THERE IS A UNIQUE REAL TIGHT CONTACT 3-BALL

FERİT ÖZTÜRK AND NERMİN SALEPCİ

ABSTRACT. We prove that there is a unique real tight contact structure on the 3-ball with convex boundary up to isotopy through real tight contact structures. We also give a partial classification of the real tight solid tori with the real structure being antipodal map along longitudinal and the identity along meridional direction. For the proofs, we use the real versions of contact neighborhood theorems and the invariant convex surface theory in real contact manifolds.

1. INTRODUCTION

Motivated by the complex conjugation on complex manifolds, a *real structure* on an oriented $2n$-manifold X with boundary (possibly empty) is defined as an involution c_X on X which is orientation preserving if n is even and orientation reversing if n is odd and which has the fixed point set of half dimension, if it is not empty. We extend the definition of a real structure to smooth oriented $2n - 1$ dimensional manifolds. A *real structure* on an oriented $(2n - 1)$-manifold is an involution which is orientation preserving if n is even and orientation reversing if n is odd and the fixed point set of which is of dimension $n - 1$, if it is not empty. In particular, if M is the oriented boundary of an oriented 4-manifold X, it can be shown that a real structure on X restricts to a real structure on M. For a manifold W with arbitrary dimension, the pair (W, c_W) is called a *real (or c_W-real) manifold*. The set fix(c_W) of fixed points of the real structure is called the *real part* of the real manifold.

If there is a symplectic (or a contact) structure on a real manifold then one can talk about the compatibility of the structure with the real structure. As the simplest example, consider B^4 in $\mathbb{C}^2$ with the real structure given by the complex conjugation *conj* and the symplectic structure given by the standard symplectic form ω_{std}. Observe that *conj* is orientation preserving and $conj^*\omega_{std} = -\omega_{std}$. On $S^3 = \partial B^4$, the map $c_{S^3} = conj|_{S^3}$ is an orientation preserving involution. It is straightforward to verify that the induced contact form α_{std} on S^3 satisfies $c_{S^3}^*\alpha_{std} = -\alpha_{std}$ too. This toy example exhibits a general fact. Assume that (M, α) is a contact $(2n-1)$-manifold which is strongly symplectically fillable by (X, ω) and that there is a real structure c_X on X preserving M and satisfying $c_X^*\omega = -\omega$. Then $c_X^*|_M \alpha = -\alpha$. Thus we are motivated to give the following

Definition. (1) (due to C. Viterbo [13]) Let (X, ω) be a symplectic manifold of dimension $2n$ and let c_X be a real structure on X. If $c_X^*\omega = -\omega$, the form ω is called a *real (or c_X-real) symplectic form* with respect to c_X and c_X is called an *anti-symplectic involution*. The triple (X, ω, c_X) is called a *real symplectic manifold*.

(2) Let (M, ξ) be a contact manifold of dimension $2n - 1$ with contact distribution ξ. Let c_M be a real structure on M. If $(c_M)_* \xi = -\xi$, the contact structure ξ is called a *real (or c_M-real) contact structure* with respect to c_M. The triple (M, ξ, c_M) is called a *real contact manifold*.

In this note, we always assume that ξ is cooriented so that it is globally defined by the kernel of a contact form α and hence the real condition reads $c_M^* \alpha = -\alpha$ as well. Moreover, if M is oriented we assume that ξ is positive, viz. the orientation defined by $\alpha \wedge d\alpha$ agrees with the given orientation of M.

Real algebraic varieties with the symplectic form induced from the standard one on the ambient projective space are natural examples for real symplectic manifolds while links of real algebraic singularities constitute natural examples for real contact manifolds.

In dimension 3, the standard examples for real contact manifolds are
(i) for $M = \mathbb{R}^3$ or B^3, (M, ξ_{std}, c_{std}) with ξ_{std} given by $\alpha_{std} = dz + xdy - ydx$ and c_{std} rotation by π around y-axis;
(ii) $(T = S^1 \times D^2, \eta_{std}, c_2)$ with $c_2 : (\theta, (u, v)) \mapsto (\theta, (-u, -v))$ and η_{std} given by the 1-form $\cos\theta du - \sin\theta dv$ (u, v rectangular coordinates).

Given a real structure c on B^3, it can be isotoped to c' near S^2 so that c' coincides with c_{std} over S^2 ([2] or [12]). Furthermore, it follows from [8, Lemma 2.1] that c' is isotopic to c_{std} on B^3. Hence, the real structure in the first example is unique on B^3 up to isotopy through real structures. Whereas on a solid torus, there are four real structures up to isotopy through real structures [8]. Let us consider a solid torus $S^1 \times D^2$. We choose an oriented identification of T^2 with $\mathbb{R}^2_{(x,y)}/\mathbb{Z}^2$, where x direction corresponds the meridional direction of T^2 and fix the coordinates of $S^1 \times D^2$ as (y, x, t) where t is the radius direction of D^2 (in accordance with the convention in [9]). In these coordinates, the four possible real structures on the solid torus are:
(1) $c_1 : (y, (x, t)) \mapsto (-y, (-x, t))$;
(2) $c_2 : (y, (x, t)) \mapsto (y, (x + \frac{1}{2}, t))$;
(3) $c_3 : (y, (x, t)) \mapsto (y + \frac{1}{2}, (x, t))$;
(4) $c_4 : (y, (x, t)) \mapsto (y + \frac{1}{2}, (x + \frac{1}{2}, t))$.
Moreover any orientation preserving involution on T^2 can be extended to an involution on $S^1 \times D^2$. Such an extension is unique up to isotopy and fixes a core of the solid torus setwise [8]. We will refer the real structures c_1, c_2, c_3, c_4 frequently in the sequel. Likewise there are three involutions on the core circle up to isotopy through involutions: reflection, identity and rotation by π (antipodal map). We also denote them by c_1, c_2 (or id) and c_3 respectively. Note that $c_4|_{\text{core}} = c_3$ too.

Before we state the main result of the present work, recall that every tight contact structure on B^3 can be isotoped to a tight structure through tight contact structures so that it coincides with ξ_{std} on the boundary [5]. Moreover, up to isotopy relative boundary through tight contact structures, there is a unique such structure on B^3 and ξ_{std} is a representative [3]. With respect to ξ_{std}, ∂B^3 is convex and a dividing curve on it is the equator $\mathcal{E} = \{x^2 + y^2 = 1\}$. Also we remarked above that c_{std} is the unique real structure on B^3. However, it is not at all obvious that any real tight structure on B^3 is isotopic to ξ_{std} through real tight structures. We prove the uniqueness:

Theorem 1. *Up to isotopy through real contact structures making ∂B^3 convex, there is a unique real tight contact structure on B^3.*

To prove the theorem, we divide the 3-ball into real tight pieces. All are *standard* except one which is a c_3-real tight solid torus. Then we use our partial classification of c_3-real tight structures on the solid torus to conclude.

In Section 2, we prove real contact neighborhood theorems for real isotropic and contact submanifolds in real tight manifolds of arbitrary odd dimension. To do that we follow the classical proofs. In Section 3, we prove fundamental theorems of invariant convex surface theory in real tight 3-manifolds following [5]. We devote Section 4 to the study of c_3-real tight solid tori where we present a partial result towards the classification of c_3-real tight solid tori up to isotopy through real contact structures. The proof of Theorem 1 is also included in this section. The main approach in this section is convex surface theory involving dividing sets, bypass attachments and edge-rounding (see [5], [9]).

In what follows, when we say that a map $f : (M, c) \to (M', c')$ is *equivariant* we mean $f \circ c = c' \circ f$. In particular f is *c-equivariant* means $f \circ c = c \circ f$. Besides, when objects are sent to themselves under a real structure c, we call them *invariant* (or *c-invariant*) or sometimes *symmetric*. We say *antiequivariant* or *antisymmetric* (or sometimes *c-antisymmetric*) if they are sent to *minus* themselves. For example a submanifold might be invariant (symmetric) as a set while with an orientation it might be antisymmetric.

2. Real contact neighborhood theorems

In this section, we give several real contact neighborhood theorems. Although we will not use all in the present work, we present them to refer in our future work. We first observe that *Gray stability* is still valid in real setting. In its proof one can still employ Moser trick, which is valid in real setting too. Indeed, to find an equivariant isotopy ψ_t of contact forms, we consider it as the flow of a time-dependent antiequivariant vector field X_t. Since the differential equation for X_t is antiequivariant, the solution for X_t must be symmetric (see e.g. the proof of [4, Theorem 2.2.2]). It is easy to check that an appropriate version of Equivariant Darboux-Weinstein Theorem ([7, Theorem 22.1]) for symplectic manifolds is valid in the real contact setting. To do that, we go through the proofs of contact neighborhood theorems, minutely presented in [4], and make sure that the real versions of those theorems are valid. Below, we do not repeat the proofs; instead we point out where the proofs in op.cit. have to be altered to apply to our setting. Hence the following discussion is not self-contained so it must be read in parallel with the related proofs in [4].

In the sequel we will use the notion of a real symplectic bundle the definition of which is given here:

Definition. Let (E, B, π, ω) be a $2k$-dimensional symplectic vector bundle over a manifold B and the bundle map π. Assume there is an involution c_E on E which is a bundle isomorphism covering the identity and which gives a real structure on each fiber. The triple (E, ω, c_E) is called a *real symplectic bundle*.

Theorem 2. *(Real Contact Neighborhood Theorem for Real Parts) Let $(M_i, \xi = \ker \alpha_i, c_i)$, $(i = 0, 1)$, be two real contact manifolds, and L_i closed isotropic submanifold of M_i lying in the real part of M_i. Suppose that an equivariant real symplectic bundle isomorphism*

$$\Phi : ((TL_0)^{\perp d\alpha_0}/TL_0, d\alpha_0, Tc_0) \to ((TL_1)^{\perp d\alpha_1}/TL_1, d\alpha_1, Tc_1)$$

is given and that Φ covers a diffeomorphism $\phi : L_0 \to L_1$. Then there exist neighborhoods $\nu(L_i)$ of L_i and a diffeomorphism $F : \nu(L_0) \to \nu(L_1)$ such that:
(i) $F|_{L_0} = \phi$;
(ii) $F^*\alpha_1 = \alpha_0$;
(iii) F is equivariant.

Proof. We follow the proof of [4, Theorem 2.5.8]. Recall that (dropping the index i) there is a decomposition

$$T_L M = \langle R_\alpha \rangle \oplus TL \oplus (\xi|_L)/(TL)^{\perp d\alpha} \oplus (TL)^{\perp d\alpha}/TL$$

where R_α is the Reeb vector field of α and $\perp_{d\alpha}$ denotes the orthogonal complement with respect to $d\alpha$. We first observe that $(TL \oplus (\xi|_L)/(TL)^{\perp d\alpha}, d\alpha, Tc)$ is a real symplectic bundle. Also consider the real symplectic bundle $(TL \oplus T^*L, \Omega_L, s)$ where $s = \mathrm{id} \oplus -\mathrm{id}$ and Ω_L is defined by:

$$(\Omega_L)_p(X + \eta, X' + \eta') = \eta(X') - \eta'(X); \quad X, X' \in T_pL, \eta, \eta' \in T_p^*L.$$

It is straightforward to confirm that $s^*\Omega_L = -\Omega_L$.

Now recall (see [4, Lemma 2.5.7]) that there is a symplectic bundle isomorphism

$$(1) \qquad \begin{aligned} \mathrm{id}_{TL} \oplus \Psi : \left(TL \oplus (\xi|_L)/(TL)^{\perp d\alpha}, d\alpha, Tc\right) &\to (TL \oplus T^*L, \Omega_L, s), \\ (X, Y) &\mapsto (X, (\iota_Y d\alpha)|_{TL}) \end{aligned}$$

where we have already identified $J(TL)$ with $(\xi|_L)/(TL)^{\perp d\alpha}$ ([4, Proposition 2.5.5])

Thus having symplectic bundle isomorphisms onto $TL_i \oplus T^*L_i$ for each i, and using the symplectic bundle isomorphism Φ for the remaining symplectic part of $T_{L_i} M_i$, one can build a symplectic bundle isomorphism from $T_{L_0} M_0$ to $T_{L_1} M_1$. This finishes the linear algebra part of the proof in the lack of a real structure.

In the real setting, we first check the validity of [4, Lemma 2.5.7].

Lemma 3. *The bundle map in (1) is an equivariant isomorphism of real symplectic bundles.*

Proof. We just need to check that the map $\mathrm{id}_{TL} \oplus \Psi$ is equivariant. In fact,

$$\begin{aligned} (\mathrm{id}_{TL} \oplus \Psi)_p \circ (Tc)_p(X, Y) &= (\mathrm{id}_{TL} \oplus \Psi)_p(X, (Tc)_p(Y)) \\ &= \left(X, (\iota_{(Tc)_p(Y)} d\alpha)|_{T_pL}\right) \\ &= \left(X, d\alpha|_{T_pL}((Tc)_p(Y), (Tc)_p(\cdot))\right) \\ &= \left(X, (\iota_Y c^* d\alpha)|_{T_pL}\right) \\ &= \left(X, -\iota_Y d\alpha|_{TL}\right)_p \\ &= (s \circ \mathrm{id}_{TL} \oplus \Psi)_p (X, Y) \end{aligned}$$

For the third equation, we use the facts that $d\alpha$ is restricted on TL and Tc restricted to TL is the identity map, L lying in the real part. The fifth equation follows from the fact that $d\alpha$ is real with respect to c. $\qquad\square$

Since ϕ is equivariant, we have already showed that the map

$$T\phi \oplus (\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0) : \begin{aligned} &\left(TL_0 \oplus (\xi_0|_{L_0})/(TL_0)^{\perp d\alpha_0}, d\alpha_0, Tc_0\right) \\ &\qquad \to \left(TL_1 \oplus (\xi_1|_{L_1})/(TL_1)^{\perp d\alpha_1}, d\alpha_1, Tc_1\right) \end{aligned}$$

is an equivariant isomorphism of real symplectic vector bundles.

Let $\Phi_R : \langle R_{\alpha_0} \rangle \to \langle R_{\alpha_1} \rangle$ be the bundle map taking $R_{\alpha_0}(p)$ to $R_{\alpha_1}(\phi(p))$. It is equivariant, i.e. $\Phi_R \circ Tc_0 = Tc_1 \circ \Phi_R$; in fact $Tc_i(R_i) = -R_i$, since α_i is c_i-real. Then the bundle map

$$\tilde{\Phi} = \Phi_R \, \oplus (\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0) \, \oplus \Phi :$$
$$\begin{aligned} &\langle R_{\alpha_0} \rangle \oplus (\xi|_{L_0})/(TL_0)^{\perp d\alpha_0} \oplus (TL_0)^{\perp d\alpha_0}/TL_0 \\ &\qquad \to \langle R_{\alpha_1} \rangle \oplus (\xi|_{L_1})/(TL_1)^{\perp d\alpha_1} \oplus (TL_1)^{\perp d\alpha_1}/TL_1 \end{aligned}$$

is an equivariant bundle isomorphism of real bundles, preserving the structures induced from the contact structure on each subspace. Hence we are done with the linear algebraic part of the proof.

Now we extend this real symplectic vector bundle isomorphism to neighborhoods of L_0 and L_1. As in the proof of [4, Theorem 2.5.8], we choose similar tubular maps $\tau_i : (NL_i, Tc) \to (M_i, c)$ requiring one more condition that τ_i is equivariant. This can be achieved by choosing an equivariant Riemannian metric in a neighborhood of L_i in M_i. Hence we obtain a diffeomorphism $F := \tau_1 \circ \tilde{\Phi} \circ \tau_0^{-1} : \nu(L_0) \to \nu(L_1)$ which satisfies the conditions *(i), (iii)* of the theorem on L_i's. Finally, the pullback of α_1 in $\nu(L_0)$ can be equivariantly isotoped to α_0 thanks to the real version of Gray stability. Composing the time-1 diffeomorphism with the diffeomorphism above, condition *(ii)* is satisfied too. $\qquad\square$

In particular, since the 1-dimensional real part in any contact real 3-manifold is Legendrian, we obtain

Corollary 4. *(Real Contact Neighborhood Theorem for Real Knots) A real knot in an arbitrary real contact 3-manifold has a neighborhood equivariantly contactomorphic to the real contact 3-manifold $(S^1 \times D^2, \eta_{std}, c_2)$, defined in the Introduction, with L having the image $S^1 \times \{0\}$.*

We now state a similar theorem for neighborhoods of real knots up to *equivariant contact isotopy*. Its proof lies in the proof of Theorem 2 and essentially the same as the proof for the standard neighborhood theorem for Legendrian knots up to contact isotopy. The idea is to classify the real plane bundles over the knot as in the proof of Theorem 2 and then to use the exponential map corresponding to an equivariant Riemannian metric.

We denote by $tw(L, s)$ the twisting number of ξ along L with respect to a fixed trivialization s of the normal bundle of the Legendrian knot L in M. Similarly we can define the twisting number of a Legendrian arc relative end points. Note that in that case $tw(L, s)$ would be a real number. If s is understood or its omission from the notation does not affect the discussion, we write simply $tw(L)$.

Corollary 5. *(Real Contact Neighborhoods of Real Knots up to Equivariant Contact Isotopy) Let L be a real knot in a real contact 3-manifold (M, ξ, c). Given*

an identification with $S^1 \times D^2$ and the longitudinal framing along L, an invariant neighborhood of L is isotopic through real contact structures to the real contact 3-manifold

$$(S^1 \times D^2, \ker(\cos(tw(L)\theta)du + \sin(tw(L)\theta)dv), c_2)$$

where $(\theta, (u, v)) \in S^1 \times D^2$. If L is a real arc, then an invariant neighborhood of L is isotopic relative end points through real contact structures to

$$([0, 1] \times D^2, \ker(\cos(2\pi tw(L)\theta)du + \sin(2\pi tw(L)\theta)dv), c_2).$$

Proof. The proof of Theorem 2 shows that isotopy classification of the contact structures near L turns out to be topological so the first part of the corollary is obvious. As for the second part, we need to explain how to construct a smooth neighborhood of the arc L.

Consider an auxiliary Legendrian arc L' with end points p_0 and p_1 (such an L' exists; if necessary, to get L' one can add an auxiliary 3-dimensional 2-handle equipped with a real structure such that the core is real). By the previous part of the corollary, the union $L \cup L'$ is a Legendrian knot whose contact neighborhood is isotopic to $(S^1 \times D^2, \cos(tw(L \cup L')\theta)du + \sin(tw(L \cup L')\theta)dv, c_2)$. We can assume that by a rescaling of $L \cup L'$ the contact planes $\xi(p_0)$ and $\xi(p_1)$ stay fixed. We consider the neighborhood $\nu(L)$ of L given by $[p_0, p_1] \times D^2$. $\qquad\square$

One can prove a theorem similar to Theorem 2 for invariant contact submanifolds. We will use an equivariant version of the set-up for [4, Theorem 2.5.15]. Let $(M, \xi = \ker\alpha, c)$ be a real contact manifold and $(K, \xi|_K)$ a closed invariant contact submanifold of M. Then we have

$$TM|_K = TK \oplus NK = TK \oplus (TK)^{\perp d\alpha}/TK$$

as smooth vector bundles.

Theorem 6. *(Real Contact Neighborhood Theorem for Invariant Contact Submanifolds) Let (M_i, ξ_i, c_i), $(i = 0, 1)$ be two real contact manifolds, and $(K_i, \xi_i|_{K_i})$ closed invariant contact submanifolds of M_i. Suppose there is an isomorphism of equivariant conformal symplectic normal bundles*

$$\Phi : ((TK_0)^{\perp d\alpha_0}/T_0, Tc_0) \to ((TK_1)^{\perp d\alpha_1}/T_1, Tc_1)$$

covering a given equivariant contactomorphism $\phi : (K_0, \xi_0|_{K_0}) \to (K_1, \xi_1|_{K_1})$. Then there exist neighborhoods $\nu(K_i)$ of K_i and a diffeomorphism $F : \nu(K_0) \to \nu(K_1)$ such that:
(i) $F|_{K_0} = \phi$;
(ii) $F^*\alpha_1 = \alpha_0$;
(iii) F is equivariant;
(iv) $TF|_{(TK_0)^{\perp d\alpha_0}/TK_0}$ and Φ are equivariantly bundle homotopic as conformal symplectic bundle isomorphisms.

Proof. As in the proof of Theorem 2 above, it is enough to build an equivariant bundle map from $TM_0|_{K_0}$ to $TM_1|_{K_1}$ covering ϕ and inducing Φ.

For this it suffices for us to secure that the choices made in the proof of [4, Theorem 2.5.15] can still be made in the real setting. The rest of the proof is exactly the same as in the proof of that theorem. Now, we go into that proof. We first choose a real contact form α_i' for $\xi_i|_K$ $(i = 0, 1)$. Whatever the choice is, we want to scale α_i' to a *real* contact form α_i such that α_i gives 1 when contracted with

the Reeb vector field R_i' corresponding to α_i'. The scaling function f_i is found to satisfy $df_i = \iota_{R_i'} d\alpha_i$ on $TM_i|_{K_i}$. Since R_i' and α_i are antisymmetric, the function f_i is found to be equivariant. $\square$

3. Invariant convex surfaces

Let S be a closed connected oriented invariant surface in a real 3-manifold (M, c). If $c|_S$ is orientation preserving (i.e. S is symmetric), then the real points of S (if any) are isolated whereas if $c|_S$ is orientation reversing (i.e. S is antisymmetric), then the set of real points on S is a disjoint union (possibly empty) of circles.

Theorem 7. *Let S be an oriented, symmetric (respectively antisymmetric) convex surface in (M, ξ, c). Then there exists a symmetric (respectively antisymmetric) transverse contact vector field for ξ defined near S and thus an invariant dividing set on S.*

Proof. Since S is convex, there is a contact vector field Y over S by definition, i.e. $\mathcal{L}_Y \alpha = \mu \alpha$ for some contact form α defining ξ in a neighborhood νS of S and some function $\mu : \nu S \to \mathbb{R}$. We claim that if $c|_S$ preserves (respectively reverses) the orientation of S then the symmetric vector field $X = Y + c_* Y$ (respectively the antisymmetric vector field $X' = Y - c_* Y$) is a contact vector field. Below, calculations are done for X, the case of X' is similar.

If $c|_S$ preserves the orientation then Y and $c_* Y$ are on the same side of S. This assures that $Y + c_* Y$ is never zero and is transverse to S. Moreover we have:

$$
\begin{aligned}
\mathcal{L}_{Y + c_* Y} \alpha &= \mathcal{L}_Y \alpha + \mathcal{L}_{c_* Y} \alpha \\
&= \mu \alpha + c^* \mathcal{L}_Y c^* \alpha \\
&= \mu \alpha - c^* (\mu \alpha) \\
&= (\mu + \mu \circ c) \alpha,
\end{aligned}
$$

where $\mu + \mu \circ c : \nu S \to \mathbb{R}$.

Let $\Gamma \subset S$ denote the dividing set for X. By definition, $p \in \Gamma$ if and only if X_p is tangent to ξ_p or equivalently $\alpha_p(X_p) = 0$. The set Γ is invariant if $\alpha_{c(p)}(X_{c(p)}) = 0$. Since $c^* \alpha = -\alpha$, we have $0 = \alpha_p(X_p) = -(c^* \alpha)_p(X_p) = -\alpha_{c(p)} c_*(X_p)$. It was shown above that $(c_*(X))_p = X_{c(p)}$, thus $c(\Gamma) = \Gamma$. $\square$

Remark 1. Recall that the characteristic foliation of a contact structure inherits an orientation from the orientation of S and the coorientation of the contact structure ξ. Namely, suppose Ω is a volume form on S and α a 1-form defining ξ. The characteristic foliation of ξ is defined as the integral curves of the vector field X satisfying $\iota_X \Omega = \alpha|_S$ [4, Lemma 2.5.20]. This equality determines the orientation of the characteristic foliation. It follows from the equality $c^* \alpha = -\alpha$ that the real structure reverses (respectively preserves) the orientation of the characteristic foliation if $c|_S$ is orientation preserving (respectively reversing). Let us note that the dividing curve is also canonically oriented using the orientation of the characteristic foliation and of S. Thus, the orientation of the dividing curve is reversed by c regardless of $c|_S$ being orientation reversing or preserving. Notice that Theorem 7 showed that the nonoriented dividing set is invariant.

Now we prove an existence theorem for symmetric oriented convex surfaces.

Proposition 8. *Let S be an invariant closed surface in a real contact 3-manifold (M, ξ, c). Assume S is oriented and $c|_S$ is orientation preserving. Then there is a surface S' equivariantly isotopic and C^∞-close to S such that S' is convex for ξ.*

Proof. Consider the quotient $q : S \to \bar{S} = S/c$. Since $c|_S$ is orientation preserving, $\text{fix}(c|_S)$ is either empty or a finite number of points, say $p_1, \dots, p_k$. Let us denote by X the antisymmetric vector field on S which is the infinitesimal generator of the restriction of the characteristic foliation $S|_\xi$. Note that the fixed points p_i (if they exist) are regular points of X. In fact, at a singular point r of X, ξ_r is tangent to S. If r were fixed then $c_* \xi_r$ would be ξ_r contradicting with ξ being c-real. Observe that $S|_\xi$ does not project to an orientable foliation on $\bar{S}$; nevertheless, $q_*(S|_\xi)$ describes a line field $\mathfrak{F}$ on $\bar{S}$. Since each p_i is regular, $q(p_i) \in \bar{S}$ is a *nontrivial singular point*, around which the behavior of $\mathfrak{F}$ is as depicted in Figure 1.12 of [1]. Hence our situation satisfies the hypothesis of [1, Theorem 2]. From its proof and the openness of the contact condition, it follows that there is a c-antisymmetric Morse-Smale vector field X_{ξ_1} on S arbitrarily close to X which corresponds to a contact structure ξ_1, and which is isotopic to X through c-antisymmetric vector fields X_{ξ_t}, $t \in [0,1], \xi_0 = \xi$. Using Gray stability in real setting, we deduce that there is a surface S' equivariantly isotopic to S such that the characteristic foliation $S'|_\xi$ is Morse-Smale. Since a surface in a contact 3-manifold with Morse-Smale characteristic vector field is convex, the proof follows [5, Proposition II.2.6].   □

In the hypothesis of the theorem above, it is more problematic if we assume that $c|_S$ is orientation reversing. In that case there are two main difficulties. First, the quotient $\bar{S}$ may be nonorientable. The denseness of Morse-Smale vector fields is still an open question for nonorientable surfaces (see e.g. [11]). Second, $\bar{S}$ may have boundary. The boundary curves are in general not Legendrian; nor the field of line elements on $\bar{S}$ is in a general position with respect to the boundary. These make it difficult to conclude the denseness of Morse-Smale vector fields in our setting.

From the existence of an invariant transverse contact vector field, it follows immediately that S has an invariant neighborhood with the contact form vertically invariant (see e.g. [4, Lemma 4.6.19] and the discussion following it). Furthermore, we have:

Proposition 9. *(Real version of Giroux's Flexibility Theorem) Suppose S is an invariant closed convex surface in $(M, \xi = \ker \alpha, c)$; S_ξ the (oriented) characteristic foliation and Γ an antisymmetric oriented dividing set. If F is another oriented symmetric (or antisymmetric) singular foliation with the same oriented dividing set, and if the orientations of F and S_ξ agree in a neighborhood of Γ, then there is an isotopy ϕ_s ($s \in [0,1]$) of S satisfying: (i) $\phi_0 = id$, (ii) ϕ_s is equivariant; (iii) $\phi_s|_\Gamma = id$; (iv) $\phi_s(S)$ is convex and (v) $\xi|_{\phi_1(S)} = F$.*

To prove this version of the theorem, it is enough to observe that every step of the original proof (see [5, Proposition II.3.6]) can be made c-invariant, leading an antisymmetric differential equation in the standard Moser-type argument. That differential equation has a unique solution which is necessarily symmetric. Note that the conclusion (iii) of Proposition 9 can be obtained following [4, Remark 4.8.12] or the proof of [9, Theorem 3.4.].

Corollary 10. *The surface S being as above, let ξ and ξ' be two contact structures with respect to which S is convex. Let Γ and Γ' be the corresponding antisymmetric dividing sets on S. If there is an equivariant orientation preserving diffeomorphism $f : S \to S$ which is equivariantly isotopic to identity and which sends Γ to Γ' preserving the orientation then ξ and ξ' are isotopic through real contact structures in a sufficiently small neighborhood of S.*

Proof. Via the isotopy Γ' is made to coincide with Γ while ξ' is changed to ξ''. We can assume that in a neighborhood of Γ the oriented characteristic foliations for ξ and ξ'' coincide, since f preserves the orientation of S. Now ξ and ξ'' have Γ as their common oriented dividing curve. The proof of Proposition 9 shows that ξ'' can be isotoped to ξ through real contact structures in a sufficiently small neighborhood of S. $\qquad\square$

4. CLASSIFICATION OF REAL TIGHT CONTACT STRUCTURES ON $(S^1 \times D^2, c_3)$

We consider a contact solid torus $S^1 \times D^2$ whose boundary T^2 is convex. Let us denote the dividing set of T^2 by Γ which is determined up to isotopy by the number of connected components $\#\Gamma$, and their slope s. We choose coordinates on solid torus as described in the Introduction. Using the fact that every curve is isotopic to a linear curve on $\mathbb{R}^2/\mathbb{Z}^2$ the slope of the dividing curve is defined as the slope of the corresponding linear curve on $\mathbb{R}^2/\mathbb{Z}^2$.

In this section, we prove the following classification theorem for real tight contact structures on $S^1 \times D^2$ with $\#\Gamma = 2$ up to c_3-equivariant isotopy. Note that with respect to the above coordinates the real structure c_3 is given by $(y, (x, t)) \mapsto (y + \frac{1}{2}, (x, t))$.

Theorem 11. *Up to equivariant isotopy through tight structures relative boundary there is a unique real tight contact structure on $(S^1 \times D^2, c_3)$ having convex boundary with $\#\Gamma = 2$ and with slope $\pm\frac{1}{2k+1}$ ($k \in \mathbb{Z}, k \geq 0$). If the slope is $\pm\frac{1}{2k}$ ($k \geq 0$) or $-k$ ($k > 1$) there is no real tight contact structure on $(S^1 \times D^2, c_3)$.*

In the proof of the theorem, we will see that one can peel off $S^1 \times D^2$ by means of c_3-invariant double slices and possibly a single basic slice. The existence of these leads to an overtwisted structure except in the case when the slope equals $\frac{1}{2k+1}, k \in \mathbb{Z}$. In these cases, the real tight structure is unique and a representative is the form $\cos((2k + 1)\theta)du + \sin((2k + 1)\theta)dv$.

Using the above theorem we can prove the main theorem of this article:

Proof of Theorem 1. Without loss of generality we may assume that the real structure on the given 3-ball B^3 is c_{std}. With this real structure, the arc $L = B^3 \cap y$-axis is the real part.

On the convex boundary S^2 of (B^3, c_{std}) the dividing set Γ is connected [6] and can be chosen symmetric (Theorem 7). Any such oriented symmetric circle on S^2 can be equivariantly ambiently isotoped (preserving the orientation) to the equator $\mathcal{E} = \{z = 0\}$ oriented counterclockwise in x-y plane. Note that $\mathcal{E}$ is an invariant dividing set for ξ_{std}. Let $\alpha = dz + xdy - ydx$ so that $\xi_{std} = \ker \alpha$. Thus we can use Corollary 10 to deduce that ξ can be equivariantly isotoped to ξ' in a neighborhood of S^2 so that in that neighborhood ξ' and ξ_{std} coincide, with the dividing set $\{z = 0\}$ defined by the contact vector field $X = 2z\partial_z + y\partial_y + x\partial_x$. Furthermore, ξ' in a symmetric neighborhood U of L in B^3 can be equivariantly isotoped relative end points so that around L, it has the standard form prescribed in Corollary 5. Guided by the condition on the end points of L, we may choose the form $\beta = -\sin(2\pi(m - \frac{1}{4})y)dx + \cos(2\pi(m - \frac{1}{4})y)dz$, $m \in \mathbb{Z}$ as the contact form defining ξ' around U. Note that with respect to β, the vector field $Y = -x\partial_x - z\partial_z$ is a symmetric transverse contact vector field with the dividing set $x \sin(2\pi(m - \frac{1}{4})y) - z \cos(2\pi(m - \frac{1}{4})y) = 0$.
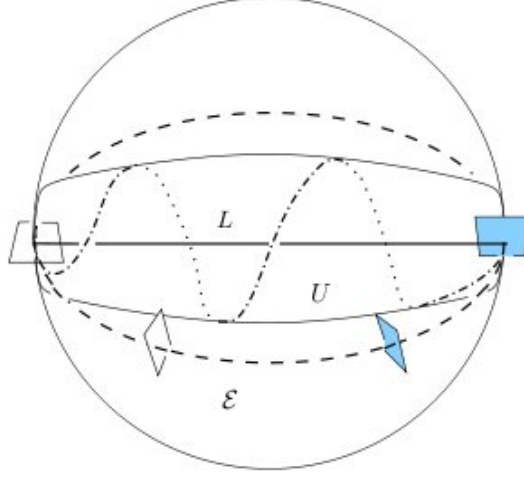
FIGURE 1. The 3-ball with a neighborhood of its boundary and of the real arc L in standard form.

Topological boundary ∂U of U meets S^2 at small disks $D_\pm$ around the real points $p_\pm = (0, \pm 1, 0)$ on S^2. Near these disks, β and ξ' can be made to coincide and the boundary of $T = B^3 \setminus (int(U) \cup D_\pm)$ can be smoothened out and be made convex.

To be precise, let us demonstrate how we smooth ∂T out and make it convex. Observe that $\partial D_\pm$ is not Legendrian (nor transverse). Therefore the usual edge-rounding argument (see [9, Lemma 3.11]) does not work to make ∂T smooth. Nevertheless, we can use the model below around D_-. The model can easily be adapted for smoothening around D_+.

First of all, we observe that on U near D_- (respectively near D_+) contact forms α and β are equivariantly isotopic (by a linear isotopy) and the vector field Y is a contact vector field not only for β but also for α, thus for each contact structure obtained through the isotopy. It is not hard to see that throughout this isotopy each component of the dividing set defined by $\beta(Y) = 0$ connects to the corresponding component of the dividing set defined by $\alpha(Y) = 0$. Therefore, it is enough to work with α near D_- (respectively near D_+) all through the smoothening. Consequently, at the end of the smoothening process the change in the dividing set does not depend on β.

Let us build a local model for smoothening around D_-. In the standard real tight $(\mathbb{R}^3, \ker \alpha, c)$, with $c(x, y, z) = (-x, y, -z)$, we consider the plane $\{y = -1, x^2 + z^2 \geq r^2, r \ll 1\}$ (representing a part of S^2) and the cylinder $\{x^2 + z^2 = r^2, y \in [-1, -1 + \epsilon], 0 < \epsilon \ll r\}$ (representing a part of $\partial U \setminus int(D_-)$ near ∂D_-). They intersect along the circle $\{x^2 + z^2 = r^2, y = -1\}$. We round the edge by the *quarter* torus τ obtained as the rotation about the y-axis the quarter circle $\{(y + 1 - \delta)^2 + (z - r - \delta)^2 = \delta^2, y \leq -1 + \delta, z \leq r + \delta\}$ $(0 < \delta < \epsilon)$. The vector field $W = kx\partial_x + s\partial_y + (sx + kz)\partial_z$ is a symmetric contact vector field with respect to ξ_{std} and is transverse (pointing outward) to the smoothened surface for $s, k < 0$, k sufficiently greater than s. By a straightforward calculation it follows that the dividing set around smoothened region can be obtained by connecting each

component of the dividing set on $\partial U \setminus int(D_-)$ to the corresponding component of the dividing curve on $S^2 \setminus int(D_-)$ regardless of the choices of k, s.

To conclude the proof, we note that around D_- contact vector fields W and X as well as W and Y are equivariantly isotopic through contact vector fields since their dividing sets are equivariantly ambiently isotopic. Thus, near D_- we can replace X with W around S^2 (respectively Y with W around ∂U). Moreover, by keeping track of the dividing set under the linear isotopy it can be seen that each component of the dividing set acquires a rotation by $-\frac{3\pi}{4}$. Similarly under the linear isotopy from W to Y, the rotation in the dividing set is 0. The computation near D_+ is similar (with $k < 0, s > 0$).

As a result, T with rounded edges is a real tight solid torus $(S^1 \times D^2, c_3)$ with symmetric convex boundary whose dividing set has 2 components with slope $\frac{m-1}{1}$. We claim that m cannot be positive. Otherwise, there would be $L' \subset L$ with $tw(L') = 1 - \frac{1}{4}$. Take an invariant ball around L'. The above discussion shows that we get a c_3-real solid torus with symmetric convex boundary whose dividing set has 2 components with slope 0 but there is no tight solid torus with zero slope. Now, for m nonpositive, it follows from Theorem 11 above that such a c_3-real tight solid torus exists if and only if $m = 0$. Moreover in that case the c_3-real tight structure on the solid torus is unique up to isotopy through real contact structures. ∎

Theorem 11 emerges from propositions and lemmas presented below. In the following sections we fix c_3 as the real structure and sometimes do not refer it explicitly. We always assume that $\#\Gamma = 2$ on the tori below.

A convex torus in a tight contact manifold has a standard form as follows [10]. The dividing set Γ has an even number of homotopically essential, parallel curves. In between each pair of dividing curves there is a Legendrian curve (called *Legendrian divide*) parallel to the dividing curves. The leaves of the characteristic foliation are transverse to Γ and are Legendrian; they define a family of closed Legendrian curves (called *Legendrian rulings*) exhausting the torus. Thanks to Theorem 7, the standard form for tori above can be made symmetric, so that the dividing set and the set of Legendrian divides become symmetric. Furthermore, using the real version of Giroux's Flexibility Theorem (Proposition 9) the characteristic foliation can be made invariant and the slope of the Legendrian rulings can have any value except that of Legendrian divides. We say that such a torus is in *symmetric standard from*.

4.1. **Non-existence results for real contact solid tori.** First, let us note that there is no tight contact structure on $S^1 \times D^2$ with convex boundary with slope 0 and $\#\Gamma = 2$. It is simply because each Legendrian divide, being of slope 0, bounds a meridional disk in $S^1 \times D^2$. This disk is an overtwisted disk since along L the contact planes remain tangent to T^2, so the surface framing and the contact framing are the same, i.e. the twisting of L is 0 with respect to the meridional disk.

Proposition 12. *There is no real tight contact structure on $(S^1 \times D^2, c_3)$ with convex boundary with slope ∞ and $\#\Gamma = 2$.*

Proof. Consider a real convex torus in the standard form whose dividing curves have slope ∞. With respect to c_3, each component of Γ is fixed. Moreover, this action preserves the orientation of Γ which is not possible (as already discussed above in Remark 1). □

Remark 2. A diffeomorphism of $T^2 = S^1 \times \partial D^2$ takes meridians to meridians if and only if it extends to a diffeomorphism of $S^1 \times D^2$ (see e.g. [8, Theorem 2.4]). In particular, a Dehn twist along a meridian of T^2 extends to a diffeomorphism of $S^1 \times D^2$. Thus by changing $S^1 \times D^2$ using a diffeomorphism, which is a Dehn twist along a meridian when restricted to T^2, one can change the slope, in particular, make the slope negative. A similar idea applies in the real case. However, in order not to spoil the symmetry one should consider a pair of Dehn twists along symmetric meridional curves. Such a pair of Dehn twists will be called an *equivariant meridional Dehn twist*. Note that by means of equivariant Dehn twists we can always adjust the slope s so that $s = -\frac{p}{q} \leq -\frac{1}{2}$ with $p, q \in \mathbb{Z}^+$, $(p, q) = 1$.

Corollary 13. *There is no real tight contact structure on $(S^1 \times D^2, c_3)$ with convex boundary with slope $\frac{1}{2k}$, $(k \in \mathbb{Z}, k \neq 0)$ and $\#\Gamma = 2$.*

Proof. As it is discussed in Remark 2, by sufficiently many equivariant meridional Dehn twists, we can change the slope to ∞ without changing the number of dividing curves. Hence, the result follows from Proposition 12.   □

4.2. **Existence results for real contact solid tori.**

Proposition 14. *Up to equivariant isotopy through real tight structures relative boundary there is a unique c_3-real tight contact structure on $S^1 \times D^2$ with convex boundary with $s = \frac{1}{2k+1}$ $(k \in \mathbb{Z})$, and $\#\Gamma = 2$.*

Proof. The proof is the c_3-invariant version of the proof of [9, Proposition 4.3]. Here, we outline that proof and point out required modifications. First let us note that by sufficiently many equivariant meridional Dehn twists we can make the slope -1. Hence it is enough to consider just the case $s = -1$.

By the real version of Giroux's Flexibility Theorem (Proposition 9), we suppose that the Legendrian rulings have slope 0 so that each bounds a meridional disk in $S^1 \times D^2$. Let L be a Legendrian ruling and D be the disk it bounds. By [9, Proposition 3.1], we can perturb D relative L to make it convex. Since the slope is -1, we have $\mathrm{tw}(L, D) = -1$ and so the dividing set on D must be connected. It is a path with end points on the boundary.

Now, we consider $S^1 \times D^2 \setminus (D \cup T^2 \cup c_3(D))$ which consists of two disjoint open 3-balls that are mapped to each other via c_3. Consider the closure of one of those balls. After edge rounding and perturbing we obtain a 3-ball whose boundary is convex and has a connected dividing set. Up to isotopy relative boundary, there is a unique tight contact structure on it [3]. Hence define an equivariant isotopy on $S^1 \times D^2$ relative boundary as follows: apply the isotopy on one of the 3-balls relative its boundary to obtain the model of the unique tight structure on the ball and perform the corresponding isotopy on the other ball antisymmetrically.   □

We have an interesting corollary:

Corollary 15. *Suppose L is a Legendrian knot in a real tight contact manifold (M, ξ, c) and in a neighborhood of L, c is isotopic to c_3 through real structures. Then the twisting number of L with respect to any c-invariant framing is always odd.*

Proof. Take a small invariant neighborhood of L with convex boundary on which c acts as c_3. Identify the boundary with $\mathbb{R}^2/\mathbb{Z}^2$ so that the longitude determined

by the framing corresponds to $(0,1)$ while meridian corresponds to $(1,0)$. With respect to this identification, the slope of the dividing curves on the boundary becomes $1/\mathrm{tw(L)}$. It follows from Corollary 13 and Proposition 14 that such a real tight contact solid torus exists if and only if $\mathrm{tw(L)}$ is odd. $\qquad\square$

To cover the remaining case of Theorem 11, we first split the solid torus into two pieces as follows. Without loss of generality, we may assume that its core is c_3-invariant. There is an invariant Legendrian curve L isotopic to the core. In fact, we can always perturb the core to get a Legendrian curve, via creating small zigzags in the front projection. Since each zigzag creation is local, we can create each zigzag and its symmetric counterpart invariantly to obtain an invariant Legendrian L at the end.

By Corollary 15 the twisting number of L (with respect to the longitude $S^1 \times \{(1,0)\} \subset S^1 \times D^2$) is odd. Take an invariant neighborhood $\nu(L)$ of this curve such that the boundary is invariant and convex (Theorem 2 and Proposition 8). Consider the decomposition of the solid torus into two c_3-invariant pieces: $\nu(L)$ and the closure of the complement of $\nu(L)$, which is the thick torus $T^2 \times I$, $(I = [0,1])$, whose boundary is the disjoint union of two convex tori with slopes $\pm\frac{1}{2k+1}, k \in \mathbb{N}$ and s respectively for inner and outer boundaries. Indeed, as discussed in Remark 2 we can assume that $s = -\frac{p}{q} \le -\frac{1}{2}$ with $p,q \in \mathbb{Z}^+$, $(p,q) = 1$, in which case the slope of the dividing curve on $\partial\nu(L)$ is necessarily $-\frac{1}{2k+1}$. (This can be shown by an argument similar to that presented in the proof of Lemma 16 below.)

By Proposition 14 we know that there is a unique real tight structure on $\nu(L)$. In the next section our aim is to figure out the possible real tight contact structures on the thick torus with convex boundary with slopes $-\frac{p}{q}$ and $-\frac{1}{2k+1}$. The tool we use is the method of layering thick tori by means of bypasses introduced in [9].

4.3. **Factoring c_3-real tight solid tori.** Now consider the real thick torus $(T^2 \times I, c_3)$ with T^2 identified by $\mathbb{R}^2/\mathbb{Z}^2$ as before and suppose that we are given a real tight contact structure ξ on $T^2 \times I$ making the boundary convex. We denote by s_k ($k \in \{0,1\}$) the slope of the dividing curve Γ_k on $T_k = T^2 \times \{k\}$. We also denote by $T(s_0, s_1)$ a c_3-real tight thick torus that has a convex boundary $T_0 \cup T_1$ with slopes s_0 and s_1 respectively and with dividing sets Γ_0 and Γ_1, $\#\Gamma_0 = 2$, $\#\Gamma_1 = 2$. We will show that if $T^2 \times I$ lies in a solid torus then it is *minimally twisting* and it can be decomposed into *symmetric basic slices* and *symmetric double slices* (see [9] for the definitions of minimally twisting, basic slice and bypass). A double slice is essentially two basic slices glued together:

Definition. Given the real tight solid torus $T(s_0, s_1)$ as above, it is called a *symmetric double slice* if on the Farey tessellation of the unit disk one can connect s_0 and s_1 in such a way that the minimal number of edges of the paths in the counterclockwise direction from s_1 to s_0 is two.

Let us note the following fact from [9] for the sake of completeness and for reference in the sequel. Its proof does not require the real setting.

Lemma 16. *Let $S^1 \times D^2$ be a tight contact solid torus whose boundary is convex with slope $s < 0$ and $\#\Gamma = 2$. Denote by L a Legendrian curve which is isotopic to the core and by s_L the slope of the convex boundary of the contact neighborhood of L in $S^1 \times D^2$. Then s_L lies in $[s,0)$.*

Proof. The idea of the proof is similar to that of Proposition 4.15 in [9]. Assume the contrary: there is a Legendrian curve such that s_L is outside the interval $[s, 0)$. As it is discussed in [9] by successive bypasses inwards, we can peel off the solid torus to get solid tori with convex boundary T_i of slope s_i. Let s_i denote the slope of the convex boundary of the solid torus after i^{th} bypass. Then on the Farey tessellation of the unit disk there is an edge from s_i to s_{i+1} in the counterclockwise direction [9, Lemma 3.15]. Thus a counterclockwise path from s to s_L on the Farey tessellation, obtained as a result of some number of bypasses, necessarily passes through 0 because s_L is outside $[s, 0)$. This gives a contradiction since there is no tight solid torus with slope 0.   □

Let us reconsider the real tight thick torus $T(-\frac{1}{2k+1}, -\frac{p}{q}) \subset S^1 \times D^2$. It follows from the proof above that $-\frac{1}{2k+1} \geq -\frac{p}{q}$; in fact otherwise there would exist a tight solid torus with slope 0. It follows similarly that $T(-\frac{1}{2k+1}, -\frac{p}{q})$ is minimally twisting. By Proposition 9 we can assume that the invariant Legendrian rulings on the boundary tori have slope $-\frac{1}{2j}$ for some $j \in \mathbb{Z}^+$ with $2j > 2k + 1$. Consider an annulus A bounded by a pair of Legendrian rulings on T_0 and T_1. By [9, Proposition 3.1], A can be made convex relative its boundary. Set $S_i = \partial A \cap T_i$, $i = 0, 1$. If the difference between the intersection numbers of S_1 and S_0 with a component of the dividing curve of T_1 is greater than or equal to 3, namely if

$$(2) \qquad ((q, -p) - (2k+1, -1)) \cdot (2j, -1) = 2jp - 2j + 2k + 1 - q \geq 3,$$

then there is at least a pair of bypasses that can be performed on A consecutively. Furthermore, after performing the first, the second bypass disk can be moved to the symmetric position of the first bypass region so that a pair of c_3-symmetric bypass operations can be performed one after the other. Given $p \neq 1$ and q, the inequality at (2) can always be satisfied for a sufficiently large j. Note that if $p = 1$, we must have $2k + 1 - q \geq 3$.

For $2j > 2k + 1$, i.e. $-\frac{1}{2j}$ outside $[-\frac{p}{q}, -\frac{1}{2k+1}]$, a new pair of bypasses can always be performed as long as (2) is satisfied for an appropriately chosen j. Note that when equality is attained then we have a symmetric double slice. Hence we have proven

Proposition 17. *Suppose $T(s_0, s_1)$ ($s_0 \neq s_1$) is a real tight thick torus with ξ minimally twisting and $\#\Gamma_0 = 2$, $\#\Gamma_1 = 2$. Then $T(s_0, s_1)$ assumes a factoring into a number of symmetric double slices and possibly a symmetric basic slice. If the minimal number of edges on the paths from s_1 to s_0 in the counterclockwise direction on the Farey tessellation of the unit disk is odd, there is a single symmetric basic slice in the factoring. If that number is even, there is no basic slice.*

In particular, if $s_1 \in \mathbb{Z}^-$ and $s_0 = -1$ then the last slice is either the thick torus $T(-1, -3)$ or $T(-1, -2)$.

If $s_1 \in \mathbb{Z}^-$ and $s_0 = -\frac{1}{2k+1}$ ($k \in \mathbb{Z}^+$) then the last slice is either $T(-\frac{1}{2k+1}, -\frac{1}{2k})$ or $T(-\frac{1}{2k+1}, -\frac{1}{2k-1})$.

4.4. **Some nonexistence results for c_3-real basic and double slices.** Proposition 17 suggests the investigation of the real tight structures on symmetric basic and double slices. This is what we are going to do now.

The peeling-off process above shows that there is a single basic slice occurring if the total number of bypasses performed is odd. This basic slice has to be

$T(-\frac{1}{2k+1}, -\frac{1}{2k})$. However there can be no such thick torus in the initially given solid torus since there is no solid torus with slope $-\frac{1}{2k}$ (Corollary 13).

For double slices, we observe that if the total number of bypasses performed during the peeling-off process above is even, then the last thick torus is the double slice $T(-\frac{1}{2k+1}, -\frac{1}{2k-1})$ or $T(-1, -3)$. For $T(-1, -3)$, we have the following

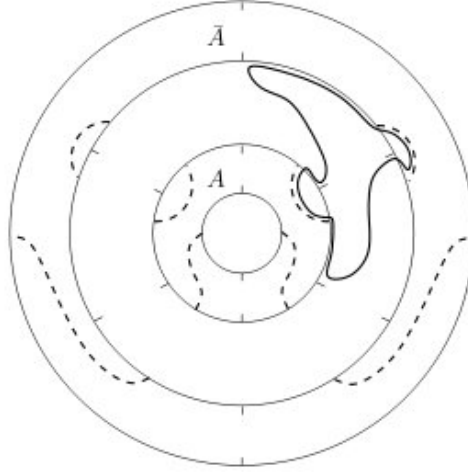**Proposition 18.** *There is no c_3-real tight $T(-1, -3)$.*

Proof. We assume that there exists a real contact thick torus with the required properties and conclude that in each case possible, the contact structure must be overtwisted. Suppose the slope of Legendrian rulings is set to be 0 on both T_0 and T_1. Let A denote the convex annulus obtained by a small perturbation relative boundary of an annulus bounded by Legendrian rulings on T_0 and T_1. Set $S_1 = \partial A \cap T_1$ and $S_2 = \partial A \cap T_2$. We observe that $tw(S_1, T_1) = -3$ and $tw(S_0, T_0) = -1$ Therefore on A there are at least two arcs of the dividing set which start and end at S_1 [9, Proposition 3.17].

The real structure c_3 gives a disjoint copy $\bar{A} = c_3(A)$ of A which has the same characteristic foliation (and the same dividing set) as A except the signs of the singularities and the orientation of the foliation reversed. The complement of $\bar{A} \cup A$ consists of two connected components, each of which has as closure a topological solid torus. These solid tori are symmetric with respect to c_3. Let S denote one of those solid tori oriented outwards. Then we consider the contact solid torus $\hat{S}$ obtained by rounding the edges of S [9, Lemma 3.11]. While rounding the edges the dividing set is constructed by going towards right at each surface change and following the dividing curve first encountered [9].

We have several possible configurations of the dividing set on the annulus A. First let us note that the dividing set on A can not have a closed component which bounds a disk on A since in that case the contact structure would be overtwisted [6]. Note also that by an argument similar to Step 1 of the proof of [9, Proposition 4.7] one can show that every dividing arc starting from the inner boundary of A must end at the outer boundary of A. Furthermore, those arcs can be put in a *standard* position in which the number of their rotations around the inner boundary can be made 0 (this is called *holonomy* in [9]; there the interior of the chosen annulus is pushed one turn around the torus to change the holonomy by ± 1.) We do not present how to attain 0 holonomy in or setting since the discussion below is insensitive to the holonomy. Those said, there are three possible configurations for the dividing curve on A (up to holonomy), depicted in Figure 2.

With those configurations, Figure 3 proves that there is no real tight $T(-1, -3)$ leading to the configuration in Figure 2(a). Indeed, we observe that the contact structure around $\hat{S}$ must be overtwisted since the dividing curve in Figure 3 obtained after edge-rounding bounds a disk on the outer boundary torus T_1. In Figure 3 the innermost and outermost annuli depict A and $\bar{A}$ respectively. The annulus in between shows half of T_1. Slope on T_1 is -3, therefore going from A to $\bar{A}$ the dividing curve rotates $-\frac{1}{6}$ of full turn. The annulus corresponding to half of T_0 is on the reverse of the figure, which we do not see. The torus in the figure thus obtained is oriented in accordance with the orientation of T_1.

Simlarly, Figure 4 proves that there is no real tight $T(-1, -3)$ leading to the configuration in Figure 2(b). Case (c) is similar and we do not draw a figure for it. $\qquad\square$
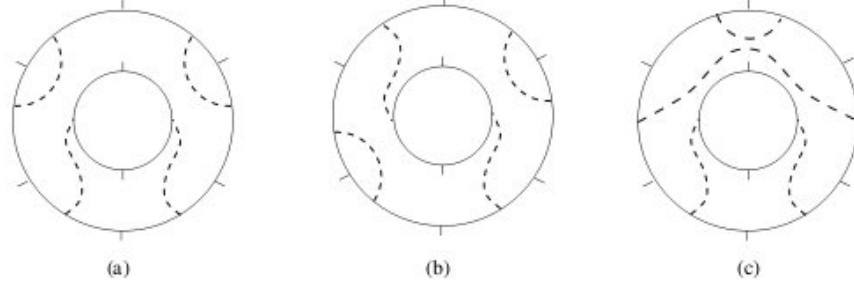
FIGURE 2. Possible dividing sets on A in $T(-1,-3)$

FIGURE 3. The dividing set on $\hat{S}$ for the configuration in the case (a).

In a similar way one can prove the following

Proposition 19. *There is no c_3-real tight $T(-1,-2)$.*

Proof. The proof is exactly as the previous. The only configuration on A up to holonomy is as in Figure 5, which leads an overtwisted disk as shown.   □

Proof of Theorem 11.

Given a c_3-real solid torus with slope $-\frac{p}{q}$, the case $p = 1$ is covered by Corollary 13 and Proposition 14. If $q = 1$ and $p > 1$ then we split the solid torus into two pieces as in the discussion following Corollary 15. An invariant isotopic Legendrian copy L of the core has a standard symmetric tight neighborhood with the boundary invariant and convex. We decompose the solid torus into two invariant pieces: $\nu(L)$ and $T(-\frac{1}{2k+1}, -\frac{p}{1})$. Following Lemma 16, the latter was observed to be minimally twisting. After c_3-invariantly factoring the thick torus, among the
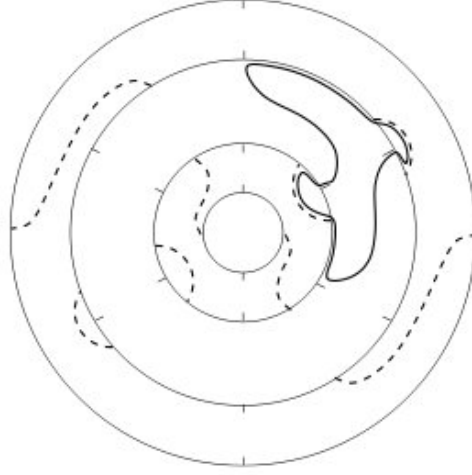
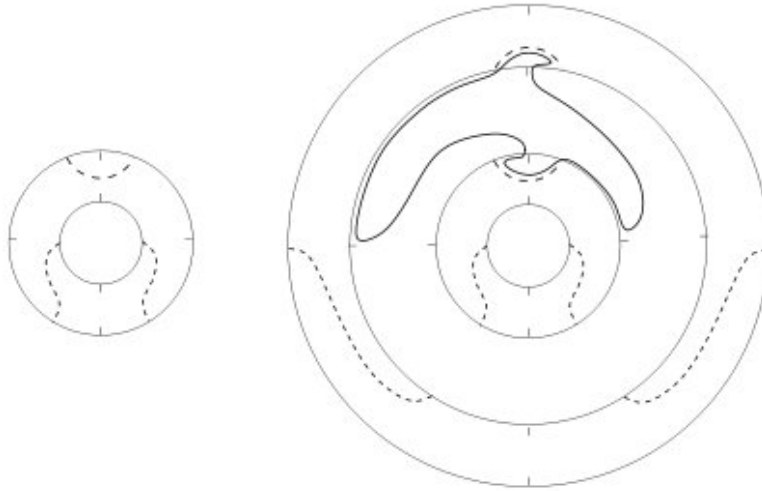FIGURE 4. The dividing set on $\hat{S}$ for the configuration in the case (b).

FIGURE 5. The single possible configuration (left) of the dividing curve on A in $T(-1,-2)$ begets an overtwisted disk (right).

slices we obtain either a final slice $T(-\frac{1}{2k+1}, -\frac{1}{2k})$ or $T(-1,-2)$ or an intermediate slice $T(-1,-3)$ (the first two and the last cases are mutually exclusive). None of these cases is possible thanks to the first paragraph of this section, Proposition 19 and Proposition 18 respectively. ∎

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Boğaziçi Üniversitesi, Department of Mathematics, TR-34342 Bebek, İstanbul, Turkey
E-mail address: `ferit.ozturk@boun.edu.tr`

Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstr. 9-11 (Lorenzenhof), 77709 Oberwolfach-Walke, Germany
E-mail address: `salepci@mfo.de`