Invariants of totally real Lefschetz fibrations

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Abstract. In this note we introduce certain invariants of real Lefschetz fibrations. We call these invariants *real Lefschetz chains*. We prove that if the fiber genus is greater than 1, then the real Lefschetz chains are complete invariants of totally real Lefschetz fibrations. If however the fiber genus is 1, real Lefschetz chains are not sufficient to distinguish real Lefschetz fibrations. We show that by adding a certain binary decoration to real Lefschetz chains, we get a complete invariant.

1. Introduction

This note is devoted to a topological study of Lefschetz fibrations equipped with certain $\mathbb{Z}_2$ actions compatible with the fiber structure. The action is generated by an involution, which is called a *real structure*. Intuitively, real structures are topological generalizations of the complex conjugation on complex algebraic varieties defined over the reals. Real Lefschetz fibrations appear, for instance, as blow-ups of pencils of hyperplane sections of complex projective algebraic surfaces defined by real polynomial equations. Regular fibers of real Lefschetz fibrations are compact oriented smooth genus-g surfaces while singular fibers have a single node. The invariant fibers, called the *real fibers*, inherit a real structure from the real structure of the total space. We focus on fibrations whose critical values are all fixed by the action and call such fibrations *totally real*. We also assume that the fixed point set of the base space is oriented. We use the term *directed* to indicate such fibrations.

The main results of this article are exhibited in Section 6 and Section 8 in which we treat the cases of fiber genus $g > 1$ and $g = 1$, respectively. In Section 6, we introduce *real Lefschetz chains* and prove that if $g > 1$, then real Lefschetz chains are complete invariants of directed genus-g totally real Lefschetz fibrations over the disk (Corollary 6.4). The case of $g = 1$ (elliptic fibrations) is considered in Theorem 8.1. We show that directed totally real elliptic Lefschetz fibrations over D^2 are determined uniquely by their decorated real Lefschetz chains. Furthermore, in both cases we study extensions of such fibrations to fibrations over a sphere and obtain complete invariants of directed totally real Lefschetz fibrations over a sphere.

It is possible to give a purely combinatorial shape to decorated real Lefschetz chains. We will discuss such combinatorial objects (which we call *necklace diagrams*) and their applications in [9] (see also [1] for other applications of necklace diagrams).

The present work is organized as follows. In Section 2, we settle the definitions and introduce basic notions. Section 3 is devoted to the topological classification of equivariant neighborhoods of real singular fibers. We show that real Lefschetz fibrations around real singular fibers are determined by the pair consisting of the inherited real structure on one of the nearby regular real fibers and the vanishing cycle which is invariant under the action of the real structure. We call such a pair a *real code*.

In Section 4, we compute the fundamental group of the components of the space of real structures on a genus-g surface. These computations are applied in Section 5 where we define a *strong boundary fiber sum* (that is, the boundary fiber sum of $\mathbb{C}$-marked real Lefschetz fibrations) and show that if the fiber genus is greater than 1, then the strong boundary fiber sum is well-defined. Section 6 is devoted to $\mathbb{C}$-marked genus-$g > 1$ fibrations. We show that directed $\mathbb{C}$-marked genus-$g > 1$ totally real Lefschetz fibrations are classified by their *strong real Lefschetz chains*. As a corollary, we obtain the result for non-marked fibrations.

Because of the different geometric nature of the surfaces of genus $g > 1$ and $g = 1$, we apply slightly different techniques to deal with the case of $g = 1$. In Section 7, we define a *boundary fiber sum* of non-marked real elliptic Lefschetz fibrations. We observe that the boundary fiber sum is not always well-defined. This observation leads to a decoration of directed totally real Lefschetz chains. In the last section, we introduce *decorated real Lefschetz chains* and prove that they are complete invariants of real elliptic Lefschetz fibrations. We also study extensions of such fibrations to fibrations over a sphere.

Let us note that real Lefschetz chains are, indeed, sequences of real codes each of which is associated to a neighborhood of a real singular fiber. Obviously, each real Lefschetz fibration with real critical values defines a real Lefschetz chain which is, by definition, invariant of the fibration. The natural question to ask is to what extent real Lefschetz chains determine the fibration. This note explores an answer to this question.

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2. Basic definitions

Throughout the paper X will stand for a compact connected oriented smooth 4-manifold and B for a compact connected oriented smooth 2-manifold.

Definition 2.1. A *real structure* c_X on a smooth 4-manifold X is an orientation preserving involution, $c_X^2 = id$, such that the set of fixed points, $Fix(c_X)$, of c_X is empty or of the middle dimension.

Two real structures c_X and c'_X are considered *equivalent* if there exists an orientation preserving diffeomorphism $\psi : X \to X$ such that $\psi \circ c_X = c'_X \circ \psi$.

A real structure c_B on a smooth 2-manifold B is an orientation reversing involution $B \to B$. Such structures are similarly considered up to conjugation by orientation preserving diffeomorphisms of B.

The above definition mimics the properties of the standard complex conjugation on complex manifolds. Actually, around a fixed point every real structure defined as above behaves like complex conjugation.

We will call a manifold together with a real structure a *real manifold* and the fixed point set the *real part*.

Remark 2.2. It is well known that for given g there is a finite number of equivalence classes of *real structures* on a genus-g surface Σ_g. These classes can be distinguished by their *types* and the number of real components. Namely, one distinguishes two types of real structures: separating and non-separating. A real structure is called *separating* if the complement of its real part has two connected components,

otherwise we call it *non-separating* (indeed, in the first case the quotient surface Σ_g/c is orientable while in the second case it is not). The number of real components of a real structure (note that the real part forms the boundary of Σ_g/c), can be at most $g+1$. This estimate is known as *Harnack inequality* [6]. By looking at the possible number of connected components of the real part, one can see that on Σ_g there are $1+[\frac{g}{2}]$ separating real structures and $g+1$ non-separating ones. A significant property of the case of genus-1 surfaces is that the number of real components, which can be 0, 1 or 2, is enough to distinguish the real structures.

In this article we stick to the following definition of Lefschetz fibrations.

Definition 2.3. A Lefschetz fibration is a surjective smooth map $\pi : X \to B$ such that:

- $\pi(\partial X) = \partial B$ and the restriction $\partial X \to \partial B$ of π is a submersion;
- π has only a finite number of critical points (that is, the points where $d\pi$ is degenerate), all the critical points belong to $X \setminus \partial X$ and their images are distinct points of $B \setminus \partial B$;
- around each of the critical points one can choose orientation-preserving charts $\psi : U \to \mathbb{C}^2$ and $\phi : V \to \mathbb{C}$ so that $\phi \circ \pi \circ \psi^{-1}$ is given by $(z_1, z_2) \to {z_1}^2 + {z_2}^2$.

When we want to specify the genus of the non-singular fibers, we prefer calling them *genus-g Lefschetz fibrations*. In particular, we will use the term *elliptic Lefschetz fibrations* when the genus is equal to one. For each integer g, we will fix a closed oriented surface of genus g, which will serve as a model for the fibers, and denote it by Σ_g. In what follows we will always assume that a Lefschetz fibration is *relatively minimal*; that, is none of its fibers contains a self intersection -1 sphere.

Definition 2.4. A real structure on a Lefschetz fibration $\pi : X \to B$ is a pair of real structures (c_X, c_B) of X and B such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{c_X} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{c_B} & B. \end{array}$$

A Lefschetz fibration equipped with a real structure is called a *real Lefschetz fibration* and is sometimes referred as $\mathcal{RLF}$. When the fiber genus is 1, we call it a *real elliptic Lefschetz fibration* (abbreviated $\mathcal{RELF}$).

Definition 2.5. An $\mathbb{R}$-*marked* $\mathcal{RLF}$ is a triple (π, b, ρ) consisting of a real Lefschetz fibration $\pi : X \to B$, a real regular value b and a diffeomorphism $\rho : \Sigma_g \to F_b$ such that $c_X|_{F_b} \circ \rho = \rho \circ c$ where $c : \Sigma_g \to \Sigma_g$ is a real structure. Let us note that if $\partial B \neq \emptyset$, then b will be chosen in ∂B.

A $\mathbb{C}$-*marked* $\mathcal{RLF}$ is a triple $(\pi, \{m, \bar{m}\}, \{\rho, \bar{\rho}\})$ including a real Lefschetz fibration $\pi : X \to B$ a pair of regular values $m, \bar{m} = c_B(m)$ and a pair of diffeomorphisms $\rho : \Sigma_g \to F_m$, $\bar{\rho} = c_X|_{F_m} \circ \rho : \Sigma_g \to F_{\bar{m}}$ where F_m and $F_{\bar{m}} = c_X(F_m)$ are the fibers over m and $\bar{m}$, respectively. As in the case of $\mathbb{R}$-marking, if $\partial B \neq \emptyset$, then we choose m in ∂B. When precision is not needed we will denote $F_m, F_{\bar{m}}$ by F and $\bar{F}$, respectively.

Two real Lefschetz fibrations $\pi : X \to B$ and $\pi' : X' \to B'$ are said to be *isomorphic* if there exist orientation preserving diffeomorphisms $H : X \to X'$ and $h : B \to B'$ such that the following diagram is commutative

$$\begin{tikzcd}
 & X \arrow[rr, "H"] \arrow[dd, "\pi"] & & X' \arrow[dd, "\pi'"] \\
X \arrow[ur, "c_X"] \arrow[rr, "H"] \arrow[dd, "\pi"'] & & X' \arrow[ur, "c_{X'}"'] \arrow[dd, "\pi'"] & \\
 & B \arrow[rr, "h"] & & B' \\
B \arrow[ur, "c_B"] \arrow[rr, "h"] & & B'. \arrow[ur, "c_{B'}"'] &
\end{tikzcd}$$

Two $\mathbb{R}$-marked $\mathcal{RLF}s$ are called isomorphic if they are isomorphic as $\mathcal{RLF}s$ such that $h(b) = b'$, and the following diagram is commutative

$$\begin{tikzcd}
F \arrow[rrr, "H"] \arrow[ddd, "c_X"'] & & & F' \arrow[ddd, "c_{X'}"] \\
 & & \Sigma_g \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] \arrow[dd, "H"] & \\
 & & & \\
F \arrow[rrr, "e"] & & & F' \\
 & & \Sigma_g. \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] & \\
\end{tikzcd}$$

Two $\mathbb{C}$-marked $\mathcal{RLF}s$ are called isomorphic if they are isomorphic as $\mathcal{RLF}s$ and the following diagram is well-defined and commutative

$$\begin{tikzcd}
F \arrow[rrr, "H"] \arrow[ddd, "c_X"'] & & & F' \arrow[ddd, "c_{X'}"] \\
 & & \Sigma_g \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] \arrow[dd, "H"] & \\
 & & & \\
\bar{F} \arrow[rrr, "id"] & & & \bar{F}' \\
 & & \bar{\Sigma}_g. \arrow[ul, "\bar{\rho}"] \arrow[ur, "\bar{\rho}'"'] & \\
\end{tikzcd}$$

Definition 2.6. A real Lefschetz fibration $\pi : X \to B$ is called *directed* if the real part of (B, c_B) is oriented. (If c_B is separating, then we consider an orientation on the real part inherited from one of the halves $B \setminus Fix(c_B)$.)

Two directed $\mathcal{RLF}s$ are isomorphic if they are isomorphic as $\mathcal{RLF}s$ with the additional condition that the diffeomorphism $h : B \to B$ preserves the chosen orientation on the real part.

Unless otherwise stated all fibrations we consider are directed.

Remark 2.7. The notion of Lefschetz fibration can be slightly generalized to cover the case of fibrations whose fibers have non-empty boundary. Then, X turns into a manifold with corners and its boundary, ∂X, becomes naturally divided into two parts: the *vertical boundary* $\partial^v X$ which is the inverse image $\pi^{-1}(\partial B)$, and the *horizontal boundary* $\partial^h X$ which is formed by the boundaries of the fibers. We call such fibrations *Lefschetz fibrations with boundary.*

3. Elementary real Lefschetz fibrations

In this section, we classify real structures on a neighborhood of a real singular fiber of a real Lefschetz fibration. Such a neighborhood can be viewed as a Lefschetz fibration over a disc D^2 with a unique critical value $q = 0 \in D^2$. We call such a fibration an *elementary real Lefschetz fibration*. Without loss of generality, we may assume that the real structure on D^2 is the standard one, *conj*, induced from $\mathbb{C} \supset D^2$.

Let $\pi : X \to D^2$ be an elementary $\mathcal{RLF}$. By definition, there exist equivariant local charts (U, ϕ_U), (V, ϕ_V) around the critical point $p \in \pi^{-1}(0)$ and the critical value $0 \in D^2$ respectively such that U and V are closed discs and $\pi|_U : (U, c_U) \to (V, conj)$ is equivariantly isomorphic (via ϕ_U and ϕ_V) to either of $\xi_\pm : (E_\pm, conj) \to (D_\epsilon, conj)$, where

$$E_\pm = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| \leq \sqrt{\epsilon},\ \left|z_1^2 \pm z_2^2\right| \leq \epsilon^2\}$$

and

$$D_\epsilon = \{t \in \mathbb{C} \,:\, |t| \leq \epsilon^2\},\ 0 < \epsilon < 1$$

with $\xi_\pm(z_1, z_2) = z_1^2 \pm z_2^2$.

The above real local models, $\xi_\pm : E_\pm \to D_\epsilon$, can be seen as two real structures on the neighborhood of a critical point. These two real structures are not equivalent. The difference can be seen already at the level of the singular fibers: in the case of ξ_+ the two branches are imaginary and they are interchanged by the complex conjugation; in the case of ξ_- the two branches are both real (see Figure 1).

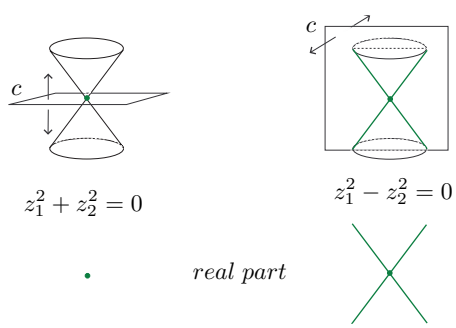


Fig. 1. Actions of real structures on the singular fibers of $\xi_\pm$.

To understand the action of the real structures on the regular real fibers of $\xi_\pm$, we can use the branched covering defined by the projection $(z_1, z_2) \to z_1$. Thus, we have:

- in the case of ξ_+, there are two types of real regular fibers; the fibers F_t with $t < 0$ have no real points, their vanishing cycles have invariant representatives (that is, $c(a_t) = a_t$ set-theoretically), and in this case, c acts on the invariant vanishing cycles as an antipodal involution; the fibers F_t with $t > 0$ has a circle as their real part and this circle is an invariant (pointwise fixed) representative of the vanishing cycle;
- in the case of ξ_-, all the real regular fibers are of the same type and the real part of such a fiber consists of two arcs each having its endpoints on

the two different boundary components of the fiber; the vanishing cycles have invariant representatives, and c acts on them as a reflection.

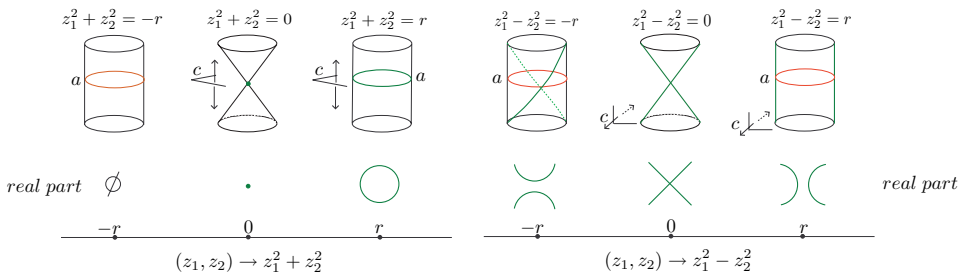


Fig. 2. Nearby regular fibers of $\xi_\pm$ and the vanishing cycles.

Using the ramified covering $(z_1, z_2) \to z_1$, we observe that the horizontal boundary of the fibration $\xi_\pm$ is equivariantly trivial and has a distinguished equivariant trivialization. Moreover, since the complement of U in $\pi^{-1}(V)$ does not contain any critical point, X can be written as union of two $\mathcal{RLF}s$ with boundary: one of them, $U \to V$, is isomorphic to $\xi_\pm : E_\pm \to D_\epsilon$, and the other one is isomorphic to the trivial real fiber bundle $R \to D_\epsilon$ whose real fibers are equivariantly diffeomorphic to the complement of an open regular neighborhood of the vanishing cycle $a \subset F_b$. The action of the complex conjugation on the boundary components of the real fibers of $R \to D_\epsilon$ determines the type of the model, $\xi_\pm : E_\pm \to D_\epsilon$ glued to $R \to D_\epsilon$: in the case of ξ_+ it switches the boundary components while in the case of ξ_- boundary components are preserved (and the complex conjugation acts as a reflection on each of them).

We use the above decomposition to get first a classification of directed $\mathbb{R}$-marked elementary $\mathcal{RLF}$, and then we discuss the cases of $\mathbb{C}$-marked and non-marked fibrations.

Let $\mathcal{A}_g^c$ denote the set of equivariant isotopy classes of non-contractible curves on a real surface (Σ_g, c) and $\mathcal{V}_g^c$ the set of equivariant isotopy classes of non-contractible embeddings $\nu : S^1 \times I \to \Sigma_g$ such that $c \circ \nu = \nu$ and $\mathcal{L}_g^{\mathbb{R},c}$ the set of isomorphism classes of directed $\mathbb{R}$-marked elementary real Lefschetz fibrations whose distinguished fiber is identified with (Σ_g, c).

We consider the map $\Omega : \mathcal{V}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ defined as follows. Let $[\nu]$ be a class in $\mathcal{V}_g^c$ with a representative ν. As $c \circ \nu = \nu$, the closure, Σ_g^ν, of $\Sigma_g \setminus \nu(S^1 \times I)$ inherits a real structure from (Σ_g, c). Let $R^\nu = \Sigma_g^\nu \times D_\epsilon \to D_\epsilon$ be the trivial real fibration with the real structure $c_{R^\nu} = (c, conj) : R^\nu \to R^\nu$ and let $E_\pm^\nu \to D_\epsilon$ denote the model $\xi_\pm : E \to D_\epsilon$ whose marked fiber is identified with $\nu(S^1 \times I)$. Depending on the real structure on the horizontal boundary $S^1 \times D_\epsilon \to D_\epsilon$ (where the real structure on $S^1 \times D_\epsilon$ is taken as $(c_{\partial\Sigma_g^\nu}, conj)$) of $R^\nu \to D_\epsilon$, we choose either of $E_\pm^\nu \to D_\epsilon$. We then glue $R^\nu \to D_\epsilon$ and the suitable model $E_\pm^\nu \to D_\epsilon$ along their horizontal trivial boundaries to get a fibration in $\mathcal{L}_g^{\mathbb{R},c}$.

Lemma 3.1. *$\Omega : \mathcal{V}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ is well-defined.*

Proof: Let $\nu_t : S^1 \times I \to \Sigma_g$ be an isotopy between ν_0 and ν_1. Then, there exists an equivariant ambient isotopy $\Psi_t : \Sigma_g \to \Sigma_g$ such that $\Psi_0 = id$ and $\nu_t = \Psi_t \circ \nu_0$

with $\Psi_t \circ c = c \circ \Psi_t$ for all t. The diffeomorphism, Ψ_1, induces equivariant diffeomorphisms $\Psi_1^R : R^{\nu_0} \to R^{\nu_1}$ and $\Psi_1^E : E_\pm^{\nu_0} \to E_\pm^{\nu_1}$ that respect the fibrations and the gluing; thus, it gives an isomorphism of the images, $\Omega([\nu_0])$ and $\Omega([\nu_1])$, as $\mathbb{R}$-marked fibrations. $\square$

Since $c \circ \nu = \nu$, we have $c(\nu(S^1 \times \{\frac{1}{2}\})) = \nu(S^1 \times \{\frac{1}{2}\}))$. Hence, we can define $\varepsilon : \mathcal{V}_g^c \to \mathcal{A}_g^c$ such that $\varepsilon([\nu]) = [\nu(S^1 \times \{\frac{1}{2}\})]$. This mapping is two-to-one. Since the monodromy does not depend on the orientation of the vanishing cycle, there exists a well-defined mapping $\hat{\Omega}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V}_g^c \arrow[r, "\varepsilon"] \arrow[d, "\Omega"'] & \mathcal{A}_g^c \arrow[dl, "\hat{\Omega}"] \\ \mathcal{L}_g^{\mathbb{R},c}. & \end{tikzcd}$$

Theorem 3.2. *$\hat{\Omega} : \mathcal{A}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ is a bijection.*

Proof: As it is discussed in the beginning of the section, any elementary $\mathcal{RLF}$ can be divided equivariantly into two $\mathcal{RLF}s$ with boundary: an equivariant neighborhood of the critical point (isomorphic to one of the models, $\xi_\pm$), and the complement of this neighborhood (isomorphic to a trivial real Lefschetz fibration). Such a decomposition defines the equivariant isotopy class of the vanishing cycle. Thus, $\hat{\Omega}$ is surjective.

To show that $\hat{\Omega}$ is injective, let us consider the classes, $[a], [a'] \in \mathcal{A}_g^c$, such that $\hat{\Omega}([a]) = \hat{\Omega}([a'])$. Let $\pi : X \to D_\epsilon$ (respectively, $\pi' : X' \to D_\epsilon$) denote the image of $[a]$ (respectively, the image of $[a']$). Since $\hat{\Omega}$ is well-defined, there exist equivariant orientation preserving diffeomorphisms $H : X \to X'$ and $h : D_\epsilon \to D_\epsilon$ such that we have the following commutative diagrams

$$\begin{tikzcd} & X \arrow[r, "H"] \arrow[d, "\pi"] & X' \arrow[d, "\pi'"] \\ X \arrow[ur, "c_X"] \arrow[r, "H"] \arrow[d, "\pi"'] & X' \arrow[ur, "c_{X'}"'] \arrow[d] & \\ D_\epsilon \arrow[r, "h"] \arrow[d, "conj"'] & D_\epsilon \arrow[r, "\pi'"] & D_\epsilon \\ D_\epsilon \arrow[ur] \arrow[r, "h"] & D_\epsilon \arrow[ur, "conj"'] & \end{tikzcd} \qquad \begin{tikzcd} F \arrow[rr, "HI_F"] \arrow[dd, "c_X"'] & & F' \arrow[dd, "c_{X'}"] \\ & \Sigma_g \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] \arrow[d, "H_F"] & \\ F \arrow[rr, "e"] & & F' \\ & \Sigma_g. \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] & \end{tikzcd}$$

Clearly, $H(\rho(a))$ is equivariantly isotopic to $\rho'(a')$ where a and a' are representatives of $[a]$ and $[a']$, respectively. Moreover, since $HI_F \circ \rho = \rho'$, we have $H(\rho(a)) = \rho'(a)$, so $\rho'(a)$ is equivariant isotopic to $\rho'(a')$.

Let $\psi_t : F' \to F'$, $t \in [0,1]$ such that $\psi_0 = id$ and $\psi_1(\rho'(a)) = \rho'(a')$ and that $\psi_t \circ c' = c' \circ \psi_t$ for all $t \in [0,1]$. Then, $\Psi_t = \rho'^{-1} \circ \psi_t \circ \rho' : \Sigma_g \to \Sigma_g$ is the required isotopy between a and a'. $\square$

Theorem 3.2 shows that c-equivariant isotopy classes of vanishing cycles on (Σ_g, c) classify directed $\mathbb{R}$-marked elementary $\mathcal{RLF}s$. To obtain a classification for directed $\mathbb{C}$-marked $\mathcal{RLF}s$, we study the difference between two $\mathbb{C}$-markings.

Let $(\{m, \bar{m}\}, \{\rho_m, c_X \circ \rho_m\})$ be a $\mathbb{C}$-marking on a directed $\mathcal{RLF}$, $\pi : X \to D^2$. The complement, $\partial D^2 \setminus \{m, \bar{m}\}$, has two pieces $S_\pm$ (left/ right semicircles)

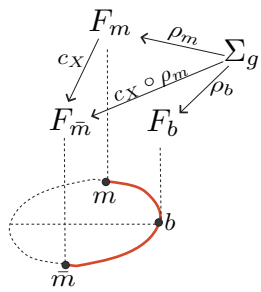


Fig. 3. Relation between $\mathbb{R}$-marking and $\mathbb{C}$-marking.

distinguished by the direction. By considering a trivialization of the fibration over the piece of S_+ connecting m to the marked real point b (the trivialization over the piece connecting $\bar{m}$ to the real point obtain by the symmetry), we can pull the marking $\rho_m : \Sigma_g \to F_m$ to F_b in order to obtain a marking $\rho_b : \Sigma_g \to F_b$ and a real structure $c = \rho_b^{-1} \circ c_X \circ \rho_b : \Sigma_g \to \Sigma_g$. Any other trivialization results in another marking isotopic to ρ_b and a real structure isotopic to $c : \Sigma_g \to \Sigma_g$. Hence, a directed elementary $\mathbb{C}$-marked $\mathcal{RLF}$ defines a vanishing cycle defined up to c-equivariant isotopy where the real structure c is also considered up to isotopy.

Definition 3.3. A pair (c, a) of a real structure $c : \Sigma_g \to \Sigma_g$ and a non-contractible simple closed curve $a \in \Sigma_g$ is called a *real code* if $c(a) = a$.

Two real codes, (c_0, a_0), (c_1, a_1), are said to be isotopic if there exist a pair of isotopies, (c_t, a_t), of real structures and vanishing cycles such that $c_t(a_t) = a_t$, $\forall t \in [0, 1]$. Two real codes, (c_0, a_0) and (c_1, a_1), are called conjugate if there is an orientation preserving diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi \circ c_0 = c_1 \circ \phi$ and that $\phi(a_0)$ is isotopic to a_1.

We denote the isotopy class of the real code by $[c, a]$ and the conjugacy class by $\{c, a\}$.

Proposition 3.4. *There is a one-to-one correspondence between the isomorphism classes of directed $\mathbb{C}$-marked elementary $\mathcal{RLF}$s and the isotopy classes of real codes.*

Proof: Above we discuss how to assign a real code to a directed $\mathbb{C}$-marked elementary $\mathcal{RLF}$. It is straightforward to check that this map is well-defined and surjective.

To show that it is injective, we consider two isotopy classes $[c_i, a_i]$, $i = 1, 2$ such that $[c_1, a_1] = [c_2, a_2]$. Let $(\pi_1 : X_1 \to D^2, \{m_1, \bar{m}_1\}, \{\rho_{m_1}, \bar{\rho}_{m_1}\})$ and $(\pi_2 : X_2 \to D^2, \{m_2, \bar{m}_2\}, \{\rho_{m_2}, \bar{\rho}_{m_2}\})$ be two directed $\mathbb{C}$-marked elementary $\mathcal{RLF}$s, associated to the classes $[c_1, a_1]$ and $[c_2, a_2]$, respectively. We need to show that π_1 and π_2 are isomorphic as directed $\mathbb{C}$-marked $\mathcal{RLF}$s.

Note that we can always choose a representative c for both $[c_1]$ and $[c_2]$ such that $[a_1] = [a_2] \in \mathcal{A}_g^c$. Then, by Theorem 3.2, π_1 is isomorphic to π_2 as $\mathbb{R}$-marked $\mathcal{RLF}$s. An isomorphism of $\mathbb{R}$-marked $\mathcal{RLF}$s may not preserve the $\mathbb{C}$-markings; however, it can be modified to preserve them.

Up to homotopy one can identify X_2 with a subset, $\mathring{X}_2$, of X_1. Since the difference $X_1 \setminus \mathring{X}_2$ has no singular fiber, one can transform the marking $\mathring{m}_2$ of $\mathring{X}_2$ to m_1 preserving the real marking and the trivializations over the corresponding paths,

S_+ and $\mathring{S}_+$ (see Figure 4). This way we get an isomorphism of $\mathbb{C}$-marked $\mathcal{RLF}s$ preserving the isomorphism class of $\mathbb{R}$-marked $\mathcal{RLF}s$.

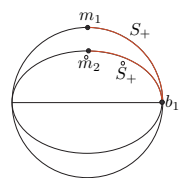


Fig. 4. The difference of two $\mathbb{C}$-markings.

□

For fibrations without marking we allow $[c, a]$ to change by an equivariant diffeomorphism. Hence, we have the following proposition.

Corollary 3.5. *There is a one-to-one correspondence between the set of conjugacy classes of real codes and the set of classes of directed non-marked elementary real Lefschetz fibrations.* □

Remark 3.6. As the classification of real structures on a genus-g surface is known, it is possible to enumerate the conjugacy classes, $\{c, a\}$ of real codes. In the case when a is non-separating, there are 6 classes if $g = 1$; $8g - 3$ classes if $g > 1$ and is odd; $8g - 4$ classes otherwise. The formulas for separating curves can be found in [7].

Remark 3.7. Note that there is no preferable real fiber over the boundary of the disk if the fibration is not directed. Thus, to an elementary non-directed $\mathcal{RLF}$, we can associate two real codes, (c_-, a_-), (c_+, a_+), extracted from the "left" and "right" real fibers, respectively. It is a fundamental property of the monodromies of real Lefschetz fibrations that the real structures c_-, c_+ are related by the monodromy such that $c_+ \circ c_- = t_{a_-} = t_{a_+}$ (cf. [8]).

4. Equivariant diffeomorphisms and the space of real structures

In this section we compute the fundamental group of the space of real structures on a genus-g surface. The computations will be essential in next sections.

Let $\mathcal{C}^c(\Sigma_g)$ denote the space of real structures on Σ_g which are isotopic to a fixed real structure c, and let $\mathit{Diff}_0(\Sigma_g)$ denote the group of orientation preserving diffeomorphisms of Σ_g which are isotopic to the identity. We consider two subgroups of $\mathit{Diff}_0(\Sigma_g)$: the one, denoted $\mathit{Diff}_0^c(\Sigma_g)$, consists of those diffeomorphisms which commute with c, and the other, $\mathit{Diff}_0(\Sigma_g, c)$, is the group of diffeomorphisms which are c-equivariantly isotopic to the identity. The group $\mathit{Diff}_0(\Sigma_g)$ acts transitively on $\mathcal{C}^c(\Sigma_g)$ by conjugation. The stabilizer of this action is the group $\mathit{Diff}_0^c(\Sigma_g)$. Hence, $\mathcal{C}^c(\Sigma_g)$ can be identified with the homogeneous space $\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0^c(\Sigma_g)$.

Lemma 4.1. *The space $\mathit{Diff}_0^c(\Sigma_g)$ is connected for all $c : \Sigma_g \to \Sigma_g$ if $g > 1$, and for $c : \Sigma_g \to \Sigma_g$ which has one real component if $g = 1$.*

Proof: (We will use different techniques for the cases $g > 1$ and $g = 1$.)

The case of $g > 1$: we consider the fiber bundle description of conformal structures on Σ_g, introduced in [2]. Let $Conf_{\Sigma_g}$ denote the space of conformal structures on Σ_g equipped with C^∞-topology. The group $Diff_0(\Sigma_g)$ acts on $Conf_{\Sigma_g}$ by composition from right. This action is proper, continuous, and effective; hence, $Conf_{\Sigma_g} \to Conf_{\Sigma_g}/Diff_0(\Sigma_g)$ is a principle $Diff_0(\Sigma_g)$-fiber bundle (cf. [2]). The quotient is the Teichmuller space of Σ_g, denoted $Teich_{\Sigma_g}$. Note that conformal structures can be seen as equivalence classes of Riemannian metrics with respect to the relation that two Riemannian metrics are equivalent if they differ by a positive function on Σ_g. Let $Riem_{\Sigma_g}$ denote the space of Riemannian metrics on Σ_g. Then, we have the following fibrations

$$\begin{array}{ccc} \{u : \Sigma_g \to \mathbb{R} : u > 0\} & \longrightarrow & Riem_{\Sigma_g} \\ & & \Big\downarrow p_2 \\ Diff_0(\Sigma_g) & \longrightarrow & Conf_{\Sigma_g} \\ & & \Big\downarrow p_1 \\ & & Teich_{\Sigma_g}. \end{array}$$

The real structure c acts on $Diff_0(\Sigma_g)$ by conjugation. This action can be extended to $Conf_{\Sigma_g}$ and $Riem_{\Sigma_g}$ as follows. We fix a section $s : Teich_{\Sigma_g} \to Conf_{\Sigma_g}$ of the bundle p_1 and we consider a family of diffeomorphisms $\phi^s_\zeta : Diff_0(\Sigma_g) \to p_1^{-1}(\zeta)$ parametrized by $Teich_{\Sigma_g}$ such that $\phi^s_\zeta(id) = s(\zeta)$. Let $s(\zeta) = [\mu_x]$ for some Riemannian metric μ_x on Σ_g. Then, we define $\phi^s_\zeta(f(x)) = [\mu_{f(x)}]$ for all $f \in Diff_0(\Sigma_g)$. The action of the real structure, thus, can be written as $c.[\mu_{f(x)}] = [\mu_{c\circ f\circ c(x)}]$. Clearly the definition does not depend on the choice of the representative of the class $[\mu_{f(x)}]$, so the action extends to $Riem_{\Sigma_g}$.

Let $Fix_{Conf_{\Sigma_g}}(c)$ denote the set of fixed points of the action of c on $Conf_{\Sigma_g}$ and $Fix_{Riem_{\Sigma_g}}(c)$ be the set of fixed points on $Riem_{\Sigma_g}$. Note that $s(\zeta) = \phi^s_\zeta(id) \in Fix_{Conf_{\Sigma_g}}(c)$ for all $\zeta \in Teich_{\Sigma_g}$. Indeed, each $[\mu_{f(x)}]$ where $f \in Diff_0^c(\Sigma_1)$ is in $Fix_{Conf_{\Sigma_g}}(c)$.

The space $Fix_{Conf_{\Sigma_g}}(c)$ is connected. If $Fix_{Conf_{\Sigma_g}}(c)$ were disconnected, then the inverse image $Fix_{Riem_{\Sigma_g}}(c)$ would also be disconnected in $Riem_{\Sigma_g}$. However, it is known that $Riem_{\Sigma_g}$ is convex; thus, $Fix_{Riem_{\Sigma_g}}(c)$ is convex, so it is connected. Therefore, $Fix_{Conf_{\Sigma_g}}(c) \cap Diff_0(\Sigma_g) = Diff_0^c(\Sigma_g)$ is connected since $Fix_{Conf_{\Sigma_g}}(c)$ is a union of sections.

The case of $g = 1$: if c has one real component, then the quotient Σ_1/c is the Möbius band. The space of diffeomorphisms of the Möbius band has two connected components [4]: the identity component and the component of the diffeomorphism induced (if the Möbius band is obtained from $I \times I$, by identifying the points $t \times 0$ with the points $1 - t \times 1, t \in I = [0, 1]$) from the reflection of $I \times I$ with respect to $I \times \frac{1}{2}$. This diffeomorphism is not isotopic to the identity because before identifying the ends it reverses the orientation of $I \times I$, and it lifts to a diffeomorphism of Σ_1 (considered as the obvious quotient of $[-1, 1] \times [-1, 1]$) induced from the the central symmetry of $[-1, 1] \times [-1, 1]$. This diffeomorphism is not isotopic to the identity on Σ_1 since it reverses the orientation of the real curve.

Therefore, we have

$$\{f : \Sigma_1/c \to \Sigma_1/c : \ \hat{f} : \Sigma_1 \to \Sigma_1 \text{ is isotopic to } id\} = \{f : \Sigma_1/c \to \Sigma_1/c : f \cong id\}.$$

The former is identified by $\mathit{Diff}_0^c(\Sigma_1)$ and the latter is connected. □

Lemma 4.2. *For any real structure $c : \Sigma_g \to \Sigma_g$,*

$$\pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c), id) = \begin{cases} 0 & \text{if } g > 1 \\ \mathbb{Z} & \text{if } g = 1. \end{cases}$$

Proof: Note that the subgroup $\mathit{Diff}_0(\Sigma_g, c)$ acts on $\mathit{Diff}_0(\Sigma_g)$ by composition from left. Such an action is free, so $\mathit{Diff}_0(\Sigma_g) \to \mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c)$ is a $\mathit{Diff}_0(\Sigma_g, c)$-fiber bundle. The fibers, $\mathit{Diff}_0(\Sigma_g, c)$, can be identified with the group $\mathit{Diff}_0(\Sigma_g/c)$ because the lifting of diffeomorphisms of Σ_g/c can always be assured by means of the orientation double cover of Σ_g/c. (Note that if c is non-separating, then Σ_g/c is non-orientable. In this case, $\mathit{Diff}_0(\Sigma_g/c)$ denotes the space of all diffeomorphisms of Σ_g/c and $\mathit{Diff}_0(\Sigma_g/c)$ is component of the identity.)

Now, we consider the long exact homotopy sequence of this fibration.

$$\cdots \to \pi_2(\mathit{Diff}_0(\Sigma_g)) \to \pi_2(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c)) \to \pi_1(\mathit{Diff}_0(\Sigma_g, c)) \to \\ \pi_1(\mathit{Diff}_0(\Sigma_g)) \to \pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c)) \to \pi_0(\mathit{Diff}_0(\Sigma_g)) \to \cdots$$

The case of $g > 1$: the space $\mathit{Diff}_0(\Sigma_g)$ is contractible for $g > 1$ [2], so is $\mathit{Diff}_0(\Sigma_g/c)$ [3]. Therefore, from the homotopy long exact sequence of the fibration we obtain $\pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c), id) = 0$.

The case of $g = 1$: it is known that Σ_1 is deformation retract of $\mathit{Diff}_0(\Sigma_1)$ [5], so the space $\mathit{Diff}_0(\Sigma_1)$ can be considered as a group generated by the rotations which lift to the standard translations on the universal cover.

To understand $\mathit{Diff}_0(\Sigma_g, c)$, we first consider the case when c has two real components. Note that, in this case, the quotient Σ_1/c is topologically an annulus, so $\pi_1(\mathit{Diff}_0(\Sigma_1/c), id) = \mathbb{Z}$ [5]. We fix an identification of $\varrho : \mathbb{C}/\mathbb{Z}^2 \to \Sigma_1$ such that the real structure c is the one induced from the standard complex conjugation on $\mathbb{C}$. We consider the following family of diffeomorphisms

$$\begin{array}{llll} R'^1_t : & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_{\mathbb{Z}^2} & \to & (x+t+iy)_{\mathbb{Z}^2} \end{array} \qquad \begin{array}{llll} R'^2_t : & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_{\mathbb{Z}^2} & \to & (x+i(y+t))_{\mathbb{Z}^2} \end{array}$$

where $t \in [0,1]$ and $(x+iy)_{\mathbb{Z}^2}$ denotes the equivalence class of $x+iy$ in $\mathbb{C}/\mathbb{Z}^2$. Clearly $R'^j_0 = R'^j_1 = id$ and for each $t \in [0,1]$, $R'^j_t, j = 1,2$ is isotopic to identity. The homotopy classes of $R^1_t = \varrho \circ R'^1_t \circ \varrho^{-1}$ and $R^2_t = \varrho \circ R'^2_t \circ \varrho^{-1}$ form a basis of $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$. Moreover, with respect to the identification ϱ, each diffeomorphism R^1_t is in $\mathit{Diff}_0(\Sigma_1, c)$, so the loop R^1_t is a generator of $\pi_1(\mathit{Diff}_0(\Sigma_1, c), id)$. Thus, from the homotopy exact sequence we get $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$.

If c has no real component, then the quotient Σ_1/c is a Klein bottle, so the group $\mathit{Diff}_0(\Sigma_1/c)$ is isomorphic to S^1 and is generated by the rotation which lifts to a translation in the universal cover of the Klein bottle [4]. Let us now fix an identification $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$ such that the real structure c is induced from the real structure

$$\begin{array}{lll} \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}/\mathbb{Z}^2 \\ (x,y)_{\mathbb{Z}^2} & \to & (x+\frac{1}{2}, -y)_{\mathbb{Z}^2}. \end{array}$$

The classes of family of diffeomorphisms $R_t^j = \varrho \circ R'^j_t \circ \varrho^{-1}$, $j = 1, 2$ where

$$\begin{array}{llll} R'^1_t: & \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ & (x,y)_{\mathbb{Z}^2} & \to & (x+t,y)_{\mathbb{Z}^2} \end{array} \qquad \begin{array}{llll} R'^2_t: & \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ & (x,y)_{\mathbb{Z}^2} & \to & (x,y+t)_{\mathbb{Z}^2}. \end{array}$$

form a basis of $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$. Moreover, with respect to the identification ϱ each diffeomorphism R_t^1 is in $\mathit{Diff}_0(\Sigma_1, c)$, and so R_t^1 is a generator of $\pi_1(\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$. Therefore, we get $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$.

If c has a unique real component, C, then the restriction $f|_C$ of $f \in \mathit{Diff}_0(\Sigma_1, c)$ defines a diffeomorphism of C. Such a restriction defines a fibration, $\mathit{Diff}_0(\Sigma_1, c) \to \mathit{Diff}_0(C)$, whose fibers isomorphic to $\mathit{Diff}_0(\Sigma_1, C) = \{f \in \mathit{Diff}_0(\Sigma_1, c) : f|_C = id\}$. Note that $\mathit{Diff}_0(\Sigma_1, C) \cong \mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ where $\overline{\Sigma_1 \setminus C}$ denotes the closure of $\Sigma_1 \setminus C$ and $\mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ the group diffeomorphisms of $\overline{\Sigma_1 \setminus C}$ which are identity on the boundary.

Topologically $\Sigma_1 \setminus C$ is an annulus, so $\mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ is contractible [5]. From the homotopy long exact sequence of the following fibration

$$\begin{array}{ccc} \mathit{Diff}_0(\Sigma_1, C) & \longrightarrow & \mathit{Diff}_0(\Sigma_1, c) \\ & & \downarrow \\ & & \mathit{Diff}_0(C) \end{array}$$

we get $\pi_k(\mathit{Diff}_0(\Sigma_1, c), id) \cong \pi_k(\mathit{Diff}_0(C), id)$, $\forall k$.

Let us now choose an identification $\varrho : \mathbb{C}/\Lambda \to \Sigma_1$ where Λ is the lattice generated by $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $v_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then, the real structure c can be taken as the one induced from the complex conjugation on $\mathbb{C}$.

We consider $R'_i(t) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, $t \in [0, 1]$ such that

$$\begin{array}{llll} R'^1_t: & \mathbb{C}/\Lambda & \to & \mathbb{C}/\Lambda \\ & (x+iy)_\Lambda & \to & (x+t+iy)_\Lambda \end{array} \qquad \begin{array}{llll} R'^2_t: & \mathbb{C}/\Lambda & \to & \mathbb{C}/\Lambda \\ & (x+iy)_\Lambda & \to & (x+i(y+t))_\Lambda. \end{array}$$

Again, the classes of $R_t^j = \varrho \circ R'^j_t \circ \varrho^{-1}, j = 1, 2$ form a basis for $\mathit{Diff}_0(\Sigma_1)$ while R_t^1 can be taken as a generator for $\pi_1(\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$. Therefore, $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$.

□

Proposition 4.3. *For any real structure $c : \Sigma_g \to \Sigma_g$,*

$$\pi_1(\mathcal{C}^c(\Sigma_g)) = \pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0^c(\Sigma_g), id) = \begin{cases} 0 & \text{if } g > 1 \\ \mathbb{Z} & \text{if } g = 1. \end{cases}$$

Proof: By Lemma 4.1, $\mathit{Diff}_0^c(\Sigma_g)$ is connected for all real $c : \Sigma_g \to \Sigma_g$, $g > 1$ and for the real structure $c : \Sigma_1 \to \Sigma_1$ which has one real component. Hence, in these cases $\mathit{Diff}_0^c(\Sigma_1) = \mathit{Diff}_0(\Sigma_1, c)$, so the result follows from Lemma 4.2.

In the case when $c : \Sigma_1 \to \Sigma_1$ has 2 real components, the space $\mathit{Diff}_0^c(\Sigma_1)$ has two connected components. Note that the diffeomorphism $R^2_{\frac{1}{2}}$ (induced from the translation, $(x+iy)_{\mathbb{Z}^2} \to (x+i(y+\frac{1}{2}))_{\mathbb{Z}^2}$, on $\mathbb{C}/\mathbb{Z}^2$) is equivariant; however, it is not equivariantly isotopic to the identity. Hence, $\mathit{Diff}_0^c(\Sigma_1)$ has two components: the component, $\mathit{Diff}_0(\Sigma_1, c)$, of the identity and the component of the rotation $R^2_{\frac{1}{2}}$. (In what follows, we denote $R^2_{\frac{1}{2}}$ by $R_{\frac{1}{2}}$.)

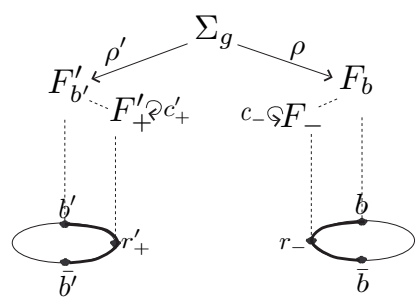
We can identify rotations in $\mathit{Diff}_0(\Sigma_1)\setminus \mathit{Diff}_0(\Sigma_1, c)$ with S^1 by letting $R^2_t \to 2\pi t$. Then, rotations in the quotient $\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}^c_0(\Sigma_1)$ are identified with $S^1/_{\theta\sim(\theta+\pi)}$, so we have $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}^c_0(\Sigma_1), id) = \mathbb{Z}$.

The case when $c : \Sigma_1 \to \Sigma_1$ has no real component can be treated similarly using the identification $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$. □

5. Boundary fiber sum of $\mathbb{C}$-marked real Lefschetz fibrations

Let $(D^2, conj)$ be a real disk with oriented real part. We denote by $S^{\pm}$ the upper/ lower semicircles of ∂D^2. We consider also left/ right semicircles, denoted by $S_{\pm}$, and the quarter-circles $S^{\pm}_{\pm} = S^{\pm} \cap S_{\pm}$. (Here directions right/ left and up/ down are determined by the orientations D^2 and of the real part.) Let $r_{\pm}$ be the real points of $S_{\pm}$, and $c_{\pm}$ the real structures on $F_{\pm} = \pi^{-1}(r_{\pm})$.

Definition 5.1. Let $(\pi' : X' \to D^2, \{b', \bar{b}'\}, \{\rho', \bar{\rho}'\})$ and $(\pi : X \to D^2, \{b, \bar{b}\}, \{\rho, \bar{\rho}\})$ be two directed $\mathbb{C}$-marked real Lefschetz fibrations such that the real structures c'_+ on F'_+ and c_- on F_- induce (via the markings) isotopic real structures on Σ_g. Then, we define the *strong boundary fiber sum* (the boundary fiber sum of $\mathbb{C}$-marked $\mathcal{RLF}s$) as follows.



We choose trivializations of $\pi'^{-1}(S^+_+)$ and $\pi^{-1}(S^+_-)$ such that the pull backs of c'_+ and c_- give the same real structure c on Σ_g. The trivialization of $\pi'^{-1}(S_+)$ can be obtained as a union $\Sigma_g \times S^+_+ \cup \Sigma_g \times S^-_+ /_{(x,1_+)\sim(c(x),1_-)}$ and similarly $\pi^{-1}(S_-) = \Sigma_g \times S^+_- \cup \Sigma_g \times S^-_- /_{(x,-1_+)\sim(c(x),-1_-)}$. The strong boundary fiber sum $X' \natural_{\Sigma_g} X \to D^2 \natural D^2$ is, thus, obtained by gluing $\pi'^{-1}(S_+)$ to $\pi^{-1}(S_-)$ via the identity map.

Remark 5.2. (1) In fact, the construction described above creates a manifold with corners, but there is a canonical way to smooth the corners; hence, the strong boundary fiber sum is the manifold obtained by smoothing the corners.

(2) By definition, the strong boundary fiber sum is associative but not commutative.

(3) The strong boundary fiber sum of $\mathbb{C}$-marked $\mathcal{RLF}s$ is naturally $\mathbb{C}$-marked.

Proposition 5.3. *If $g > 1$, then the strong boundary fiber sum, $X' \natural_{\Sigma_g} X \to D^2$, of directed $\mathbb{C}$-marked genus-g real Lefschetz fibrations is well-defined up to isomorphism of $\mathbb{C}$-marked $\mathcal{RLF}s$.*

Proof: Note that the boundary fiber sum does not affect the fibrations outside a small neighborhood of the interval where the gluing is made. Let us choose a neighborhood N which is real and far from the critical set. Obviously, the real

structures on the fibers over the real points of N are isotopic. Therefore, each fiber sum defines a path in the space of real structures on Σ_g, and the difference of two strong boundary fiber sums gives a loop in this space. Thus, the result follows from the contractibility (shown in Proposition 4.3) of this loop in the case of $g > 1$. $\square$

6. Strong real Lefschetz chains associated to $\mathbb{C}$-marked real Lefschetz fibrations

Let us now consider a directed $\mathbb{C}$-marked totally real Lefschetz fibration $\pi : X \to D^2$. We slice D^2 into smaller discs, $D_1, D_2, \ldots, D_n$ (ordered with respect to the orientation of the real part of $(D^2, conj)$) such that each D_i contains only one critical value and the base point b (which is chosen to be the "north pole", see Figure 5). Let $r_1, r_2, \ldots, r_n, r_{n+1}$ be the real points of $\cup_{i=1}^{n} \partial D_i$ and let c_i be the real structure on Σ_g pulled back from the inherited real structure of F_{r_i}.

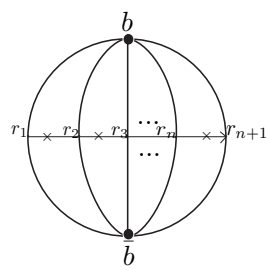


Fig. 5. Slicing D^2 into small discs having one critical value.

As asserted in Remark 3.7, for each fibration over D_i we have $c_{i+1} \circ c_i = t_{a_i}$ where a_i denotes the corresponding vanishing cycle. Moreover, as shown in Proposition 3.4, each $\mathbb{C}$-marked real Lefschetz fibration over D_i is determined by the isotopy class $[c_i, a_i]$ of a real code. Therefore, the fibration $\pi : X \to D^2$ yields a sequence of real codes $[c_i, a_i]$ satisfying $c_{i+1} \circ c_i = t_{a_i}$. Obviously this sequence is an invariant of π.

Definition 6.1. A sequence $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$ of isotopy classes of real codes is called a *strong real Lefschetz chain* if we have $c_{i+1} \circ c_i = t_{a_i}$ for all $i = 1, ..., n$.

Theorem 6.2. *If $g > 1$, then there is a one-to-one correspondence between the strong real Lefschetz chains $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$ and the isomorphism classes of directed $\mathbb{C}$-marked genus-g totally real Lefschetz fibrations over D^2.*

Proof: Necessity is clear. As for the converse, we consider the unique class (assured by Proposition 3.4) of directed $\mathbb{C}$-marked elementary real Lefschetz fibration associated to each real code $[c_i, a_i]$. We then glue these elementary fibrations (from left to right respecting the order determined by the chain) using the strong boundary fiber sum. The result, thus, follows from Proposition 5.3. $\square$

Let us note that if we consider non-marked fibrations, then the real codes around real singular fibers are defined up to conjugation. Thus, we are motivated to give the following definition and state the immediate corollary of Theorem 6.2.

Definition 6.3. A sequence $\{c_1, a_1\}, \{c_2, a_2\}, ..., \{c_n, a_n\}$ of conjugacy classes of real codes is called a *real Lefschetz chain* if $t_{a_i} \circ c_i$ is conjugate to c_{i+1} for all $1 \leq i \leq n$.

Corollary 6.4. *If $g > 1$, then there is a one-to-one correspondence between the real Lefschetz chains $\{c_1, a_1\}, \{c_2, a_2\}, ..., \{c_n, a_n\}$ and the isomorphism classes of non-marked directed genus-g totally real Lefschetz fibrations over D^2.* □

If the total monodromy of the fibration $\pi : X \to D^2$ is the identity, then we can consider the extension of π to a fibration $\hat{\pi} : \hat{X} \to S^2$. Two such extensions, $\hat{\pi} : \hat{X} \to S^2$ and $\check{\pi} : \check{X} \to S^2$, are considered *isomorphic* if there is an equivariant orientation preserving diffeomorphism $H : \hat{X} \to \check{X}$ such that $\hat{\pi} = \check{\pi} \circ H$.

Proposition 6.5. *Let $\pi : X \to D^2$ be a $\mathbb{C}$-marked genus-g real Lefschetz fibration whose total monodromy is the identity. If $g > 1$, then π can be extended uniquely up to isomorphism to a real Lefschetz fibration over S^2.*

Proof: Once again, the difference of two extensions corresponds to a loop in the space of real structures. Hence, the result follows from Proposition 4.3. □

Corollary 6.6. *If $g > 1$, then there is a one-to-one correspondence between the strong real Lefschetz chains $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$ such that $c_{n+1} \circ c_1 = (t_{a_n} \circ c_n) \circ c_1 = id$ and the isomorphism classes of directed $\mathbb{C}$-marked genus-g totally real Lefschetz fibrations over S^2.* □

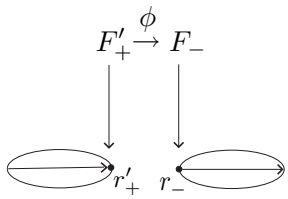
Remark 6.7. It is known that the components of the space of diffeomorphisms of the torus fixing a point is contractible [2], so Theorem 6.2 can be adapted to $\mathbb{C}$-marked real elliptic Lefschetz fibration admitting a real section (a section compatible with the real structures). Details can be found in [7, Section 5.4]. In the next section, we treat the case of non-marked elliptic Lefschetz fibrations, which possibly do not admit a real section.

7. Boundary fiber sum of non-marked real elliptic Lefschetz fibrations

To deal with the case of elliptic fibrations, we introduce the boundary fiber sum for non-marked fibrations. (Although we concentrate on the case of $g(F) = 1$, the definition applies to any genus.)

Definition 7.1. Let $\pi' : X' \to D^2$ and $\pi : X \to D^2$ be two directed non-marked $\mathcal{RLF}s$. We consider the real fibers, F'_+ and F_- of π' and π over the real points r'_+ and r_-, respectively. Let us assume that the real structure $c'_+ : F'_+ \to F'_+$ is conjugate to $c_- : F_- \to F_-$. Namely, there is an orientation preserving equivariant diffeomorphism $\phi : F'_+ \to F_-$. Then, the *boundary fiber sum* of $X' \natural_{F,\phi} X \to D^2$ is obtained by identifying the fibers F'_+ and F_- via ϕ.

The boundary fiber sum does depend on the choice of ϕ in such a way that the two boundary fiber sums defined by the equivariant diffeomorphisms $\phi, \psi : F'_+ \to F_-$ are isomorphic, if $\psi \circ \phi^{-1} : F_- \to F_-$ can be extended to an equivariant diffeomorphism of $X \to D^2$ (or similarly if $\phi^{-1} \circ \psi : F'_+ \to F'_+$ can be extended to an equivariant diffeomorphism of $X' \to D^2$). The necessary and sufficient condition



for $\psi \circ \phi^{-1} : F_- \to F_-$ to extend to an equivariant diffeomorphism of the fibration $X \to D^2$, is that $\psi \circ \phi^{-1}$ takes the unique vanishing cycle a of $X \to D^2$ to a curve equivariantly isotopic to a.

Note that if $c(a) = a$, then c induces an action on a. Such an action can be the identity, a reflection or an antipodal involution. It is not hard to show that if $c : \Sigma_1 \to \Sigma_1$ has one real component, then Σ_1 contains a unique c-equivariant isotopy class of non-contractible curves on which c acts as a reflection, a unique class of curves where the action of c is an antipodal involution, and a unique real curve; if c has 2 real components, then Σ_1 contains no c-equivariant isotopy class of curves on which c acts as an antipodal involution, a unique class of curves on which c acts as a reflection, and two classes of real curves (in which case, we call a pair of representatives of different classes *c-twin curves*); if c has no real components, then there exist two c-equivariant isotopy classes where c acts as an antipodal involution (as above, a pair of representatives of different classes are called *c-twin curves*) and no classes of other types. The boundary fiber sum is, therefore, well-defined unless the real structure c has no real component or c has two real components one of which is the vanishing cycle a.

Recall that the rotation $R_{\frac{1}{2}}$ (introduced in the proof of Proposition 4.3) switches the c-twin curves. Hence, c-twin curves can be carried to each other via equivariant diffeomorphisms although they are not equivariantly isotopic, so in the case of existence of c-twin curves, there is an ambiguity in the definition of the boundary fiber sum $X' \natural X \to D^2$ (it can be defined in two ways). To resolve the ambiguity, we should specify how we identify the c'_+-twin curves on the fiber F'_+ in X' with the c_--twin curves on the fiber F_- in X. In a certain case, namely if the real structure c'_+ has two real components and acts on the vanishing cycle a' as a reflection, the problem of switching c-twin curves can be eliminated via the transformation introduced below.

Let $\pi : X \to D^2$ be an elementary directed real elliptic Lefschetz fibration such that the real structure $c_+ : F_+ \to F_+$ acts on the vanishing cycle as a reflection. As a result, one of $c_\pm : F_\pm \to F_\pm$ has 1 real component while the other has 2 real components. Without loss of generality, we can assume that the real structure c_- has 1 real component. Our aim is to construct a transformation, T_{sing}, of X that does not change the isomorphism class of the fibration $\pi : X \to D^2$ and that is identity over $S_- \subset \partial D^2$ and interchanges the real components of F_+. To construct T_{sing}, we consider the following well known model for elementary elliptic fibrations.

Let $\hat{\Omega} = \{z | \, |Re(z)| \leq \frac{1}{2}, Im(z) \geq 1\} \cup \infty$, (the subset bounded by $Im(z) \geq 1$ of the one point compactification of the standard fundamental domain $\{z | \, |Re(z)| \leq \frac{1}{2}, |z| \geq 1\}$ of the modular action on $\mathbb{C}$, see Figure 6).

We consider the real structure $c_{\hat{\Omega}} : \hat{\Omega} \to \hat{\Omega}$ such that $c_{\hat{\Omega}}(\omega) = \overline{-\omega}$. Let Ω denote the quotient $\hat{\Omega} \diagup_{\frac{1}{2}+iy \sim -\frac{1}{2}+iy}$. The real structure $c_{\hat{\Omega}}$ induces a real structure on Ω.

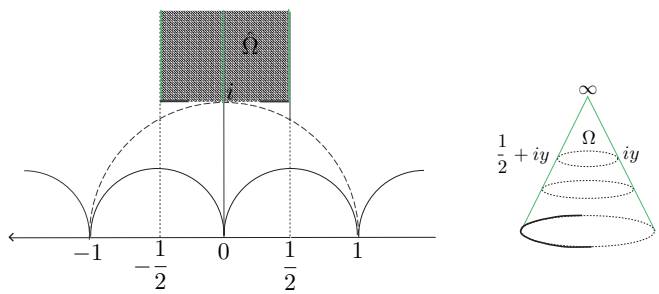


Fig. 6. Moduli space of prescribed $\mathcal{RELF}s$

Note that Ω is a topological real disc and can be identified with D^2 so that the real part of D^2 corresponds to the union of the half-lines iy and $\frac{1}{2}+iy$ where $y \geq 1$. For any $\omega \in \Omega$, the fiber over ω is given by $F_\omega = \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$, where the fiber F_∞ has the required nodal type singularity.

Let $\pi_\Omega : X_\Omega \to \Omega$ denote the fibration such that $\pi_\Omega^{-1}(\omega) = F_\omega = \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$, $\forall \omega \in \Omega$. Then, we consider the translation T'_Ω defined by

$$\begin{array}{rlcl} T'_\Omega : & X_\Omega & \to & X_\Omega \\ & (z)_{\mathbb{Z}+\omega\mathbb{Z}} \in F_\omega & \to & (z+\tau(w))_{\mathbb{Z}+\omega\mathbb{Z}} \in F_\omega \end{array}$$

where $(.)_{\mathbb{Z}+\omega\mathbb{Z}}$ denotes the equivalence class in $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$.

We consider $\tau : \Omega \to \Omega$ such that

$$\tau(\omega) = -\frac{1}{2} + (\frac{1}{2} - f(Re(\omega)) + i)exp(-Im(\omega)+1)$$

where $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a smooth mapping whose graph is as shown in Figure 7 and which satisfies the following properties:

- $f(0) = \frac{1}{2}$ (modulo $\mathbb{Z}$),
- $f(1-x) = 1 - f(x)$, ($\Rightarrow f(\frac{1}{2}) = \frac{1}{2}$) (modulo $\mathbb{Z}$),
- f is linear on $[\frac{1}{4}, \frac{3}{4}]$ (modulo $\mathbb{Z}$).

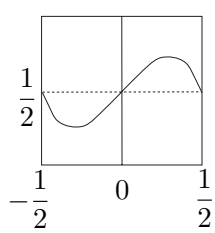


Fig. 7. The graph of f.

Note that τ has the following properties. (Equations are considered modulo the relation $-\frac{1}{2}+iy \sim \frac{1}{2}+iy$, $y \geq 1$.)

- $\tau(\overline{-\omega}) = \overline{-\tau(\omega)}$,

- $\tau(\infty) = \frac{1}{2}$,
- $\tau(\frac{1}{2} + iy) = -\frac{1}{2} + iexp(-y+1) = \frac{1}{2} + iexp(-y+1)$,

in particular, if $y = 1$, then $\tau(\frac{1}{2} + i) = \frac{1}{2} + i$,

- $\tau(iy) = -\frac{1}{2} + iexp(-y+1) = \frac{1}{2} + iexp(-y+1)$,

in particular, if $y = 1$, then $\tau(i) = \frac{1}{2} + i$.

Let $T_{sing} : X \to X$ denote the transformation induced from $T'_{sing} : X_\Omega \to X_\Omega$. By definition T_{sing} is equivariant and the identity over $S_- \subset \partial D^2$, and its restriction to F_+ is the rotation $R_{\frac{1}{2}}$. (Figure 8 shows the action of T_{sing} on the real part.)

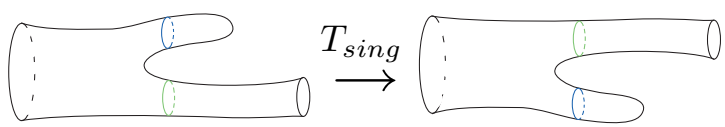


Fig. 8. The action of T_{sing} on the real part.

Lemma 7.2. *Let $\pi' : X' \to D^2$ and $\pi : X \to D^2$ be two non-marked elementary $\mathcal{RELF}$s such that both c'_+ and c_- have 2 real components. We assume that the vanishing cycle a of π is real with respect to c_-. Then, the boundary fiber sum $X' \natural_F X \to D^2$ is well-defined if c'_+ acts on the vanishing cycle a' as a reflection.*

Proof: The boundary fiber sums $X' \natural_{F,\phi} X \to D^2$ and $X' \natural_{F,\psi} X \to D^2$ are not isomorphic if $\phi \circ \psi^{-1}(a)$ and a are c-twin curves. However, in the case when c'_+ acts on the vanishing cycle a' as a reflection, we can apply T_{sing} to X' so that $T_{sing}(F'_+)$ differs from the fiber F'_+ by the rotation $R_{\frac{1}{2}}$. Therefore, $X' \natural_{F,\phi} X \to D^2$ is isomorphic to $T_{sing}(X') \natural_{F,\phi \circ R_{\frac{1}{2}}} X \to D^2$ which is isomorphic to $X' \natural_{F,\psi} X \to D^2$. □

8. Real Lefschetz chains associated to non-marked real elliptic Lefschetz fibrations

We now consider a non-marked directed totally real elliptic Lefschetz fibrations $\pi : X \to D^2$, $q_1 < q_2 < ... < q_n$. Around each critical value q_i we choose a small real disc D_i such that $D_i \cap \{q_1, q_2, ..., q_n\} = \{q_i\}$ and $D_i \cap D_{i+1} = \{r_{i+1}\} \subset [q_i, q_{i+1}]$, see Figure 9. Let c_i be the real structures on the fibers F_{r_i}, $1 \leq i \leq n$ (where r_1 is the left real point of ∂D^2) and a_i be the corresponding vanishing cycle.

By Proposition 3.5, each directed (non-marked) fibration over D_i is classified by the conjugacy class $\{c_i, a_i\}$ of the real code. Thus, we can encode the fibration $\pi : X \to D^2$ by the *real Lefschetz chain*, $\{c_1, a_1\}, \{c_2, a_2\}, ..., \{c_n, a_n\}$.

Clearly, real Lefschetz chains are invariants of directed non-marked totally real elliptic Lefschetz fibrations over D^2, but they are not sufficient for classifying such fibrations. Additional information is needed, if for some i the real structure c_i has 2 real components and vanishing cycles corresponding to the critical values q_i and q_{i+1} are real, respectively, or if c_i has no real component. Indeed, in these cases the vanishing cycles corresponding to the critical values q_i and q_{i+1} can be the same curve, or they can be c_i-twin curves. If they are c_i-twin curves, then we mark $\{c_i, a_i\}^R$ the corresponding real code $\{c_i, a_i\}$ by adding R (here R refers to

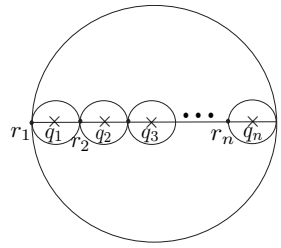


Fig. 9. Subdividing D^2 into smaller discs.

the rotation $R_{\frac{1}{2}}$ which interchanges c-twin curves). The real Lefschetz chain we obtain is called the *decorated real Lefschetz chain*. Figure 10 shows all possible configurations of the real locus associated to $\{c_i, a_i\}$ and $\{c_i, a_i\}^R$.

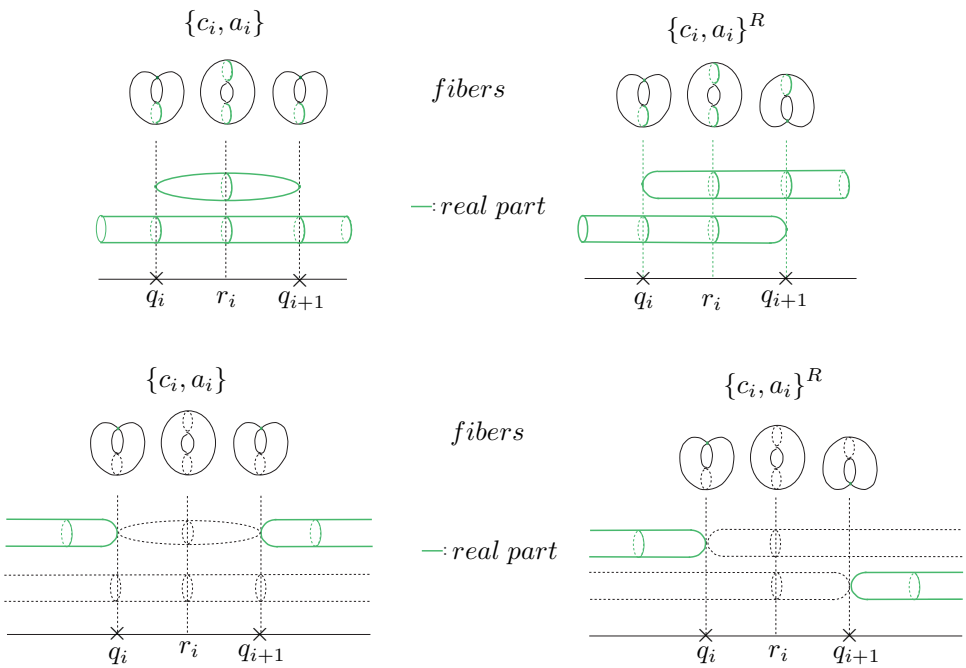


Fig. 10. Real parts of the fibrations associated to $\{c_i, a_i\}$ and $\{c_i, a_i\}^R$.

Theorem 8.1. *There exists a one-to-one correspondence between the decorated real Lefschetz chains and the isomorphism classes of directed non-marked totally real elliptic Lefschetz fibrations over D^2.*

Proof: Necessity is clear. As for the converse, we consider the unique class of directed non-marked elementary $\mathcal{RELF}$ (assured by Proposition 3.5) associated to each real code $\{c_i, a_i\}$. Then, we construct the required fibration by gluing elementary fibrations (from left to right) using the boundary fiber sum. As is discussed above, the boundary fiber sum is uniquely defined in the case when the real structure on the fiber where the sum is performed has 1 real component or when it

has 2 real components and acts on the vanishing cycle of the elementary fibration glued to right as a reflection. In the case when the real structure has 2 real components and acts on the last vanishing cycle of the already constructed fibration $\pi' : X' \to D^2$ as a reflection, the two possible boundary fiber sums are isomorphic by Lemma 7.2 since in this case we can apply T_{sing} to X' (by considering T_{sing} on a neighborhood N of the last critical value, as shown in Figure 11, and extending it to X' as the identity outside of $\pi'^{-1}(N)$). In all the other cases, the boundary fiber sum is defined uniquely by the decoration. □

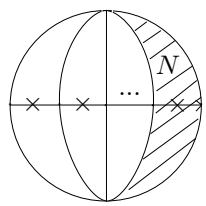


Fig. 11. Neighborhood over which T_{sing} is applied.

If c_1 is conjugate to c_{n+1}, then we can consider an extension of $\pi : X \to D^2$ to a fibration over S^2. As before, in the case when c_{n+1} has no real components or it has 2 real components and both a_1 and a_n are real, a decoration at infinity will be needed.

Proposition 8.2. *Let $\pi : X \to D^2$ be a real elliptic Lefschetz fibration associated to a decorated real Lefschetz chain. We assume that the real structures c_1 and c_{n+1} on the fibers over left and respectively right real point of ∂D^2 are conjugate. If c_{n+1} (and thus c_1) has 1 real component or if c_{n+1} (and thus c_1) has 2 real components and either c_{n+1} acts on the vanishing cycle a_n as a reflection, or c_1 acts on the vanishing cycle a_1 as a reflection, then π extends uniquely to a fibration over S^2. Otherwise, there are two extensions distinguished by the decoration at infinity.*

Proof: An extension of $\pi : X \to D^2$ to a fibration over S^2 defines a trivialization, $\phi : \Sigma_1 \times S^1 \to \pi^{-1}(\partial D^2)$ over the boundary ∂D^2. Two trivializations ϕ, ϕ' correspond to isomorphic real fibrations if $\phi^{-1} \circ \phi' : \Sigma_1 \times S^1 \to \Sigma_1 \times S^1$ can be extended to an equivariant diffeomorphism of $\Sigma_1 \times D^2$ with respect to the real structure $(c_{n+1}, conj) : \Sigma_1 \times D^2 \to \Sigma_1 \times D^2$. Let $\Phi_t = (\phi^{-1} \circ \phi')_t : \Sigma_1 \to \Sigma_1$, $t \in S^1$. Since there is no fixed marking, up to change of marking we assume that $\Phi_t \in \mathit{Diff}_0(\Sigma_1)$.

The real structure splits the boundary into two symmetric pieces, so instead of considering an equivariant map over the entire boundary we consider a diffeomorphism over one the symmetric pieces. Let $\Phi_t, t \in [0,1]$ denote the family of such diffeomorphisms. The family, Φ_t, $t \in [0,1]$ defines a path in $\mathit{Diff}_0(\Sigma_1)$ whose end points lie in the group $\mathit{Diff}_0^{c_{n+1}}(\Sigma_1)$; therefore, Φ_t defines a relative loop in $\pi_1(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_{n+1}}(\Sigma_1))$, and we are interested in the contractibility of this relative loop.

We consider the following exact sequence of the pair $(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_{n+1}}(\Sigma_1))$

$$... \to \pi_1(\mathit{Diff}_0^{c_{n+1}}) \to \pi_1(\mathit{Diff}_0) \xrightarrow{f} \pi_1(\mathit{Diff}_0, \mathit{Diff}_0^{c_{n+1}}) \xrightarrow{g} \pi_0(\mathit{Diff}_0^{c_{n+1}}) \xrightarrow{h} \pi_0(\mathit{Diff}_0) \to \pi_0(\mathit{Diff}_0, \mathit{Diff}_0^{c_{n+1}}) \to 0.$$

In the case when c_{n+1} has one real component, $\mathit{Diff}_0^{c_{n+1}}(\Sigma_1)$ is connected, so the map h is injective, so f is surjective. Therefore, elements of the group $\pi_1(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_{n+1}}(\Sigma_1), id)$ can be seen in $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$.

In all the other cases, $\mathit{Diff}_0^{c_{n+1}}(\Sigma_1)$ has two components. We mark one of the components to get the map h injection, when restricted to the marked component. Thus, g becomes the zero map, and so f is surjective over the marked component of $\mathit{Diff}_0^{c_{n+1}}(\Sigma_1)$. Note that decoration of real Lefschetz chains distinguishes one of the component of $\mathit{Diff}_0^{c_{n+1}}(\Sigma_1)$; hence, marking one component or other give the two extensions distinguished by the decoration. The distinctive feature of the case when c_{n+1} has 2 real components and acts a_n as a reflection (or c_1 acts on a_1 as a reflection) is that the transformation T_{sing} changes one marking to other, so the marking is not essential.

The proposition, thus, follows from Lemma 8.4 in which we show that any relative loop can be made contractible by means of some transformations T of the fibration $\pi : X \to D^2$. □

Let us first define the transformation T of real elliptic Lefschetz fibrations over D^2 that is defined over a regular slice N of D^2.

Let $\pi : X \to D^2$ be a directed $\mathcal{REL}\mathcal{F}$. We consider a real slice N of D^2 which contains no critical value, see Figure 12.

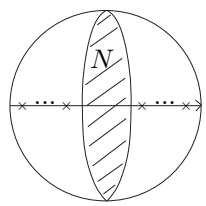


Fig. 12. Neighborhood over which T is applied.

Let $\xi : I \times I \to N$, $I = [0,1]$ be an orientation preserving diffeomorphism such that first interval correspond to the real direction on N. The fibration over N has no singular fiber; hence, it is trivializable. Let us consider a trivialization $\Xi : \Sigma_1 \times I \times I \to \pi^{-1}(N)$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_1 \times I \times I & \xrightarrow{\Xi} & \pi^{-1}(N) \\ \downarrow & & \downarrow \pi \\ I \times I & \xrightarrow{\xi} & N. \end{array}$$

Since N has no critical value, the isotopy type of the real structure on the fibers over the real part of N remains fixed. If the real structure c has 2 real components, then we consider the model $\varrho : \mathbb{C}/\mathbb{Z}^2 \to \Sigma_1$ and set $\bar{\varrho} = (\varrho, id) : \mathbb{C}/\mathbb{Z}^2 \times I \times I \to \Sigma_1 \times I \times I$ to define T as follows

$$\begin{array}{rccc} T' : & \mathbb{C}/\mathbb{Z}^2 \times I \times I & \to & \mathbb{C}/\mathbb{Z}^2 \times I \times I \\ & ((x+iy)_{\mathbb{Z}^2}, t, s) & \to & ((x+t+iy)_{\mathbb{Z}^2}, t, s). \end{array}$$

Then, we set $T = \Xi \circ (\bar{\varrho} \circ T' \circ \bar{\varrho}^{-1}) \circ \Xi^{-1}$ on $\pi^{-1}(N)$. Since T is the identity at $t = 0, 1$, we can extend T to X by the identity outside of $\pi^{-1}(N)$.

If c has 1 real component, then we construct the transformation T using $\varrho : \mathbb{C}/\Lambda \to \Sigma_1$. Similarly, if c has no real component, then we repeat the same using $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$.

Remark 8.3. 1. Since the transformation T is defined by a real rotation, T preserves the isomorphism class of the real Lefschetz fibration.

2. The map T depends only on the isotopy type of $\pi^{-1}(N)$.

Lemma 8.4. *Let $\pi : X \to D^2$ be a totally real elliptic Lefschetz fibration. We assume that there exists at least one vanishing cycle on which corresponding real structure acts as a reflection. Then, there exists a generating set for $\pi_1(\mathit{Diff}_0(\Sigma_1), id) = \mathbb{Z} + \mathbb{Z}$ consisting of transformations $T_\pm$ for some non-singular slices $N_\pm$.*

Proof: Let q_i be the critical value such that the real structure on a nearby regular real fiber acts on the vanishing cycle as a reflection. This assumption assures that the neighboring real fibers have one real component on one side and 2 real components on the other side of the critical value q_i. Without loss of generality we can assume that the real structure over a fiber over a real point which lies on the left of q_i has 2 real components. (The other case can be treated similarly.)

We choose an auxiliary $\mathbb{C}$-marking $(\{b, \bar{b}\}, \{\rho : \Sigma_1 \to F_b, \bar{\rho} : \Sigma_1 \to F_{\bar{b}}\})$ and fix an identification $\varrho : S^1 \times S^1 \to \Sigma_1$. Since the real structure has 2 real components, we can assumed that the induced real structure on $S^1 \times S^1$ is the reflection $(\alpha, \beta) \to (\alpha, -\beta)$. The real part consists of the curves $C_1 = (\alpha, 0)$ and $C_2 = (\alpha, \pi)$. Moreover, a representative of the vanishing cycle can be chosen as $(0, \beta)$. As $c_+ = t_{a_i} \circ c_-$ on $S^1 \times S^1$ the real part of c_+ is the curve, C_3, given homologically by $2\alpha - \beta$ (see Figure 13).

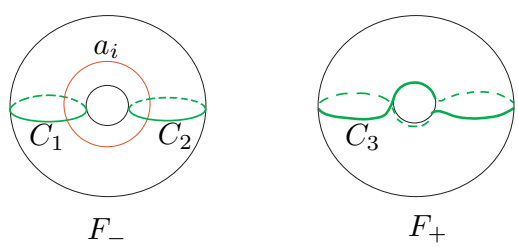


Fig. 13. Real fibers over the real points neighboring q_i.

We now consider two non-singular real slices N_-, N_+ of D^2 as shown in Figure 14. Let us suppose that the real fibers over N_- are identified to F_- while real fibers over N_+ are identified to F_+ (where $F_\pm$ are as shown Figure 13). Let C_3' and C_1' be curves on F_b obtained by pulling back $C_3 \subset F_+$ and $C_1 \subset F_-$, respectively. The curves C_3' and C_1' intersect at one point, so we can identify Σ_1 with $C_1' \times C_3'$ so that rotations along C_1' and C_3' generate the group $\mathit{Diff}_0(\Sigma_1, id)$. Hence, $\{T_+, T_-\}$ generates $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$. $\square$

Theorem 8.1 applies naturally to directed non-marked $\mathcal{REL}\mathcal{F}s$ over D^2 which admit a real section in which case real Lefschetz chain does not contain a real code (c_i, a_i) where the real structure has no real component. Besides, in the case when the real structure has 2 real components and the vanishing cycle is real, the decoration is not needed since the existence of a real section determines naturally

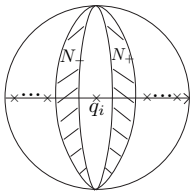


Fig. 14. Regular slices $N_\pm$.

the gluing. Moreover, the extension to a fibration over S^2 is uniquely defined by the section. Hence we have the following proposition.

Proposition 8.5. *Two directed $\mathcal{RELF}$s over S^2 admitting a real section and having the same real Lefschetz chain up to cyclic ordering are isomorphic.* $\square$

Remark 8.6. Indeed, the proposition holds even for fibrations with a fixed real section. If there are only real critical values, then the real sections are determined in a neighborhood of a real part. Moreover, over the real part one can carry one real section to another using the transformations T and *double* T_{sing}. Indeed, the *double* T_{sing} is defined for real Lefschetz fibrations with two critical values where the real structure extracted from the real fiber over a real point between the critical values acts on the vanishing cycles as a reflection. The model we use to define the *double* T_{sing} is as follows. Consider the disc D with two critical values as the double cover of a disc with one critical value branched at a regular real point. Let D_- and D_+ be two corresponding copies of the disk on the branched cover. By pulling back the fibration X_Ω over D we obtain a model fibration over $D_- \cup D_+$. Thus, we can apply T_{sing} at the same time to fibrations over D_- and D_+. The possible modifications of the section is shown in the Figure 15.

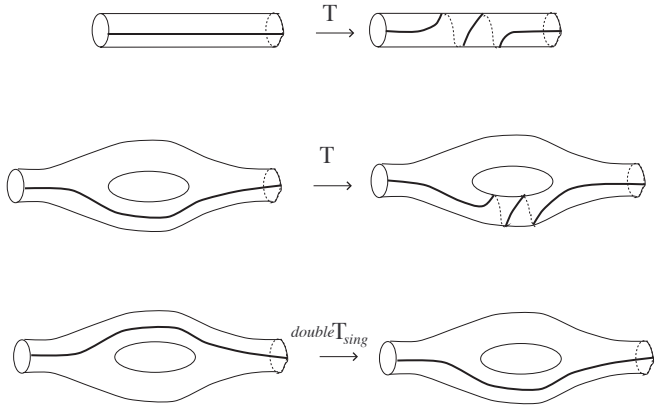


Fig. 15. Modification of the real section over the real part.

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