$a \geqslant 4$, $b \geqslant 5$, $\lambda \neq 0$. *We claim that* 1) $\mu = a + b + 1$; 2) *the system* $1, x, \ldots, x^{a-1}, y, \ldots, y^b, xy$ *of monomials is a regular basis*; 3) *conditi A is satisfied by the filtration given by the open polygon* Γ *with vertices* $(a, 0)$, $(2, 2)$, $(0, b)$.

For the proof of all these assertions it is useful to carry out certain geometrical constructions in the plane of exponents. These constructions reduce the analysis of the local ring to a sequence of geometrical operations reminis cent of the solution of crossword puzzles. This "crossword solution" technique is applicable not only in this example, and we explain it to a somewhat wider extent than is necessary for the analysis of our example.

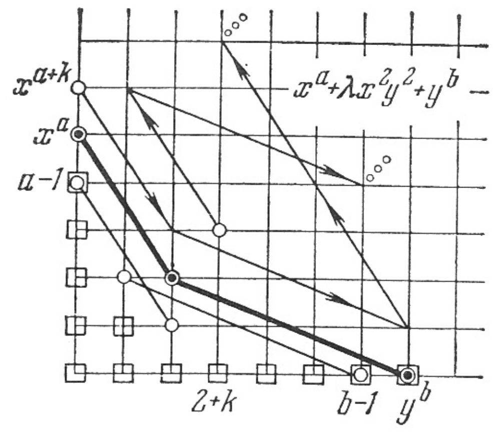

Fig. 1.

So let us assume that each partial deriva tive of the given function contains not mo than two monomials (which is obviously satisfied by f_0). We join the monomials of the partial derivative with respect to x by segment, which we call a *fundamental x-segment.* Similar fundamental segments are defined for the other variables. In our example there are two fundamental segments parallel to the open polygon Γ (Fig. 1).

The fundamental segments depict relationships between the images of their monomials in the local ring. We consider some consequences of these relatio ships. We call every translate of a fundamental segment by an integral vector with non-negative components a *permissible x-segment*. We note tha two permissible segments may lie one on the other and even coincide geometrically (if one is an x-segment and the other a y-segment). But this doe not happen in our example.

Two permissible segments are said to be *joined* if they have a common end. A *permissible chain* is a collection of permissible segments such that every one of the segments is joined to every other by a sequence of consecutive joined segments in the collection. A chain is said to be *maximal* if it is not part of any larger permissible chain.

A *cycle* is a finite sequence of consecutive joined permissible segments ir which the last segment is joined to the first. A cycle is said to be *regular* if in its journey along the x-(y-)segments each direction is traversed the same number of times as every other. (In our example cycles are trivial, but if we consider the case $a = b = 4$, we get a non-trivial cycle $(1, 3) \to (3, 1) \to (1, 3)$ with one x-segment and one y-segment.)

We can now formulate the "crossword solution" rule for a function f with a critical point of finite multiplicity.

9.7. PROPOSITION. 1. *If the exponent of a monomial lies in an infinite permissible chain, then it lies in the ideal generated by the partial* *derivatives of the function.*

2. *If all cycles are regular, then the dimension μ of the local ring is equal to the number of maximal cycles. In this case we can find a basis of the local ring by taking an (arbitrary) monomial from each of the maximal chains.*

3. *Moreover, given a filtration, we obtain a regular basis by choosing a monomial of highest filtration from each of the maximal chains.*

The proof of Proposition 9.7 follows immediately from the definitions.

9.8. PROPOSITION. *The maximal permissible chains in Example 9.6 are as follows*:

1) *each of the points* $1, x, \ldots, x^{a-2}, y, \ldots, y^{b-2}, xy$ *has an empty permissible chain*;

2) *there are three finite maximal permissible chains* $x^{a-1} \to xy^2$, $x^2y \to y^{b-1}$, $x^a \to x^2y^2 \to y^b$;

3) *each of the remaining points is the beginning of an infinite permissible chain.*

It is clear from Fig. 1 that the permissible segment $x^py^q \to y^{p-2}y^{q-1+b}$ raises the filtration for $p > q \geqslant 1$, while the segment $x^py^q \to x^{p-1+a}y^{q-2}$ raises the filtration for $q > p \geqslant 1$. The filtrations of monomials with $p = 0$ or $q = 0$ or $p = q$ rises in two steps:

$$x^{a+k} \to x^{2+k}y^2 \to x^ky^{b+1},\quad x^{2+k}y^{2+k} \to x^{k+a-1}y^{k+b-1}.$$

Consequently, apart from those described in 1) and 2), every maximal permissible chain of every monomial contains monomials of arbitrarily high degree and so is infinite.

Thus, the monomials figuring in 9.6 form a regular basis, and $\mu = a + b + 1$. To check condition A it is enough to compute the filtrations of the coefficients of the relations constructed above that describe the permissible chains. This calculation is not complicated and we omit it.

Therefore, condition A is satisfied, and the number of basis monomials of a regular basis above Γ turns out to be 0. Thus, from Theorem 9.5 we derive:

9.9. COROLLARY. *Every function f with principal part $f_0 = x^a + \lambda x^2y^2 + y^b$, where $\lambda \neq 0$, $a \geqslant 4$, $b \geqslant 5$, is equivalent to its principal part.*

REMARK. The following rule for calculating the modality of functions of two variables with a non-degenerate Newton diagram is corroborated by a large collection of diverse examples, though it has not been proved in general as yet.

From the point (2, 2) of the plane of exponents, we draw horizontal and vertical rays in the direction of increasing exponents and consider the polygon bounded by the segments of these rays in the Newton open polygon. *The modality of the function is the number of integral points inside and on the boundary of the polygon.* The non-degeneracy condition can be stated here as follows. The coefficients of the monomials corresponding to the links in the Newton open polygon must be such that the polynomial obtained on adding them is the product of a monomial and a non-degenerate quasihomogeneous function.

Kushirenko has proved that the inner modality (that is, the number of monomials in a regular basis on the Newton open polygon and above) of a function of two variables with a non-degenerate Newton diagram is, in fact, equal to the number of integral points in the polygon described above.

9.10. CONDITION FOR SUFFICIENCY OF A JET. *Suppose that, in additio, to the conditions of Theorem 9.5, all monomials of degree $s + 1$ lie in the ideal $(\partial f_0/\partial x_i)$ and are superdiagonal. Then the s-jet of f is sufficient.*

This is because the algorithm for reducing to the normal form as indicated in the proof of Theorem 7.2 does not alter the coefficients of superdiagonal monomials, no matter how we alter the terms of degree $s + 1$ and above in the series for f.

In particular, the b-jets in Example 8.6 are sufficient for $b \geqslant a$.

§ 10. Semiquasihomogeneous functions of two variables

In this section and the next we give tables in which we describe generators for local rings, normal forms of semiquasihomogeneous singularities, and sufficient jets for the simplest quasihomogeneous singularities of degree 1. The tables include all inner unimodal singularities and the first non-unimodal ones.

The non-degenerate quasihomogeneous functions of two variables fall into three (intersecting) classes. We recall that a function of two variables of corank 1 is a function whose second differential at zero is identically 0.

10.1. PROPOSITION. *Every non-degenerate quasihomogeneous function of corank 2 of two variables x, y can contain only the following monomial with non-zero coefficients: x^a and y^b, or x^a and xy^b, or $x^a y$ and y^b or $x^a y$ and $y^b x$.*

PROOF. Otherwise the function would be divisible by x^2 or y^2, and 0 would not be an isolated critical point.

We recall that all the exponents of the monomials of a quasihomogeneou function of degree 1 lie on a line Γ, the so-called diagonal (given by the equation $\langle \mathbf{k}, \boldsymbol{\alpha} \rangle = 1$).

10.2. THEOREM. *We assume that exactly two monomials of a non-degenerate function of two variables lie on the diagonal Γ. Then the follow ing system of monomials forms a basis for the local ring:*

f	$\alpha_1,\ \alpha_2$	μ	Basis monomials $x^k y^l$
x^a+y^b	$\frac{1}{a},\ \frac{1}{b}$	$(a-1)(b-1)$	$0\leqslant k\leqslant a-2,\ 0\leqslant l\leqslant b-2$
$x^a y+y^b$	$\frac{b-1}{ab},\ \frac{1}{b}$	$(a-1)b+1$	$0\leqslant k\leqslant a-2,\ \ 0\leqslant l\leqslant b-1;\ \ x^{a-1}$
$x^a y+y^b x$	$\frac{b-1}{ab-1},\ \frac{a-1}{ab-1}$	ab	$0\leqslant k\leqslant a-1,\ \ 0\leqslant l\leqslant b-1$

(*a function f can be reduced to the form shown in the table by magnification and renumbering of coordinates*).

PROOF. Consider Fig. 2. In this figure the domain of monomials contanined in the ideal (f_x, f_y) is hatched-in. The thin slanting lines

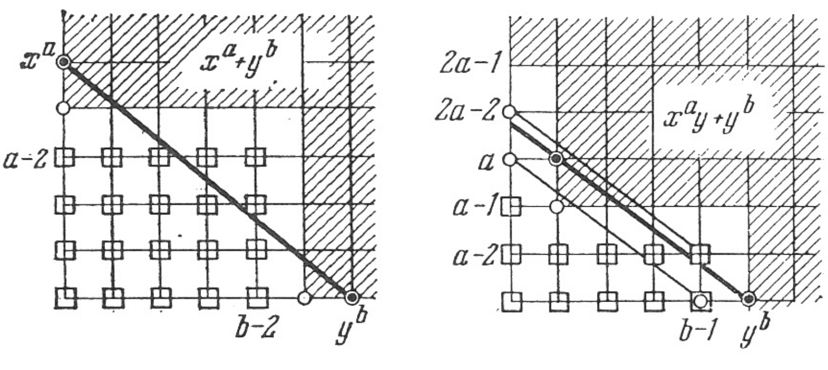

Fig. 2.

are permissible chains (see 9.6). It is clear from the figure that every monomial not contained in the ideal is joined to a monomial in the table by a permissible path. This means that the monomials in the table generate the local ring. But they are equal in number to the dimension of the local ring as calculated in 4.8. Thus they form a basis, as asserted.

By Theorem 7.2, we derive from Theorem 10.2 the normal forms for semiquasihomogeneous singularities with the quasihomogeneous parts shown in the table, and also a formula for their versal deformations. These formulae allow us to extract explicitly the characteristic values of the classical monodromy operator (see [10]).

We recall that we have called the total number μ_0 of diagonal and superdiagonal basis monomials of the local ring the *inner modality*.

10.3. THEOREM. 1) *The quasihomogeneous functions of two variables with $\mu_0 = 0$ are exhausted (up to equivalence) by the following list:*

Type	Normal form	v_1, v_2; N	Basis monomials and their weights
A_k	$x^{k+1}+y^2$	$2,\ k+1;\ 2k+2$	$1,\ x,\ \ldots,\ x^{k-1}$ $0,\ 2,\ \ldots,\ 2k-2$
D_k	x^2y+y^{k-1}	$k-2,\ 2;\ 2k-2$	$1,\ y,\ \ldots,\ y^{k-2},\ x$ $0,\ 2,\ \ldots,\ 2k-4,\ k-2$
E_6	x^3+y^4	$4,\ 3;\ 12$	$1,\ y,\ x,\ y^2,\ xy,\ xy^2$ $0,\ 3,\ 4,\ 6,\ 7,\ 10$
E_7	x^3+xy^3	$6,\ 4;\ 18$	$1,\ y,\ x,\ y^2,\ xy,\ y^3,\ y^4$ $0,\ 4,\ 6,\ 8,\ 10,\ 12,\ 16$
E_8	x^3+y^5	$10,\ 6;\ 30$	$1,\ y,\ x,\ y^2,\ xy,\ y^3,\ xy^2,\ xy^3$ $0,\ 6,\ 10,\ 12,\ 16,\ 18,\ 22,\ 28$

All non-degenerate functions with quasihomogeneous exponents U_i/N as in the table can be reduced to the normal forms of the table.

2) *The quasihomogeneous functions of two variables with $\mu_0 = 1$ are exhausted (up to equivalence) by two one-parameter families and 8 individual functions as in the following table:*

Type	Normal form	$U_1, U_2; N$	Basis monomials and their weights
X_9	$x^4+ax^2y^2+y^4$, $a^2-4\neq 0$	1, 1; 4	$1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2$ 0, 1, 1, 2, 2, 2, 3, 3, 4
Z_{11}	x^3y+y^5	4, 3; 15	$1, y, x, y^2, xy, x^2, y^3, xy^2, y^4, xy^3, xy^4$ 0, 3, 4, 6, 7, 8, 9, 10, 12, 13, 16
Z_{12}	x^3y+xy^4	3, 2; 11	$1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, xy^3, x^2y^2, x^2y^3$ 0, 2, 3, 4, 5, 6, 6, 7, 8, 9, 10, 12
Z_{13}	x^3y+y^6	5, 3; 18	$1, y, x, y^2, xy, y^3, x^2, xy^2, y^4, xy^3, y^5, xy^4, xy^5$ 0, 3, 5, 6, 8, 9, 10, 11, 12, 14, 15, 17, 20
W_{12}	x^4+y^5	5, 4; 20	$1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, xy^3, x^2y^2, x^2y^3$ 0, 4, 5, 8, 9, 10, 12, 13, 14, 17, 18, 22
W_{13}	x^4+xy^4	4, 3; 16	$1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, y^4, x^2y^2, y^5, y^6$ 0, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18
J_{10}	$x^3+ax^2y^2+y^6$, $4a^3+27\neq 0$	2, 1; 6	$1, y, x, y^2, xy, y^3, xy^2, y^4, xy^3, xy^4$ 0, 1, 2, 2, 3, 3, 4, 4, 5, 6
K_{12}	x^3+y^7	7, 3; 21	$1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, xy^4, xy^5$ 0, 3, 6, 7, 9, 10, 12, 13, 15, 16, 19, 22
K_{13}	x^3+xy^5	5, 2; 15	$1, y, \ldots, y^8; x, xy, xy^2, xy^3$ 0, 2, ..., 16; 5, 7, 9, 11
K_{14}	x^3+y^8	8, 3; 24	$1, y, \ldots, y^6; x, xy, \ldots, xy^6$ 0, 3, ..., 18; 8, 11, ..., 26

All non-degenerate functions with quasihomogeneous exponents U_i/N as in the table can be reduced to the normal forms of the table.

In these tables, the lower index in the notation for functions is the value of μ. The numbers below the monomials are their weights, that is, the numerators of the generalized degrees (the monomial $x^p y^q$ is of degree $(U_1 p + U_2 q)/N$).

PROOF OF THE THEOREM. 1) If the second differential is not identically 0, then the function is equivalent to A_k (see [2], for example). If the corank of the function is 2, then the homogeneous exponents are given by the table in 10.2. By 4.9, the generalized degree of a basis monomial of highest degree is $d_{\max} = n - 2\Sigma\alpha_i = 2 - 2(\alpha_1 + \alpha_2)$. The condition

$d_{max} < 1$ is equivalent to $\mu_0 = 0$, and in each of the three cases of Proposition 10.1 it defines the domain under a hyperbola in the (a, b)-plane. Calculation of the integral points in these domains gives the series A, D, E.

What has been said becomes a little more understandable, perhaps, if we observe that the *classification of singularities with* $\mu_0 = 0$ *reduces to an enumeration of the lines in the plane passing below the point* (2, 2), *intersecting the coordinate axes at distances not less than* 2 *from* 0, *and having integral points in the positive octant with abscissae and ordinates not exceeding* 1.

This is because the condition $d_{max} < 1$ means that the diagonal Γ lies below (2, 2).

It is easy to check that such lines are exhausted by our list (apart from re-naming the axes), and that non-degenerate quasihomogeneous functions with quasihomogeneous exponents A, D or E reduce to the forms shown in the table.

The basis monomials are easy to find from the crosswords described in 9.6; the formula for μ_i from 4.5 is also useful.

2) Suppose that $\mu_0 = 1$. Then a basis monomial of highest generalized degree lies on the diagonal or above, and the last but one strictly below it. But by 4.11, the last and last but one of the (generalized) degrees

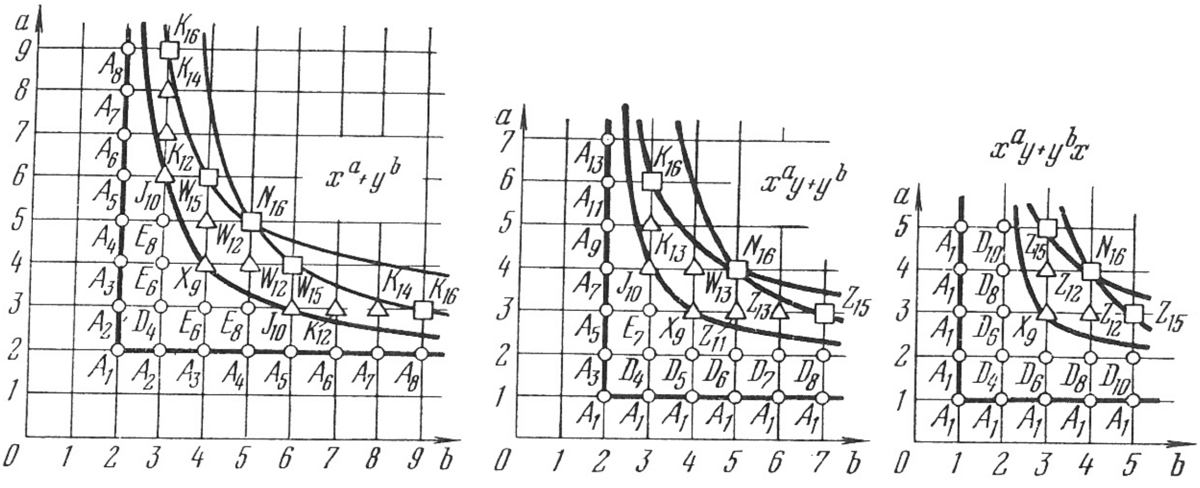

Fig. 3.

of the basis monomials are d_{max} and $d_{max} - \alpha_{min}$ (because the second degree is $\alpha_{min} = \min(\alpha_1, \alpha_2)$). Thus, the condition $\mu_0 = 1$ assumes the form

$$2\alpha_1 + 2\alpha_2 \leqslant 2, \; 3\alpha_1 + 2\alpha_2 > 2, \; 2\alpha_1 + 3\alpha_2 > 2.$$

Substituting the values of α_1 and α_2 from the table 10.2, we obtain in each of the three cases a domain between hyperbolae in the (a, b)-plane. The enumeration of the integral points in these domains (Fig. 3) gives the exponents of homogeneity shown in Table 10.3 ($X_9 - K_{14}$). For all these exponents there are exactly two integral points on the diagonal, apart from the cases X_9 and J_{10}. Magnification of coordinates reduces the function to a sum of monomials; the normal forms for X_9 and J_{10} are given in Propositions 8.3 and 8.4. The basis monomials are computed using 9.6, 4.5. This proves the theorem.

10.4. REMARK. The classification of singularities with $\mu_0 = 1$ comes to an enumeration of the lines Γ in the plane of exponents passing through the point (2, 2) or above it, but below (2, 3) and (3, 2).

We can also consider the boundary cases when Γ passes through one of the points (2, 3), (3, 2). The quasihomogeneous singularities corresponding to these boundary points are:

Type	Normal form	Non-degeneracy condition	U_1, U_2	N
Z_{15}	$x^3y+ax^2y^3+y^7$	$4a^3+27\neq 0$	2, 1	7
W_{15}	$x^4+ax^2y^3+y^6$	$a^2-4\neq 0$	3, 2	12
K_{16}	$x^3+ax^2y^3+y^9$	$4a^3+27\neq 0$	3, 1	9
N_{16}	$x^4y+ax^3y^2+bx^2y^3+y^4x$	$12(a^3+b^3)-36ab-5a^2b^2+81\neq 0$	1, 1	5

The following systems of monomials are examples of basis of the local rings:

Type	Basis monomials below the diagonal	On	Above
Z_{15}	$1, y, x, y^2, xy, y^3, x^2, xy^2, y^4, xy^3, y^5, xy^4, y^6$ 0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6	xy^5 7	xy^6 8
W_{15}	$1, y, x, y^2, xy, x^2, y^3, xy^2, x^2y, y^4, xy^3, x^2y^2, xy^4$ 0, 2, 3, 4, 5, 6, 6, 7, 8, 8, 9, 10, 11	x^2y^3 12	x^2y^4 14
K_{16}	$1, y, y^2, y^3, x, y^4, xy, y^5, xy^2, y^6, xy^3, y^7, xy^4, xy^5$ 0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8	xy^6 9	xy^7 10
N_{16}	$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3$ 0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4	x^2y^3, x^3y^2 5, 5	x^3y^3 6

In the case of N_{16}, the system of monomials written down forms a basis only when $ab\neq 9$. If we replace the last monomial (x^3y^3) by x^4y^2, the resulting system determines a basis if $35a+9b^2\neq 0$. In the cases Z_{15}, W_{15}, K_{16} one and the same system works for all values of the parameters for which the function remains non-degenerate.

10.5. COROLLARY. *Every semiquasihomogeneous function of inner modality $\mu_0 = 1$ (that is, having as principal part one of the types $X_9 - K_{14}$ or with quasihomogeneous exponents as in Table 10.3 can be reduced by a diffeomorphism to a normal form of one of the following types:*

Type	Normal form	Non-degeneracy condition	s	Type	Normal form	Non-degeneracy condition	s
X_9	$x^4+ax^2y^2+y^4$	$a^2-4\neq 0$	4	W_{13}	$x^4+xy^4+ay^6$	$a\in \mathbf{C}$	6
Z_{11}	$x^3y+y^5+axy^4$	$a\in \mathbf{C}$	5	J_{10}	$x^3+ax^2y^2+y^6$	$4a^3+27\neq 0$	6
Z_{12}	$x^3y+xy^4+ax^2y^3$	$a\in \mathbf{C}$	6	K_{12}	$x^3+y^7+axy^5$	$a\in \mathbf{C}$	7
Z_{13}	$x^3y+y^6+axy^5$	$a\in \mathbf{C}$	6	K_{13}	$x^3+xy^5+ay^8$	$a\in \mathbf{C}$	8
W_{12}	$x^4+y^5+ax^2y^3$	$a\in \mathbf{C}$	5	K_{14}	$x^3+y^8+axy^6$	$a\in \mathbf{C}$	8

Bases for the local rings of these singularities are the same as for the quasihomogeneous singularities of Theorem 10.3. The number s in the table denotes the (ordinary) degree of a sufficient jet.

10.6. COROLLARY. *Semiquasihomogeneous functions with homogeneous exponents Z_{15}, W_{15}, K_{16}, N_{16} (see 10.4) can be reduced to the following forms:*

Type	Normal form	Non-degeneracy condition	s
Z_{15}	$x^3y+ax^2y^3+y^7+bxy^6$	$4a^3+27\neq 0$	8
W_{15}	$x^4+ax^2y^3+y^6+bx^2y^4$	$a^2-4\neq 0$	7
K_{16}	$x^3+ax^2y^3+y^9+bxy^7$	$4a^3+27\neq 0$	10
N_{16}	$x^4y+ax^3y^2+bx^2y^3+xy^4+cx^3y^3$	$12(a^3+b^3)+81\neq 36ab+5a^2b^2$	6

In the case N_{16} it is assumed that $ab\neq 9$. If $ab = 9$, but $35a+9b^2\neq 0$, then cx^3y^3 must be replaced by cx^4y^2.

10.7. COROLLARY. *Each of the singularities in the list 10.5 has modality not less than 1. The singularities Z_{15}, W_{15}, K_{16} have modality not less than 2, and N_{16} not less than 3.*

For the modalities of all these singularities are exactly these values, but we do not prove this here.

§ 11. Quasihomogeneous functions of three variables

The non-degenerate quasihomogeneous functions of three variables fall into seven (intersecting) classes. The classification has been considered by Orlik and Wagreich [12], but they missed two classes (III and VI).

11.1. PROPOSITION. *Every non-degenerate quasihomogeneous function of three-variables of degree* 1 *and corank* 3 *contains at least one of the seven systems of monomials in the following tables with non-zero coefficients* (*with a suitable numbering of the variables*):

Class	Monomials	α_1,	α_2,	α_3	μ
I	$x^a,\ y^b,\ z^c$	$\frac{1}{a}$,	$\frac{1}{b}$,	$\frac{1}{c}$	$(a-1)(b-1)(c-1)$
II	$x^a,\ y^b,\ z^c y$	$\frac{1}{a}$,	$\frac{1}{b}$,	$\frac{b-1}{bc}$	$(a-1)(bc-b+1)$
III	$x^a,\ y^b x,\ z^c x$	$\frac{1}{a}$,	$\frac{a-1}{ab}$	$\frac{a-1}{ac}$	$\frac{(ab-a+1)(ac-a+1)}{a-1}$
IV	$x^a,\ y^b z,\ z^c y$	$\frac{1}{a}$,	$\frac{c-1}{bc-1}$,	$\frac{b-1}{bc-1}$	$(a-1)bc$
V	$x^a,\ y^b z,\ z^c x$	$\frac{1}{a}$,	$\frac{ac-a+1}{abc}$,	$\frac{a-1}{ac}$	$ac(b-1)+a-1$
VI	$x^a y,\ y^b x,\ z^c x$	$\frac{b-1}{ab-1}$,	$\frac{a-1}{ab-1}$,	$\frac{(a-1)b}{(ab-1)c}$	$\frac{a(abc-c-ab+b)}{a-1}$
VII	$x^a y,\ y^b z,\ z^c x$	$\frac{bc-c+1}{abc+1}$,	$\frac{ac-a+1}{abc+1}$,	$\frac{ab-b+1}{abc+1}$	abc

PROOF. We begin a classification for arbitrarily many variables $x_1, \ldots, x_n$. We fix the suffix i of the coordinate x_i. When all monomials of the form $x_i^a x_j$ are absent, the x_i-axis consists entirely of critical points.

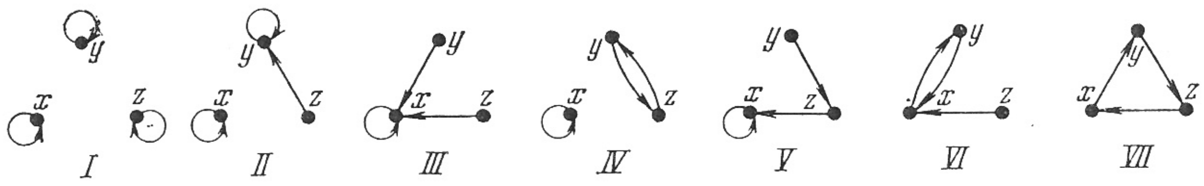

Fig. 4.