one generator of (generalized) degree $d_{\max} = \Sigma(1 - 2\alpha_s)$; all monomials of higher degree lie in the ideal $(\partial f/\partial x_1, \ldots, \partial f/\partial x_s)$.

Here are some more immediate corollaries of Theorem 4.5.

4.10. COROLLARY. *The generating polynomial of a semiquasihomogeneous mapping is always recurrent:*

$$\mu_i = \mu_{k-i} \text{ where } k = \Sigma D_s - \Sigma A_s.$$

4.11. COROLLARY. *The generalized degree of the penultimate monomial (according to filtration) in a monomial basis of the local ring of a quasihomogeneous function of type $(\alpha_1, \ldots, \alpha_n)$ and degree 1 is $d_{\max} - \alpha_{\min}$, where $\alpha_{\min} = \min(\alpha_1, \ldots, \alpha_n)$.*

4.12. COROLLARY. *A non-degenerate quasihomogeneous mapping of type $\alpha = \mathbf{A}/N$ and degree $\mathbf{d} = \mathbf{D}/N$ can exist only when the polynomial $\prod(z^{D_s} - 1)$ is divisible by $\prod(z^{A_s} - 1)$.*

4.13. COROLLARY. *A non-degenerate quasihomogeneous mapping of type $\alpha\,(\alpha_s = A_s/N)$ can exist only when $\prod(z^{N-A_s} - 1)/\prod(z^{A_s} - 1)$ is a polynomial.*

REMARK. In the case of functions of two or three variables, the fact that the fraction $\Pi(z^{N-A_s} - 1)/\Pi(z^{A_s} - 1)$ can be cancelled is sufficient as well as necessary for the existence of a non-degenerate quasihomogeneous function with exponents A_s/N (see §§ 10, 11). This is not true for four variables, as can be seen from the following example, which V. M. Izlev has shown to me:

$$N = 265, A_1 = 1, A_2 = 24, A_3 = 33, A_4 = 58.$$

In this example the quotient is a polynomial with non-negative coefficients, while all quasihomogeneous functions with exponents A_s/N are degenerate.

The results of this section have been rediscovered over and over again. At the time when this paper is going to press, Gabrielov has informed me that Theorem 4.5 and Corollaries 4.6 and 4.8 can be found in papers of Milnor, Orlik and Wagreich. Saito and Hironaka have given other proofs of Corollaries 4.7 and 4.9. Kushirenko has told me that all the results, are, in fact, in Bourbaki [14] (see Proposition 2 in Ch. 5, § 5.1, in the section "Poincaré series of a graded algebra").

PROOF OF THEOREM 4.5. It is enough to consider the case of a non-degenerate quasihomogeneous mapping F (see Corollary 3.6 and Proposition 4.2). We change T as in Proposition 4.3. It follows from the form of the generators of the local ring of $F \circ T$ that

$$\chi_{F \circ T;\, 1, \mathbf{1}}(z) = \chi_{F;\, N, \alpha}(z)\chi_{T;\, 1, \mathbf{1}}(z), \text{ where } \mathbf{1} = (1, \ldots, 1).$$

The generating polynomials of T and $F \circ T$ occurring in this formula are homogeneous in the usual sense, and can be calculated explicitly. Indeed, for the mapping $x = y^A$ we have $\chi(z) = \dfrac{z^A - 1}{z - 1}$. It follows that

$$\chi_T(z) = \prod_{s=1}^{n} \frac{z^{A_s} - 1}{z - 1}$$

(here and later the pair $(N, \alpha) = (1, 1)$ is omitted from the notation for χ).

On the other hand, $F \circ T$ is a non-degenerate mapping whose components are homogeneous functions of degree D_s. Thus (by Proposition 4.2 and Corollary 3.6), it has the same generating polynomial as every other non-degenerate homogeneous mapping with these degrees. We can take, for example, the mapping T' given by the formula

$$T'(y_1, \ldots, y_n) = (y_1^{D_1}, \ldots, y_n^{D_n}).$$

Thus,

$$\chi_{F \circ T}(z) = \chi_{T'}(z) = \prod_{s=1}^{n} \frac{z^{D_s} - 1}{z - 1}.$$

The formula for χ_F is obtained by dividing the formulae for $\chi_{F \circ T}$ and for χ_T. This proves Theorem 4.5.

§5. Quasihomogeneous diffeomorphisms and quasijets

Several Lie groups and Lie algebras are connected with the filtrations defined by a quasihomogeneous type α. In the case of ordinary homogeneity, these are the full linear group, the group of k-jets of diffeomorphisms, the subgroup of k-jets with identity $(k-1)$-jets, and their factor groups. The analogues for quasihomogeneous filtrations are defined as follows.

We consider the space $\mathbf{C}^n$, with a fixed coordinate system $(x_1, \ldots, x_n)$. The ring of formal power series[1] in these coordinates is denoted by $A = \mathbf{C}[[x_1, \ldots, x_n]]$. We assume that a type of quasihomogeneity $\alpha = (\alpha_1, \ldots, \alpha_n)$ is given. We denote by E_d the ideal of A generated by series of filtration d. Further, let $E_{>d}$ stand for the ideal of E_d consisting of series of filtration strictly greater than d.

A *formal diffeomorphism* $g: (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ is given by a collection of n power series without free terms and gives a ring isomorphism $g^*: A \to A$ by the formula $g^*f = f \circ g$, where $\circ$ denotes the substitution of a series in a series.

5.1. DEFINITION. A diffeomorphism g has *filtration* d if, for all λ,

$$(g^* - 1)E_\lambda \subset E_{\lambda + d}.$$

5.2. PROPOSITION. *Let $d \geqslant 0$. Then the set $G_d = G_d(\alpha)$ of all diffeomorphisms of filtration d is a group under $\circ$.*

PROOF. We remark, first of all, that $g^*E_\lambda = E_\lambda$ for all λ when $d \geqslant 0$ ($g^*E_\lambda \supset E_\lambda$, since $d \geqslant 0$, and $g^{*-1}E_\lambda \supset E_\lambda$, since the factor space A/E_λ is finite-dimensional). Thus, for a, b in G_d we have

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[1] The greater part of what follows carries over at once to the case when A is the ring of convergent series over $\mathbf{C}$ or $\mathbf{R}$, or the ring of germs of smooth functions.

$$[(a\circ b)^*-1]E_\lambda=[b^*(a^*-1)+(b^*-1)]E_\lambda\subset E_{\lambda+d},$$
$$(a^{-1*}-1)E_\lambda=a^{-1*}(1-a^*)E_\lambda\subset E_{\lambda+d},$$

as required.

5.3. PROPOSITION. *For $q > p \geqslant 0$, G_q is a normal subgroup of G_p.*

PROOF. The definition of G_q uses only the filtration $\{E_\lambda\}$. This filtration is invariant under G_0 and a fortiori under G_p. Thus, a subgroup defined in terms of this filtration is normal, as required.

The group G_0 is especially important, because it plays for the quasihomogeneous case the rôle played by the full group of jets of diffeomorphisms in the homogeneous case. It must be stressed that in the quasihomogeneous case certain diffeomorphisms have negative filtrations and do not lie in G_0.

5.4. DEFINITION. *The group of d-jets of type* α is the factor group of the group of diffeomorphisms by the subgroup consisting of the diffeomorphisms of filtration greater than d,

$$J_d = J_d(\alpha) = G_0/G_{>d}.$$

It is clear that J_d is a finite-dimensional Lie group. There are natural factorizations $\pi_{p,q}\colon J_p \to J_q$ $(p > q \geqslant 0)$.

Attention should be drawn to the fact that in the ordinary homogeneous case our numbering differs by 1 from the standard one: our J_0 is called the group of 1-jets, etc.

5.5. PROPOSITION. *The group J_p is obtained from J_0 by a chain of extensions with commutative factors. More accurately, let E_p be the term of the filtration immediately following E_q. Then the kernel K of the homomorphism $\pi_{p,q}$ is commutative.*

PROOF. Let A, $B \in K$; we consider any representatives a, $b \in G_0$. Then

$$(ab)^*-1=(a^*-1)+(b^*-1)+(b^*-1)(a^*-1).$$

Further, for every λ,

$$(a^*-1)E_\lambda\subset E_{\lambda+p},\quad (b^*-1)E_\lambda\subset E_{\lambda+p},$$

since the q-jets a and b are trivial. Thus, $[(ab)^*-(ba)^*]E_\lambda \subset E_{\lambda+2p}$. Therefore, ab and ba determine the same element of J_p, which is what we wanted to prove.

The group J_0 has especial value, because it is the quasihomogeneous generalization of the full linear group.

5.6. DEFINITION. A diffeomorphism $g \in G_0$ is said to be *quasihomogeneous of type* α if every space of quasihomogeneous functions of degree d (and type α) is mapped into itself by g.

The set of all quasihomogeneous diffeomorphisms (of fixed type) forms a group. We denote it by $H(= H(\alpha))$ and call it the *group of quasihomogeneous diffeomorphisms.*

We consider the natural embedding $i\colon H \to G_0$ and factorization $\pi\colon G_0 \to J_0$.

5.7. PROPOSITION. *The group J_0 is naturally isomorphic to the group of quasihomogeneous diffeomorphisms; in fact, the compound map $\pi i\colon H \to J_0$ is an isomorphism of Lie groups.*

PROOF. a) $\operatorname{Ker} \pi i = e$. For $\operatorname{Ker} \pi i = H \cap G_{>0}$. Hence for $h \in \operatorname{Ker} \pi i$ and for every monomial f of degree d, $(h^* - 1)f$ lies in the space of homogeneous functions of degree d and also has filtration greater than d. Thus, $(h^* - 1)f = 0$ for every monomial f, and so $h = e$.

b) $\operatorname{Im} \pi i = J_0$. For this proof we construct the inverse mapping $J_0 \to H$ explicitly. Let $x_1, \ldots, x_n$ be coordinates in $\mathbf{C}^n$, and let a diffeomorphism $g \in G_0$ be a representative of the jet $j \in J_0$. We consider the series $g^*x_i \in E_{\alpha_i}$. We select in it the homogeneous component y_i of degree α_i, so that $g^*x_i = y_i + z_i$, $z_i \in E_{>\alpha_i}$. We define a polynomial mapping $h^{-1}\colon \mathbf{C}^n \to \mathbf{C}^n$ by the relation $h^*x_i = y_i$. To check that h is a diffeomorphism, we calculate the Jacobian:

$$\det\left|\frac{\partial\,(y_i+z_i)}{\partial x_j}\right| = \det\left|\frac{\partial y_i}{\partial x_j}\right| + R.$$

The term R containing the derivatives with respect to z is 0 at the origin. For every summand of the determinant y is homogeneous of degree 0 in x. All other summands containing z have positive filtration since $z_i \in E_{>d_i}$. Therefore, $R \in E_{>0}$ and $R(0) = 0$. Thus, the Jacobians of g and h at 0 are the same, so that the Jacobian of h at zero is different from 0, which means that h is a diffeomorphism. The ring automorphism h^* preserves degrees of all monomials, because it preserves the degrees of the coordinates x_i. Thus, $h \in H$. It is clear that $\pi i h = j$, and this proves the assertion.

5.8. PROPOSITION. *Suppose that $d \geqslant 0$. Then the group J_d of d-jets of diffeomorphisms acts as a group of linear transformations on the space $A/E_{>d}$ of d-jets of functions.*

PROOF. Let $g \in G_{>d}$. Then application of g does not change the α-jet of a function f, since $f \circ g - f \in E_{>d}$. Thus, $(h, f) \mapsto f \circ h$ gives a mapping $J_d \times (A/E_{>d}) \to A/E_{>d}$, as required.

5.9. REMARK. In the case of ordinary homogeneity, even the group of $(d-1)$-jets of diffeomorphisms acts on the space of d-jets (all this in our notation). This fact has an analogue in the quasihomogeneous case in the action of the group of $(d - \min \alpha_s)$-jets.

§6. Quasihomogeneous vector fields

The infinitesimal analogues of the concepts we have introduced run as follows.

6.1. DEFINITION. A formal vector field $\mathrm{v} = \sum v_i \partial/\partial x_i$ *has filtration d* if the directional derivative of v raises the filtration by not less than

d: $L_{\mathrm{v}} E_\lambda \subset E_{\lambda+d}$. We denote the set of all vector fields of filtration d by g_d. Our filtration in the module of vector fields (that is, derivation of A) is compatible with the filtration of the ring:

$$a \in E_d,\ \mathrm{v} \in \mathfrak{g}_\delta \Rightarrow a\mathrm{v} \in \mathfrak{g}_{d+\delta},\ L_{\mathrm{v}} a \in E_{d+\delta}.$$

6.2. PROPOSITION. *Suppose that $d \geqslant 0$. Then* 1) *the Poisson bracket of vector fields defines on $\mathfrak{g}_d$ a Lie algebra structure*; 2) *the Poisson bracket of elements $\mathfrak{g}_{d_1}$ and $\mathfrak{g}_{d_2}$ lies in $\mathfrak{g}_{d_1+d_2}$ so that each $\mathfrak{g}_d$ is an ideal in the Lie algebra $\mathfrak{g}_0$.*

PROOF. If $f \in E_\lambda$, $\mathrm{v}_1 \in \mathfrak{g}_{d_1}$, $\mathrm{v}_2 \in \mathfrak{g}_{d_2}$ then $(L_{\mathrm{v}_1} L_{\mathrm{v}_2} - L_{\mathrm{v}_2} L_{\mathrm{v}_1}) f \in E_{\lambda+d_1+d_2}$, as required.

The filtration of a vector field is connected with the filtrations of its components in the following manner.

6.3. PROPOSITION. *The field $\mathrm{v} = \Sigma v_i \partial/\partial x_i$ has filtration d (and type α) if and only if each component v_i is a function (series) of filtration $d + \alpha_i$.*

For the proof we introduce the following notation. A vector field of the form $\mathrm{x}^{\mathrm{k}} \partial/\partial x_i$ is called a *vector monomial.* The *degree* of such a vector monomial (for given quasihomogeneous type) is the (possibly negative) rational number $\langle \mathrm{k}, \alpha \rangle - \alpha_i = \langle \mathrm{k} - 1_i, \alpha \rangle$ in the arithmetical progression containing the degrees of the ordinary monomials. A vector field is said to be *homogeneous of degree d* if all vector monomials occurring in it with non-zero coefficients are of degree d.

PROOF OF PROPOSITION 6.3. If $\mathrm{v} \in \mathfrak{g}_d$, then $v_i = L_{\mathrm{v}} x_i \in E_{d+\alpha_i}$, since $x_i \in E_{\alpha_i}$. Let $v_i = \Sigma\, \mathrm{v}_{i,\mathrm{k}} \mathrm{x}^{\mathrm{k}}$. For every monomial $f = \mathrm{x}^{\mathrm{l}}$ we have

$$L_{\mathrm{v}} f = \sum v_i \frac{\partial f}{\partial x_i} = \sum l_i v_{i,\mathrm{k}} \mathrm{x}^{\mathrm{l}+\mathrm{k}-1_i}.$$

Here, $\langle \mathrm{k}, \alpha \rangle \geqslant d + \alpha_i$ if $v_i \in E_{d+\alpha_i}$. Thus, $\langle \mathrm{l} + \mathrm{k} - 1_i, \alpha \rangle \geqslant \langle \mathrm{l}, \alpha \rangle + d$, that is, $L_{\mathrm{v}} E_\lambda \subset E_{\lambda+d}$, as required.

We obtain immediately from Proposition 5.5:

6.4. COROLLARY. *The Lie algebra $\mathfrak{j}_d$ of the group J_d of d-jets of diffeomorphisms is the factor algebra $\mathfrak{j}_d = \mathfrak{g}_0/\mathfrak{g}_{-d}$. The mapping $\pi_{p,q}$: $J_p \to J_q$ induces a homomorphism $\pi_{p,q*}$: $\mathfrak{j}_p \to \mathfrak{j}_q$ of Lie algebras. The kernel of the mapping of $\mathfrak{j}_p$ into the algebra of jets that immediately precedes it in filtration is commutative.*

Finally, from Proposition 5.7:

6.5. COROLLARY. *The quasihomogeneous vector fields of degree* 0 *form a finite-dimensional Lie subalgebra $\mathfrak{h}$ of the Lie algebra of all vector fields. The Lie algebra $\mathfrak{h}$ is naturally isomorphic to the Lie algebra of the group j_0 of* 0*-jets of diffeomorphisms.*

In what follows we sometimes identify the vector field v with the set of n functions (or series) v_i. The next two propositions are used later in the reduction of semiquasihomogeneous functions to normal form.

6.6. LEMMA. *Let F be a power series of filtration d, and let $\mathbf{v}$ be a formal vector field of positive filtration δ. Then the Taylor series*

$$F(\mathbf{x}+\mathbf{v}(\mathbf{x}))=F(\mathbf{x})+\frac{\partial F}{\partial \mathbf{x}}\mathbf{v}+R$$

has remainder term R of filtration strictly greater than $d+\delta$.

PROOF. In view of the linearity of R relative to F, it is enough to show this for the case when F is a monomial. Let $F=\mathbf{x}^{\mathbf{k}}$, $\mathbf{v}=\Sigma v_i\partial/\partial x_i$. We consider the term of the Taylor series containing $\partial^{|m|}F/\partial\mathbf{x}^{\mathbf{m}}$ $(\mathbf{m}=(m_1,\ldots,m_n))$. The monomials occurring in this term have exponents $\mathbf{p}=\mathbf{k}-\mathbf{m}+\sum_{i=1}^{n}\mathbf{l}_i$, where $\mathbf{l}_i$ is the exponent of one of the monomial functions $v_i^{m_i}$. Thus, $\mathbf{l}_i=\sum_{j=1}^{m_i}\mathbf{l}_{i,j}$, where $\mathbf{l}_{i,j}$ is one of the exponents $\mathbf{l}$ in the decomposition $v_i=\Sigma v_{i,\mathbf{l}}\mathbf{x}^{\mathbf{l}}$. So

$$\mathbf{p}=\mathbf{k}+\sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathbf{l}_{i,j}-\mathbf{1}_i).$$

But $\langle\mathbf{k},\alpha\rangle\geqslant d$, $\langle\mathbf{l}_{ij}-\mathbf{1}_i,\alpha\rangle\geqslant\delta>0$, by assumption. Therefore, $\langle\mathbf{p},\alpha\rangle\geqslant d+|\mathbf{m}|\delta$, where $|\mathbf{m}|=m_1+\ldots+m_n$. Hence all monomials occurring in terms of degree higher than 1 in the Taylor series relative to v have filtration not less than $d+2\delta$, as required.

6.7. COROLLARY. *Suppose that $F=F_0+F_1+F_2$, where $F_0\in F_d$, $F_1\in E_{>d}$, $F_2\in E_{>d+\delta}$: $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$, where $v_0\in\mathfrak{g}_\delta$, $v_1\in\mathfrak{g}_{>\delta}$, $\delta>0$. Then*

$$F(\mathbf{x}+\mathbf{v}(\mathbf{x}))=F_0(\mathbf{x})+\left[F_1(\mathbf{x})+\frac{\partial F_0}{\partial\mathbf{x}}\mathbf{v}_0\right]+R',\quad R'\in E_{>d+\delta}.$$

PROOF. We set $F_0+F_1=F'$. We have $R'=R_1+R_2+R_3+R_4$, where

$$R_1=F'(\mathbf{x}+\mathbf{v}(\mathbf{x}))-F'(\mathbf{x})-\frac{\partial F'}{\partial\mathbf{x}}\mathbf{v}\in E_{>d+\delta}$$

(Lemma 6.6),

$$R_2=\frac{\partial F_0}{\partial\mathbf{x}}\mathbf{v}_1=L_{\mathbf{v}_1}F_0\in E_{>d+\delta},$$

$$R_3=\frac{\partial F_1}{\partial\mathbf{x}}\mathbf{v}=L_{\mathbf{v}}F_1\in E_{>d+\delta},$$

$$R_4=F_2(\mathbf{x}+\mathbf{v}(\mathbf{x}))\in E_{>d+\delta},$$

as required.

§7. The normal form of a semiquasihomogeneous function

We consider the local ring of a quasihomogeneous or semiquasihomogeneous function f of degree d. We fix a system of monomials forming a basis for this ring.

7.1. DEFINITION. A monomial is said to be *upper* or *to lie above the diagonal* (or *lower*, or *diagonal*) *if it has degree greater than* d (or less than d, or equal to d) for given quasihomogeneous exponents.

Note that the *number of upper, diagonal, and lower basis monomials does not depend on the basis of the local ring* (see Remark 3.5).

Let $e_1, \ldots, e_s$ be the system of all upper basis monomials in a fixed basis of the local ring of the function f_0.

7.2. THEOREM. *Every semiquasihomogeneous function with quasihomogeneous part f_0 is equivalent to a function of the form $f_0 + \Sigma c_k e_k$, where the c_k are constants.*

The proof of Theorem 7.2 is obtained on application of the following lemma.

7.3. LEMMA. *Let f_0 be a quasihomogeneous function of degree d and $e_1, \ldots, e_r$ the set of all basis monomials of fixed degree $d' > d$ in the local ring of f_0. Then every series of the form $f_0 + f_1$, where the filtration of f_1 is greater than d, can be brought by a formal diffeomorphism to the form $f_0 + f_1'$, where the terms in f_1' of degree less than d' are the same as in f_1, and the terms of degree d' reduce to $c_1 e_1 + \ldots + c_r e_r$.*

PROOF. Let g denote the sum of the terms of degree d' in f_1. There exists a decompostion (if convenient, as far as the terms of filtration higher than d', but certainly without them)

$$g = \sum \frac{\partial f_0}{\partial x_i} v_i(\mathbf{x}) + c_1 e_1 + \ldots + c_r e_r,$$

since $e_1, \ldots, e_r$ are basis monomials. The vector field $\mathbf{v}$ occurring in this formula can be replaced by a homogeneous one of degree $\delta = d' - d > 0$ without invalidating the formula (to prove this it is enough to decompose $\mathbf{v}$ into its homogeneous parts).

We consider now the formal substitution $\mathbf{x} = \mathbf{y} - \mathbf{v}(\mathbf{y})$, where $\mathbf{v}$ is the vector field with components v_i as defined above. We claim that this is a formal diffeomorphism. For the field $\mathbf{v}$ has positive degree δ, so that if the coordinates are numbered according to decreasing exponents α_i, the Jacobian matrix of the substitution at 0 is unitriangular. Applying Corollary 6.7, we find that

$$f(\mathbf{y} - \mathbf{v}(\mathbf{y})) = f_0(\mathbf{y}) + [f_1(\mathbf{y}) + (c_1 e_1(\mathbf{y}) + \ldots + c_r e_r(\mathbf{y})) - g(\mathbf{y})] + R'(\mathbf{y})$$

(in the old notation). Since the filtration of R' is greater than d', this proves Lemma 7.3.

PROOF OF THEOREM 7.2. Applying Lemma 7.3 to the function f_0 and the monomials next to those of highest degree d', we come to the desired form of the term of degree d'. Applying the same lemma to the series $f_0 + f_1'$ so obtained and the monomials following those of degree d', we come to the desired form of the term of degree d' without changing the terms of degree d and d'. Continuing in this way, we obtain the desired normal form up to terms of degree as high as required (and even, if convenient, *we can reduce the formal series completely to formal normal form by a formal diffeomorphism*; this follows from the fact that the degrees of fields v that arise at various stages grow).

Up to this moment we have not used anywhere the finiteness of the multiplicity μ, so that the formal assertion has been proved without this assumption. If μ is finite, then a fairly long section of the Taylor series (of length bounded in terms of μ; see, for example, [5]) of the function is equivalent to the function itself, so that reduction to normal form is realized by a genuine diffeomorphism.

§8. The normal form of a quasihomogeneous function

Let f_0 be a non-degenerate quasihomogeneous function. We consider the linear space E of all quasihomogeneous functions of the same type and degree of quasihomogeneity as f_0.

8.1. THEOREM. *Every quasihomogeneous function of the type and degree of f_0 and sufficiently near to f_0 is equivalent to a function of the form $f_0 + c_1e_1 + \ldots + c_re_r$, where $e_1, \ldots, e_r$ is the collection of all diagonal basis monomials of the local ring of f_0.*

PROOF. We consider a diffeomorphism of the form $\mathrm{x} \to \mathrm{x} + \varphi(\mathrm{x})$, where φ is a quasihomogeneous vector field of degree 0. All such diffeomorphisms form a Lie group. This group acts (linearly) on E. An orbit through f_0 and one through any nearby point have one and the same dimension (see Remark 3.2 and Lemma 3.4). They intersect the plane $f_0 + \mathrm{C}e_1 + \ldots + \mathrm{C}e_r$ transversally at the point f_0 and neighbouring points. Thus, the union of the orbits intersecting this plane near f_0 contains a neighbourhood of f_0 in the whole space E, as we wanted to show.

8.2. EXAMPLE. We show that *every non-degenerate binary form of degree $n \geqslant 4$ reduces to a normal form with $n - 3$ parameters*:

$$x^{n-1}y + c_1x^{n-2}y^2 + \ldots + c_{n-3}x^2y^{n-2} + xy^{n-1}.$$

For the non-degeneracy of this form yields the existence of two simple linear factors, which can be taken as the coordinates x and y. Then the terms x^n and y^n are absent from the expression of the form; after this we make the coefficients of $x^{n-1}y$ and xy^{n-1} equal to 1. In this way the form reduces to the one above.

It is easy to check that the monomials $x^{n-2}y^2, \ldots, x^2y^{n-2}$ form a diagonal basis for every form of the type indicated (even in the degenerate case).

8.3. PROPOSITION. *A non-degenerate binary form of degree 4 can be reduced by a linear transformation to the (Legendre) normal form X_9: $x^4 + ax^2y^2 + y^4$, $a^2 \neq 4$.*

PROOF. By Proposition 8.2, the form reduces to $xy(x^2 + 2cxy + y^2)$. This is the product of the quadratic forms $a = xy$, $b = x^2 + 2cxy + y^2$. There are two *independent* degenerate forms among the linear combinations $pa + qb$ of a and b ($p{:}\,q$ is defined from the characteristic equation $p^2 + 2cpq + (c^2 - 1)q^2 = 0$, whose discriminant is not 0). Taking the square roots of these degenerate forms as coordinates, we reduce the original form to $c_1x^4 + c_2x^2y^2 + c_3y^4$. The condition for such a form to be non-degenerate is $c_1c_3(4c_1c_3 - c_2^2) \neq 0$. By a magnification of coordinates we reduce the form to that indicated in Proposition 8.3, which is thereby proved.

8.4. PROPOSITION. *A non-degenerate quasihomogeneous function of degree 1 with quasihomogeneous exponents 1/3, 1/6 can be reduced by a quasihomogeneous diffeomorphism to the normal form J_{10}: $x^3 + ax^2y^2 + y^6$, where $4a^3 + 27 \neq 0$.*

PROOF. By definition, the function has the form $c_1x^3 + c_2x^2y^2 + c_3xy^4 + c_4y^6$. From the non-degeneracy it follows that $c_1 \neq 0$. By changing x to $x + \lambda y^2$ we can achieve $c_3 = 0$. Then $c_4 \neq 0$ as a consequence of non-degeneracy, and after a magnification of coordinates the form reduces to that in Proposition 8.4, which is thereby proved.

8.5. PROPOSITION. *Every non-degenerate quasihomogeneous function of degree 1 with exponents* $\left(\frac{1}{3}, \frac{1}{3k}\right)$, $\left(\frac{k}{3k+1}, \frac{1}{3k+1}\right)$, $\left(\frac{1}{4}, \frac{1}{2k}\right)$, $\left(\frac{k}{4k+2}, \frac{1}{2k+1}\right)$ *can be reduced by a quasihomogeneous diffeomorphism to one of the respective normal forms:*

$$x^3 + ax^2y^k + y^{3k}, \quad x^3y + ax^2y^{k+1} + y^{3k+1} \quad (4a^3 + 27 \neq 0);$$
$$x^4 + ax^2y^k + y^{2k}, \quad x^4y + ax^2y^{k+1} + y^{2k+1} \quad (a^2 - 4 \neq 0).$$

The proof repeats the calculations in Propositions 8.3 and 8.4.

8.6. DEFINITION. The *inner modality* of a quasihomogeneous function is the total number of diagonal and superdiagonal monomials in some (and thus in any) monomial basis of the local ring.

8.7. THEOREM. *The modality of any semiquasihomogeneous function (in particular, of a non-degenerate quasihomogeneous function) is not less than the inner modality of its quasihomogeneous part.*

For suppose that $f = f_0 + f_1$ is the function in question and that $e_1, \ldots, e_r$ is the set of all diagonal and superdiagonal monomials in any basis of the local ring. Then the same monomials form the diagonal and superdiagonal part of a basis of the local ring of each of the functions $f + c_1e_1 + \ldots + c_re_r$, at least for sufficiently small c (see Corollary 3.3 and Lemma 3.4). Thus, there exists an s-dimensional plane through f in the space of functions with critical point and critical value 0 having the property that all non-zero tangent vectors to the orbits of points of the plane near f do not lie in the plane. Thus, the modality of f is not less than s.

8.8. REMARK. Gabrielov has shown that the modality is equal to the dimension of the submanifold of a base of versal deformations along which μ does not change. In all the examples of semiquasihomogeneous functions I know this dimension is the total number of diagonal and superdiagonal basis monomials.

§9. Piecewise filtrations

It often turns out to be useful to consider filtrations in which the rôle of the diagonal is played by a Newton open polygon (or, in the case of several dimensions, by a polyhedron convex towards 0). The formal definition is as follows.

Let $\alpha_1, \ldots, \alpha_p$ be a fixed collection of p quasihomogeneous types. We recall that the monomial $\mathbf{x}^{\mathbf{k}}$ is of degree $\langle \alpha_i, \mathbf{k} \rangle = \varphi_i(\mathbf{k})$ in the i-th filtration. We define the *piecewise degree* of $\mathbf{x}^{\mathbf{k}}$ to be $\varphi(\mathbf{k}) = \min[\varphi_i(\mathbf{k}), \ldots, \varphi_p(\mathbf{k})]$.

9.1. DEFINITION. A power series has *piecewise filtration d* if all its monomials have piecewise degree d or higher.

Note that the equation $\varphi(\mathbf{k}) = 1$ defines a hypersurface Γ in the space of exponents $\mathbf{k}$ that is convex towards 0. We call Γ an *open polyhedron*. In these terms we can say that a monomial has (piecewise) degree d if and only if its exponent vector lies on the open polyhedron $d\Gamma$ obtained from Γ by a homothety with the coefficient d. In exactly the same way a series has (piecewise) filtration d if the exponent vectors of all its monomials lie on or outside $d\Gamma$.

The sum of the terms of lowest (piecewise) degree in a given power series is called the *principal part* of the series. A (piecewise) *homogeneous function of degree d* is a polynomial whose monomials all have (piecewise) degree d.

Analogous concepts are defined for vector fields; the degree of the monomial $\mathbf{x}^{\mathbf{l}}\partial/\partial x_i$ is defined to be

$$\varphi(\mathbf{l}-\mathbf{1}_i) = \min_{1\leqslant j\leqslant p} \langle \alpha_j, \mathbf{l}-\mathbf{1}_j \rangle.$$

Note that for all functions f, g and every vector field $\mathbf{v}$ we have[1]

filtration of $fg \geqslant$ filtration of f + filtration of g,

filtration of $\sum \frac{\partial f}{\partial x_i} v_i \geqslant$ filtration of f + filtration of $\mathbf{v}$.

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[1] The filtration φ is naturally connected with the filtration arising naturally from the Koszul complex constructed from the derivative of a piecewise-homogeneous function; this was pointed out to me by Kushirenko.

The *group of diffeomorphisms of filtration d, the group of d-jets* of diffeomorphisms and the corresponding Lie algebras are defined just as in the case of quasihomogeneous filtrations. There is no analogue for piecewise filtrations, except for the group of quasihomogeneous diffeomorphisms.

9.2. DEFINITION. A piecewise-homogeneous function f_0 of degree d *satisfies condition A* if for every function g of filtration $d + \delta > d$ in the ideal spanned by the derivatives of f_0 there is a decomposition

$$g = \sum \frac{\partial f_0}{\partial x_i} v_i + g',$$

where the vector field $\mathbf{v}$ has filtration δ, and the function g' has filtration greater than $d + \delta$.

Note that a quasihomogeneous function always satisfies condition A.

We consider a basis of the local ring of a piecewise-homogeneous function f_0 of finite multiplicity μ.

9.3. DEFINITION. A basis $e_1, \ldots, e_\mu$ of homogeneous elements is said to be *regular* if, for each D, the elements of the basis of degree D are independent modulo the sum of the ideal $I = (\partial f/\partial x_0)$ and the space $E_{>D}$ of functions of filtrations greater than D.

9.4. PROPOSITION. *There always exists a regular basis, in fact, one consisting entirely of monomials.*

PROOF. The monomials whose exponent vectors lie on $D\Gamma$ generate $E_D \bmod E_{>D}$. Thus, their images in $E_D/(E_D \cap I) + E_{>D}$ generate this linear space, hence a basis for the factor space can be extracted from their images. The inverse images of the basis vectors so chosen are monomials in E_D, and we include these monomials in a basis of the local ring.

For large enough D we have $E_D \subset I$ (since $\mu < \infty$). Therefore, the system of monomials constructed is finite. It is clear from the construction that every vector in A is representable as a linear combination of the chosen monomials and of elements of this ideal. Finally, if the least degree of a monomial occurring in a relation $c_1 e_1 + \ldots \in I$ with non-zero coefficient is $D < \infty$, then the images of monomials e_i of degree D in the factor space $E_D/(E_D \cap I) + E_{>D}$ would be dependent, against the choice of the e_i. Thus, $\{e_i\}$ is a basis of the local ring, as required.

The number of elements in a regular monomial basis having given (piecewise) homogeneous degree does not depend on the choice of a basis of the local ring. A monomial in a regular basis is said to be *diagonal* (*superdiagonal*) if its degree is equal to (greater than) the degree of the function f_0 under discussion.

9.5. THEOREM. *If the principal part f_0 of a function f satisfies condition A and has finite multiplicity μ, then f can be reduced by a diffeomorphism to the form $f_0 + c_1 e_1 + \ldots + c_s e_s$, where $e_1, \ldots, e_s$ are the superdiagonal monomials in a regular basis.*

The proof of Theorem 9.5 repeats that of Theorem 7.2.

9.6. EXAMPLE. *We consider the function $f_0 = x^a + \lambda x^2 y^2 + y^b$, where*