogeneous critical points of inner modality 1: up to stable equivalence they are exhausted by three one-parameter families of parabolic singularities and 14 exceptional polynomials, 8 of which are functions of two variables, and 6 functions of three variables.

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§1. Introduction

1.1. Degenerate critical points. A critical point of a function is said to be *non-degenerate* if the second differential of the function at the point is a non-degenerate quadratic form. The behaviour of a function in the neighbourhood of a non-degenerate critical point is described by Morse's lemma: there is a system of coordinates in the neighbourhood of such a point in which the function takes a simple "normal form", namely

$$f(x_1, \ldots, x_n) = \pm x_1^2 \pm \ldots \pm x_n^2 + \text{const.}$$

A function "in general position" has only non-degenerate critical points. Every function can be brought to general position by arbitrarily small displacements, so that all its critical points become non-degenerate.

Degenerate critical points appear naturally in cases when the functions depend on parameters. For instance, in the one-parameter family

$$f(x, t) = x^3 + tx$$

of functions of a single variable x, a degenerate critical point arises for an exceptional value of the parameter ($t = 0$). This point cannot be removed by small displacement of the family: whatever the displacement, a degenerate singularity always occurs for a nearby value of the parameter.

Thus, in the investigation of critical points of functions depending on parameters we have to consider degenerate points as well as non-degenerate. The larger the number of parameters, the more complicated the critical points that can occur.

1.2. Terminology. This article is the fifth in a series ([1]–[4]) on the classification of critical points of functions. To make this paper independent of its predecessors, we give here certain definitions and results from [1]–[4].

At first glance, the problem of classifying critical points of l-parameter families of functions of n variables that are irremovable under small displacements seems hopeless for large l and n. Nonetheless, when the first piece of the classification was executed, it turned out to be not as complicated as all that, and the singularities arising were closely connected with objects that seem very remote from them at first sight: namely with Lie groups and the Weyl series A_k, D_k, E_k, Coxeter groups generated by reflections, the braid groups of Artin and Brieskorn, and finally, with the classification of regular polygons in ordinary three-space.

At first the classification of critical points is discrete: if the number l of parameters is less than six, then we can find a finite number of normal forms such that every l-parameter family of functions can be brought to "general position" by a small displacement under which every function in the family reduces to one of the given normal forms in the neighbourhood of every critical point, after a smooth change of variables.

Beginning with $l = 6$ (and the number of variables $n \geqslant 3$), "moduli" emerge, and the normal forms inevitably involve parameters.

The number of moduli (that is, the number of parameters in the normal forms) remains finite for finite l. The fact is that a smooth function is equivalent in a neighbourhood of a critical point to a fairly long segment of its Taylor series, except in the case of critical points of infinite multiplicity (which can be removed under a small displacement of an l-paramete system, for any finite l) (see [5]–[7]).

The class of functions whose Taylor series at 0 coincide up to the terms of degree k with the Taylor series of a given function is called the *k-jet* of the function at 0. A k-jet is said to be *sufficient* if all functions with this k-jet are mutually equivalent (that is, can be transformed into one another under a smooth change of variables).

In these terms we can state that *every function with a critical point of finite multiplicity has a sufficient jet.*

Finiteness of the multiplicity of a critical point is defined easily in the complex case: a function $f\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ has 0 as a critical point of finite multiplicity if 0 is an isolated critical point. The *multiplicity* μ of an isolated critical point of a complex function can be defined as the number of non-degenerate critical points into which it splits under a small displacement, or as the degree of the gradient function mapping a small sphere S^{2n-1} around 0 in $\mathbf{C}^n$ onto itself.

In the real case the multiplicity μ of a smooth mapping $F\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ at 0 is defined as the $\mathbf{R}$-dimension of the local ring

$$Q(F) = \mathbf{R}[[x_1, \ldots, x_n]]/(F_1, \ldots, F_n),$$

where $(F_1, \ldots, F_n)$ is the ideal spanned by the Taylor series of the functions specifying F. Here $\mathbf{R}[[\ldots]]$ is the ring of formal power series in the coordinates $(x_1, \ldots, x_n)$ in the spatial inverse image (instead of formal power series we can use convergent series, polynomia or smooth functions, provided that $\mu < \infty$).

The multiplicity μ of the critical point 0 of a function $f\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is defined as the multiplicity of the gradient mapping: $\mu = \dim_{\mathbf{R}} Q(F)$, where $F = \operatorname{grad} f$. We denote the local ring of the gradient mapping by Q_f, and call it the *local ring of the function* at the critical point. Of course, an analogous ring is defined in the complex case, and then $\mu = \dim_{\mathbf{C}} Q_f$ [8].

In investigating critical points of finite multiplicity, the function may be replaced by a sufficient jet or a polynomial.

Thus, the question of classifying critical points reduces to an algebraic question about the orbits of the action of a finite-dimensional Lie group o a finite-dimensional manifold.

The number of moduli (or the modality) of a function in the neighbourhood of a critical point is defined as follows.

Let G be a Lie group acting on a (finite-dimensional) manifold X. The orbits of G can generate discrete stratifications or continuous families in th neighbourhood of a given point $x \in X$. We say that a point x *has modalit* m (under the given action) if a sufficiently small neighbourhood of x in X can be covered by finitely many families of orbits, depending on not more than m parameters (and an arbitrarily small neighbourhood of x intersects some m-parameter family of orbits).

By the modality of a function at a critical point we understand the modality of the action of the group of changes of independent variables on the space of functionals of the function.

More accurately, we consider the finite-dimensional manifold of k-jets at 0 of the function f: $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with the critical point 0 and critical value 0. The group of k-jets at 0 of diffeomorphisms $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ fixing 0 acts on this manifold. By the *modality* of f at the critical point 0 we mean the modality of its k-jet under the action of the group of k-jets of diffeomorphisms, where k is sufficiently large.

1.3. 0-modal critical points. The simplest degenerate critical points are the 0-modal ones. The 0-modal critical points can be classified as follows.

Two functions with critical point 0 and critical value 0 are said to be *equivalent* if one is transformed into the other under a diffeomorphism fixing 0 of a sufficiently small neighbourhood of 0. Two functions are *stably equivalent* if they become equivalent under direct addition of non-degenerate quadratic forms. For example, the functions $f(x) = x^3$ and $g(x, y) = x^3 + y^2$ are stably equivalent.

In a neighbourhood of a 0-modal critical point 0, every function f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ is equivalent to one of the following functions [2]:

$$\begin{aligned} A_k &: f(x) = x^{k+1}, \\ D_k &: f(x, y) = x^2 y + y^{k-1}, \\ E_6 &: f(x, y) = x^3 + y^4, \\ E_7 &: f(x, y) = x^3 + xy^3, \\ E_8 &: f(x, y) = x^3 + y^5. \end{aligned}$$

The classification of 0-modal critical points has given rise to the hope that the natural classes of critical points allowing a simple description are those of small modality (and not the classes with small μ, or the classes with small codimension l, which are needed in analysis and topology).

1.4. Unimodal critical points. The classification of critical points of modality 1 turned out to be not too complicated [3]. Indeed, there is one three-index series of one-parameter families of unimodal singularities,

$$T_{p,q,r}: f(x, y, z) = axyz + x^p + y^q + z^r,$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$, and 14 "exceptional" one-parameter families. Every function f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ is equivalent in the neighbourhood of a critical point of modality 1 to one of those listed.

There are three special cases of the series T for which $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, namely (3, 3, 3), (2, 4, 4) and (2, 3, 6). These singularities are said to be *parabolic*, and the remaining singularities in the series T are *hyperbolic* (corresponding to the signature of the quadratic form given by the intersection index in the homology of average dimension of the local manifold of non-singular level of the function – see [4]; *elliptic* singularities turn up only as the 0-modal singularities A, D, E).

The 14 exceptional families of unimodal singularities are also given by simple formulae:

$$\begin{aligned} K_{12} &= x^3+y^7+axy^5, & K_{13} &= x^3+xy^5+ay^8, \\ K_{14} &= x^3+y^8+axy^6, & Z_{11} &= x^3y+y^5+axy^4, \\ Z_{12} &= x^3y+xy^4+ax^2y^3, & Z_{13} &= x^3y+y^6+axy^5, \\ W_{12} &= x^4+y^5+ax^2y^3, & W_{13} &= x^4+xy^4+ay^6, \\ Q_{10} &= x^3+y^4+yz^2+axy^3, & Q_{11} &= x^3+y^2z+xz^3+az^5, \\ Q_{12} &= x^3+y^5+yz^2+axy^4, & S_{11} &= x^4+y^2z+xz^2+ax^3z, \\ S_{12} &= x^2y+y^2z+xz^3+az^5, & U_{12} &= x^3+y^3+z^4+axyz^2. \end{aligned}$$

An analysis of these formulae shows that in each such family there is exactly one quasihomogeneous (weighted homogeneous) polynomial, and that the whole family can be considered as a one-parameter deformation of this polynomial having the special property of "semiquasihomogeneity".

The quasihomogeneous polynomials of these 14 types are connected with the 14 triangles in the Lobachevskii plane, as was shown by I. V. Dolgachev (and for K_{12} already by Klein).

1.5. Semiquasihomogeneous critical points. The aim of this paper is the investigation of quasihomogeneous functions and their semiquasihomogeneous deformations. We indicate the normal forms to which functions having non-degenerate quasihomogeneous "principal part" can be reduced. We call the number of moduli in these normal forms the *inner modality* of the quasi-homogeneous function (the exact definition of the inner modality μ_0 is in §8.6; μ_0 does not depend on the choice of the normal form).

Quasihomogeneous functions of inner modality 1 can be listed independently of the classification of all unimodal functions. The main result of our article is an algebraic classification of quasihomogeneous functions of inner modality 1.

It turns out that (up to stable equivalence) there are just three one-parameter families of parabolic singularities and 14 polynomials generating exceptional families.

The technique developed here for working with semiquasihomogeneous singularities enables us to shorten significantly the calculations necessary for classifying unimodal singularities. The technique is based on a discussion of various filtrations in the ring of functions (or power series). With every such filtration we can associate its space of jets of functions, its filtered group of jets of diffeomorphisms, and the Lie algebra of vector fields. In the case of a "quasihomogeneous" filtration there arises also a Lie group of quasihomogeneous diffeomorphisms, which plays a rôle in this theory like that of the full linear group in the case of ordinary jets (which corresponds to the filtration given by the powers of the maximal ideal).

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§2. Quasihomogeneous functions and filtrations

In this section we give the definitions of the principal objects that are studied in the sequel: quasihomogeneous and semiquasihomogeneous functions.

2.1. DEFINITION. We consider the arithmetical space $\mathbf{C}^n$ with fixed coordinates $x_1, \ldots, x_n$. *A function* $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is said to be *quasihomogeneous of degree d with exponents* $\alpha_1, \ldots, \alpha_n$ if $f(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \ldots, x_n)$ *for all* λ.

In terms of the Taylor series $f = \Sigma f_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$, quasihomogeneity of degree 1 means that all the exponents of non-zero terms lie on the hyperplane

$$\Gamma = \{\mathbf{k} : \alpha_1 k_1 + \ldots + \alpha_n k_n = 1\}.$$

In what follows, we consider quasihomogeneous functions of degree 1 with rational exponents, $0 < \alpha_s \leqslant 1/2$. Such functions are automatically polynomials. We call the hyperplane Γ the *diagonal*. The diagonal cuts the coordinate axes in segments of length $a_s = 1/\alpha_s$.

2.2. DEFINITION. A quasihomogeneous function f is said to be *non-degenerate* if 0 is an isolated critical point (that is, if the multiplicity μ of the critical point 0 is finite). The degenerate quasihomogeneous functions form an algebraic hypersurface in the linear space of all quasihomogeneous polynomials with fixed quasihomogeneous exponents.

With every type of quasihomogeneity (that is, with every set α of quasihomogeneous exponents) there is associated a filtration in the ring of power series (functions, germs etc), defined as follows:

2.3. DEFINITION. We say that a monomial $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \ldots x_n^{k_n}$ has (generalized) *degree* d if $\langle \alpha, \mathbf{k} \rangle = \alpha_1 k_1 + \ldots + \alpha_n k_n = d$.

The degree of a monomial is a rational number. The exponents of all monomials of degree d (of given type) lie on a hyperplane parallel to the diagonal Γ. We fix the quasihomogeneous type, that is, we fix the set α of exponents.

2.4. DEFINITION. A polynomial (power series, germ, function) *has filtration d* if all its monomials are of degree d or higher: when the (generalized) degree of all monomials is d, we call d the (generalized) *degree* of the polynomial; the degree of 0 is $+\infty$.

The polynomials (series, germs) of filtration d form a linear space E_d; $E_{d'} \subseteq E_d$ if $d < d'$; the filtration of a product is the sum of the filtrations of the factors, so that E_d is an ideal in the ring of polynomials (series, germs). Denoting this ring by A, we call the factor-ring A/E_d the *ring of d-jets,* and its elements *d-jets.*

By the filtration $\varphi(f)$ of a polynomial (series, germ) f we understand, a a rule, the largest d such that $f \in E_d$. The filtrations of all polynomials (series, germs) lie in a rational arithmetical progression: $\varphi(f) \in \mathbf{Z}_+ d_0$, wher d_0 is the greatest common divisor of the numbers α_s (an initial segment (the progression may not be completely filled out by the values of φ).

2.5. DEFINITION. A polynomial (power series, germ) is said to be *sem quasihomogeneous of degree d with exponents* $\alpha_1, \ldots, \alpha_s$ if it is of the form $f = f_0 + f'$, where f_0 is a non-degenerate quasihomogeneous polynomial of degree d with exponents $\boldsymbol{\alpha}$, and f' is a polynomial (series, germ of filtration strictly greater than d.

In other words, a semiquasihomogeneous function is obtained from a n(degenerate quasihomogeneous function by adding monomials whose expon ents lie *above the diagonal.* Note that a quasihomogeneous function is not semiquasihomogeneous if it is degenerate.

§3. Multiplicity and generators of the local ring of a semiquasihomogeneous function

In this section we show that a monomial basis for the local ring of a quasihomogeneous non-degenerate function is a basis for all semiquasihomogeneous functions with the given quasihomogeneous part.

3.1. THEOREM. *The multiplicity of the critical point* 0 *of a semiquasihomogeneous function f is that of its quasihomogeneous part*: $\mu(f) = \mu(f_0$

PROOF. We consider the family of topological spheres

$$S_t = \{\mathbf{x} \in \mathbf{C}^n : |x_1|^{a_1} + \ldots + |x_n|^{a_n} = t\}, \qquad a_s = 1/\alpha_s, \quad d = 1.$$

The number $\mu(f)$ is the degree of the mapping $\mathbf{x} \to (\partial f/\partial \mathbf{x})/\|\partial f/\partial \mathbf{x}\|$, $\mathbf{x} \in S_t$, for small values of t. For every point $\mathbf{x} \in S_1$, at least one of the derivatives $\partial f_0/\partial x_s$ is different from 0 (since f_0 is non-degenerate). Thus, there is a constant c such that $\max\limits_s |\partial f_0/\partial x_x| \geqslant c > 0$ on S_1.

Note that $S_t = T_t S_1$, where $T_t(x_1, \ldots, x_n) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$. Further the partial derivative $\partial f_0/\partial x_s$ is quasihomogeneous of degree $1 - \alpha_s$ and of type $\boldsymbol{\alpha}$. Therefore, at every point of S_t, $|\partial f_0/\partial x_s| \geqslant ct^{1-\alpha_s}$ for at least one s

On the other hand, the function f' has filtration not less than $1 + d_0$. Therefore there exists a constant C such that $|\partial f'/\partial x_s| \leqslant Ct^{1+d_0-\alpha_s}$ on S_t for all s.

Comparing with the preceding inequality, we see that for small enough there are no critical points of the functions $f_0 + \theta f'$, $0 \leqslant \theta \leqslant 1$, on S_t. Thus, the degree of the mappings of the sphere onto itself given by the gradients of f_0 and $f_0 + f'$ coincide, as required.

REMARK. It can be shown in a similar way that *all quasihomogeneous functions sufficiently near f_0 and having the same degree of quasihomogeneit* *have the same multiplicity* μ. Further, since the set of non-degenerate quasi-homogeneous functions of given degree is connected, *the multiplicity* μ *is the same for all non-degenerate quasihomogeneous functions of given degree (and therefore for all semiquasihomogeneous functions of given degree and given type).*

The multiplicity μ of the critical point 0 of a function f may also be defined as the dimension of the local ring

$$Q_f = \mathbf{C}[[x_1, \ldots, x_n]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

Allowing freedom of speech, we call a set of μ series (polynomials, germs) *a basis of the local ring of* f if they become a basis for Q_f over $\mathbf{C}$ after factorization by this ideal.

From the theorem just proved we deduce:

3.3. COROLLARY. *Assume that the system of monomials* $e_1, \ldots, e_\mu$ *is a basis of the local ring of the quasihomogeneous part* f_0 *of a semiquasi-homogeneous function* f. *Then this same system of monomials is a basis of the local ring of* f.

The proof is based on the following general lemma.

3.4. LEMMA. *Assume that a system of smooth functions* f *depending continuously on a finite number of parameters has* 0 *as a critical point of constant multiplicity* μ *for all values of the parameters. Then every basis of the local ring of the function corresponding to the value* 0 *of the parameter remains a basis for nearby values of the parameter.*

PROOF OF THE LEMMA. The lemma follows from this fact: given a subspace of a finite-dimensional vector space depending smoothly on some parameters, and a system of vectors forming a basis of the transversal space, then the system remains a basis of the transversal space for nearby values of the parameters.

To make the space finite-dimensional, it is enough to factor the ring $\mathbf{C}[[x_1, \ldots, x_n]]$ by a sufficiently high power of the maximal ideal (the power depending only on μ, see [5]).

PROOF OF COROLLARY 3.3. We consider a semiquasihomogeneous function $f = f_0 + f'$. We claim that the transition from f_0 to f may be regarded as a *small* deformation. We construct a one-parameter family of functions, $f_t(x) = t^{-1} f(T_t x)$, where $T_t x = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$, $d = 1$. We have $f_t(x) = f_0 + t^{-1} f'(T_t x)$, where all the coefficients in the second summand depend continuously on t, since the filtration of f' is greater than 1. By the lemma, a basis of the local ring for f_0 is a basis for f_t for sufficiently small t. A basis of the local ring for f_t goes over to a basis of that for f under the action of the diffeomorphism T_t connecting the functions f and f_t. But under T_t, every monomial goes over to a monomial proportional to it. Thus, a *monomial* basis of Q_{f_0} is not only a basis of Q_{f_t} for small t, but also of Q_f; this is what we wanted to prove.

3.5. REMARK. *The number of basis monomials of the local ring of a* *quasihomogeneous or semiquasihomogeneous function f having given generalized degree d does not depend on the choice of the basis in the local ring.*

PROOF. We consider the factor-space

$$E_\delta/(E_{>\delta} + E_\delta \cap I), \text{ where } I = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n),$$

E_δ is the space of series of filtration δ in $\mathbf{C}[[x_1, \ldots, x_n]]$,

$E_{>\delta}$ is the space of series of filtration greater than δ.

The number of basis monomials of degree δ is equal to the dimension of this factor-space, therefore, does not depend on the basis.

3.6. COROLLARY. *The number of basis monomials of the local ring of a function having given generalized degree δ (for given type α) is the same for all semiquasihomogeneous functions f of type $\boldsymbol{\alpha}$ and degree d.*

PROOF. It is enough to consider non-degenerate quasihomogeneous functions (for a semiquasihomogeneous function the basis is the same, by Corollary 3.3). The manifold of non-degenerate quasihomogeneous functions of given degree d (and type α) is arcwise connected (it is the complement of a hypersurface in a linear space). The number of basis monomials of degree δ of the local ring is locally constant along a curve joining two points of this manifold, because the same basis works for neighbouring functions (Lemma 3.4). Thus, it is constant, as required.

§4. Quasihomogeneous mappings

In this section we compute various numerical invariants of quasihomogeneous mappings, in particular, the multiplicity μ and the generating polynomial χ.

We fix a quasihomogeneous type $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ in $\mathbf{C}^n$ with a fixed coordinate system. We consider a mapping F: $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ and use the older notation $F(x_1, \ldots, x_n) = (F_1(x_1), \ldots, F_n(x_n))$. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be a vector with non-negative components.

4.1. DEFINITION. The mapping F is said to be *quasihomogeneous of degree* $\mathbf{d}$ (and type α) if every component F_s is a quasihomogeneous function of degree d_s of one and the same type $\boldsymbol{\alpha}$.

The local ring of F is the factor ring

$$Q(F) = \mathbf{C}[[x_1, \ldots, x_n]]/(F_1, \ldots, F_n).$$

F is said to be *non-degenerate* if its multiplicity at 0 is finite, that is, if the local ring $Q(F)$ has finite dimension over $\mathbf{C}$; this dimension $\mu = \dim_{\mathbf{C}} Q(F)$ is called the *multiplicity of F at* 0.

F is said to be *semiquasihomogeneous* if $F = F_0 + F'$, where F_0 is a non-degenerate quasihomogeneous mapping, and each component F'_s has filtration greater than the degree of the corresponding component F_{0s}.

If a function f is semiquasihomogeneous of degree d (and type α), then the mapping $x \mapsto \operatorname{grad}(f(x))$ is semiquasihomogeneous of degree $d_s = d - \alpha_s$.

4.2. PROPOSITION. *The assertions of Theorem 3.1, Remark 3.2, Corollary 3.3, Lemma 3.4, Remark 3.5 and Corollary 3.6 hold not only for the gradient mappings $x \mapsto \operatorname{grad}(f(x))$ as formulated above, but also for any quasihomogeneous or semiquasihomogeneous mappings.*

For example, Remark 3.2 runs as follows: *all semiquasihomogeneous mappings of the same degree* $\mathbf{d}$ *(and type* α*) have the same multiplicity* μ.

Kushirenko has proved analogues of these assertions over an arbitrary algebraically closed field. For $\mathbf{C}$ the proof proceeds as in §3.

The value of the class of quasihomogeneous mappings for our problem lies in the facts that in them homotopies can be carried out more freely, and changes of variables are easier than for gradients. In particular, the following proposition (see [10]) makes it possible to go by a simple substitution from a quasihomogeneous to a genuine homogeneous mapping.

4.3. PROPOSITION. *Suppose that $F\colon \mathbf{C}^n \to \mathbf{C}^n$ is a quasihomogeneous mapping whose type and degree have the common denominator N: $\alpha_s = A_s/N$, $d_s = D_s/N$, where now A_s, D_s, N are integers. Consider the mapping $T\colon \mathbf{C}^n \to \mathbf{C}^n$ given by the formula $T(y_1, \ldots, y_n) = (y_1^{A_1}, \ldots, y_n^{A_n})$. Then*

1) *the mapping $F \circ T\colon \mathbf{C}^n \to \mathbf{C}^n$ has as its components homogeneous functions in the usual sense, of degrees $D_1, \ldots, D_n$.*

2)
$$\mu(F \circ T) = \mu(F) \prod A_s.$$

3) *If $e_1, \ldots, e_\mu$ is a monomial basis of the local ring of F, then the following functions form a monomial basis of the local ring of $F \circ T$:*

$$e'_{i,u} = (T^* e_i) y_1^{u_1} \ldots y_n^{u_n}, \text{ where } 1 \leqslant i \leqslant \mu,\ 0 \leqslant u_s < A_s.$$

PROOF. 1) The monomial $\mathbf{x}^{\mathbf{k}}$ determines in the s-th component of $F \circ T$ the monomial $\prod y_s^{k_s A_s}$ of degree $\sum k_s A_s = N(\mathbf{k}, \alpha) = N d_s = D_s$. 2) The formula for the multiplicity is obtained from considering the system of equations $(F \circ T)(y) = \varepsilon$ in y. 3) The functions $e'_{i,u}$ generate the whole local ring. For every function of y can be written in the form $\varphi = \sum_u y^u T^* \varphi_u$, and every function of x in the form $\varphi_u = \sum c_{i,u} e_i + \sum F_s h_{s,u}$. Thus,

$$\varphi = \sum_{u,i} c_{i,u} y^u T^* e_i + \sum_{u,s} y^u h_{s,u} T^* F_s,$$

that is, the $e'_{i,u}$ generate $Q(F \circ T)$. Since the number of functions $e'_{i,u}$ is $\mu(F \circ T)$, they form a basis. This proves Proposition 4.3.

4.4. DEFINITION. The *generating function* of type α of a semiquasihomogeneous mapping F (where $\alpha_s = A_s/N$, A_s and N integers) is the polynomial $\chi_F(z) = \sum \mu_i z^i$, where μ_i is the number of basis monomials in the local ring of F having generalized degree i/N.

We remark that even for fixed quasihomogeneous type χ depends on th integer N. However, the possible values of N are multiples of one of them (the least common denominator of the fractions α_s), and $\chi_{F;\, kN,\, \alpha}(z) = \chi_{F;\, N,\, \alpha}(z^k)$.

The dimension of the local ring is given by the formula $\mu = \chi_F(1)$. Further, the degree of χ_F is the greatest of the (generalized) degrees of th monomial generators in a basis of the local ring.

4.5. THEOREM. (see [9], [14]). *The generating function of a (semi)-quasihomogeneous mapping F of degree d and type α for which $\alpha_s = A_s/N$, $d_s = D_s/N$, where A_s, D_s, N are integers, is given by the formula*

$$\chi_F(z) = \prod_{s=1}^{n} \frac{z^{D_s}-1}{z^{A_s}-1}.$$

EXAMPLE. If $F = \operatorname{grad} f$, where f is a (semi)quasihomogeneous function of type α (and degree 1), then

$$\chi_F(z) = \prod_{s=1}^{n} \frac{z^{N-A_s}-1}{z^{A_s}-1}.$$

Several useful formulae follow immediately from this theorem.

4.6. COROLLARY (see [10], [14]). *The dimension of the local ring of a semiquasihomogeneous mapping is given by the "generalized Bezout formula":*

$$\mu = \prod_{s=1}^{n} \frac{d_s}{\alpha_s}.$$

4.7. COROLLARY (see [11]). *The local ring of a semiquasihomogeneous mapping F has exactly one basis monomial of degree*

$$d_{\max} = \sum_{s=1}^{n} (d_s - \alpha_s);$$

all monomials of higher filtration lie in the ideal generated by the components $(F_1, \ldots, F_s)$.

Let us look, in particular, at the local ring of a semiquasihomogeneous function f of type $(\alpha_1, \ldots, \alpha_n)$ of degree 1. In this case $\mathbf{d}_s = 1 - \alpha_s$, and we obtain the next result.

4.8. COROLLARY (see [10], [14]). *The dimension of the local ring of a semiquasihomogeneous function f of type $(\alpha_1, \ldots, \alpha_n)$ and degree 1 is given by the formula*

$$\mu = \prod \left(\frac{1}{\alpha_s} - 1\right).$$

4.9. COROLLARY ([11]). *A monomial basis of the local ring of a semi-quasihomogeneous function f of type $(\alpha_1, \ldots, \alpha_n)$ and degree 1 has exactly*