Singularities of Hamiltonian integrable systems in physics

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Outline

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Introduction

Hamiltonian Monodromy in nonlinear waves systems

Optimal Control and Hamiltonian Singularities

Hamiltonian Singularities

Historical viewpoint

- 1850 : definition of integrable systems (J. Liouville).
- 1960 : existence of action-angle variables (V. I. Arnold)
- 1980 : hamiltonian monodromy (H. Duistermaat, N. Nekhoroshev, R. Cushman).
- 2000 : first example in spectroscopy (B. Zhilinskíi *et al.*)
- 2010 : first example in nonlinear optics (D. Sugny *et al.*)

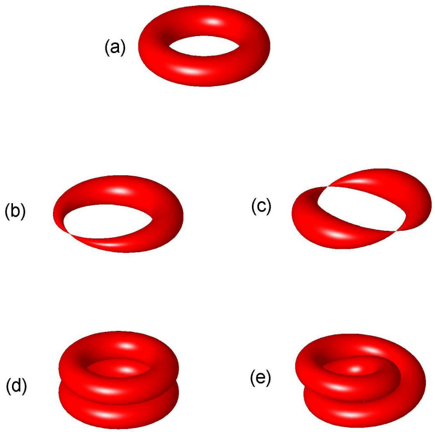
Singular Tori

- Integrable system with phase space dimension : $2n$
- Constants of motion $(K_1, ..., K_n)$

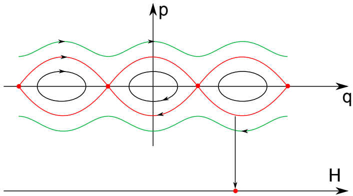
- Energy-Momentum map : $\mathcal{EM} : (q_i, p_i) \mapsto (K_1, ..., K_n)$
- Arnold-Liouville Theorem : If $(\tilde{K}_1, ..., \tilde{K}_n)$ regular, then $\mathcal{EM}^{-1}(\tilde{K}_1, ..., \tilde{K}_n)$ diffeomorphic to a regular torus T^n.

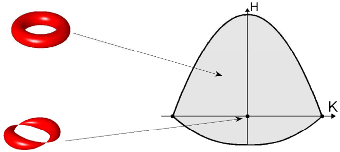
- Singular values of $\mathcal{EM}$ produce singular tori.

Two dimensional singular tori



Energy-Momentum Diagram





- Image of $\mathcal{EM}(q_1, p_1, q_2, p_2)$: (H, K) space.
- One value of (H, K) $\Leftrightarrow$ One torus in the phase space
- Singular tori $\Leftrightarrow$ Trajectories with infinite periods.

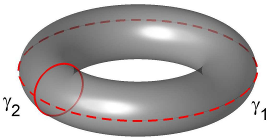
Action-angle variables

- Action-Angle variables : $(p_k, q_k) \to (I_k, \theta_k)$

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \longrightarrow \begin{cases} \dot{\theta}_i = \omega(I_i) \\ \dot{I}_i = 0 \end{cases}$$

- Arnold-Liouville Theorem : Action-Angle variables exist locally.
- Hamiltonian monodromy : an obstruction to the existence of global action-angle variables.

Cycles and Monodromy

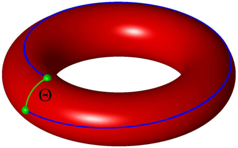


- Cycles (γ_1, γ_2) : a basis of the first homology group $H_1(T^2)$
- Monodromy : measure of the cycles deformation along a loop in (H, K)

$$\begin{pmatrix} \gamma_1^f \\ \gamma_2^f \end{pmatrix} = M \begin{pmatrix} \gamma_1^i \\ \gamma_2^i \end{pmatrix}$$

- $M \neq I_d \rightarrow$ Nontrivial monodromy (no global actions)

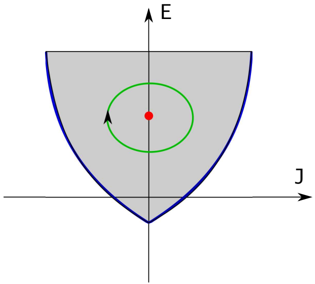
Rotation angle and first return time

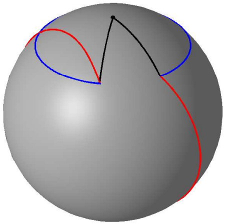


- (Θ, T) : defined by the intersection between the flows of K and H.
- Another basis of $H_1(T^2)$: $\begin{cases} X_1 = \dfrac{2\pi}{\omega_K} X_K \\ X_2 = -\dfrac{\Theta}{\omega_K} X_K + T X_H \end{cases}$
- Nontrivial Monodromy $\Leftrightarrow \Delta\Theta \neq 0$, e.g :

$$M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \to \Theta_f - \Theta_i = 2\pi$$

Spherical pendulum





- A physical interpretation of the Rotation Angle.
- Pinched torus $\rightarrow$ nontrivial monodromy :

$$M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Outline

Introduction

Hamiltonian Monodromy in nonlinear waves systems

Optimal Control and Hamiltonian Singularities

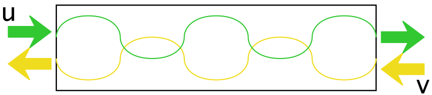
Why do we need PDE ?

- Goal : to measure a dynamical effect in a physical system.
- Hamiltonian monodromy is well defined in Hamiltonian ODE.
- One ODE trajectory $\rightarrow$ One value of (H, K).

- How to move in (H, K) ?
  - with a non integrable perturbation of the ODE
  - with an hamiltonian ODE that is a stationary solution of a PDE

- Nonlinear optics is well known to possess Hamiltonian PDE

Counterpropagating Waves



- Bragg Model : waves propagation in periodic nonlinear media

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} & = & i\kappa v + i\gamma(|u|^2 + 2|v|^2|)u \\ \frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} & = & i\kappa u + i\gamma(|v|^2 + 2|u|^2|)v \end{cases}$$

- A Hamiltonian controlled system :
$H = -\frac{\kappa}{2}(uv + u^* v^*) - \gamma |u|^2 |v|^2 - \frac{\gamma}{4}(|u|^4 + |v|^4)$
$K = \frac{1}{2}(|u|^2 - |v|^2)$

Introduction to reduction

Consider the action-angle variables (K, θ, J, ϕ). The vector field X_H that generates the flow of H :

$$X_H = \omega_K \partial_\theta + \omega_J \partial_\phi$$

Goal :

- to remove the trivial influence of K on the dynamics
- to characterize the singular tori
- to obtain a global description of the trajectories

We need a basis of the subspace invariant under the flow of K.

Reduction of the Bragg Model

Invariant polynomials such that $\{\pi_i, K\} = 0$:

$$\begin{cases} \pi_1 = -\frac{i}{2}(uv - u^*v^*) \\ \pi_2 = -\frac{1}{2}(uv + u^*v^*) \\ \pi_3 = \frac{1}{2}(|u|^2 + |v|^2) \\ \pi_4 = \frac{1}{2}(|u|^2 - |v|^2) \quad (= K) \end{cases}$$

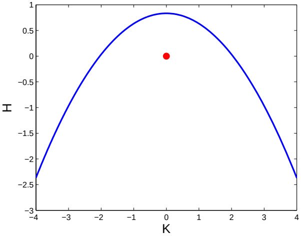
Hamiltonian :

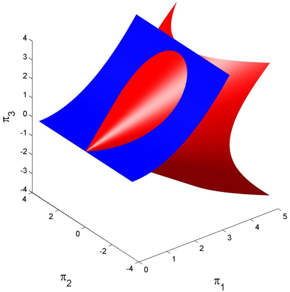
$$H = \pi_1 + \frac{\gamma}{2}(\pi_4^2 - 3\,\pi_3^2)$$

Reduced Phase Space :

$$\pi_3^2 - \pi_2^2 = \pi_1^2 + \pi_4^2,\ \pi_3 \geq 0$$

Pinched torus





- Intersection between hamiltonian surface (blue) and reduced phase space (red).
- One smooth point of the reduced phase space $\leftrightarrow$ one circle in the phase space.
- Pinched torus at $(H = 0, K = 0)$.

Nontrivial monodromy

- Polar coordinates

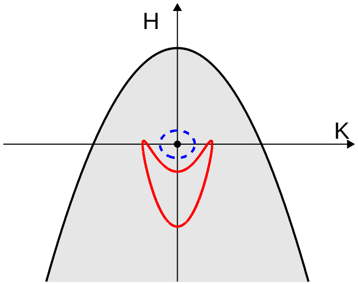
$$u = \sqrt{2I_u} \exp i\phi_u$$
$$v = \sqrt{2I_v} \exp -i\phi_v$$

- Rotation angle : $\Theta = \phi_u(L) - \phi_u(0)$
- Case $\gamma = 0$: $\Theta = \arctan(\frac{H}{K})$
- A loop around the pinched torus :

$$\begin{aligned} K &= r \cos s \\ H &= r \sin s \end{aligned} \quad , \quad s \in [0, 2\pi] \Rightarrow \Delta\theta = 2\pi \neq 0$$

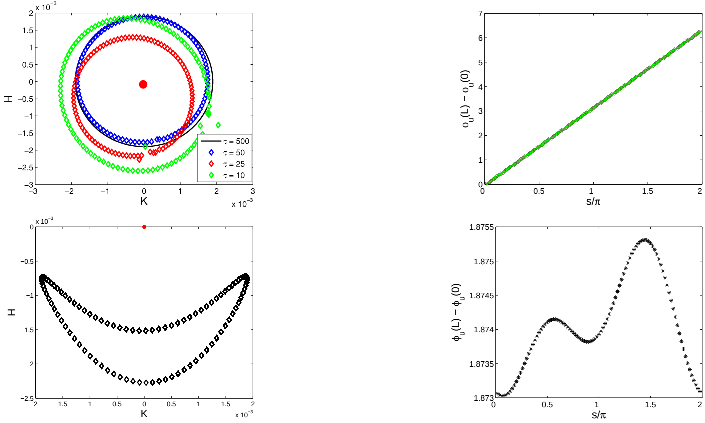
Nontrivial monodromy in nonlinear optics.

Dynamical measure



- A path in $(H, K) \Leftrightarrow$ Slow control of the pump.
- Measure of the phase difference : $\phi_u(L) - \phi_u(0)$

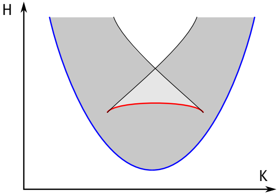
Non trivial monodromy [1]

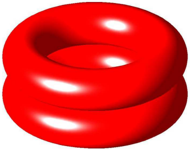


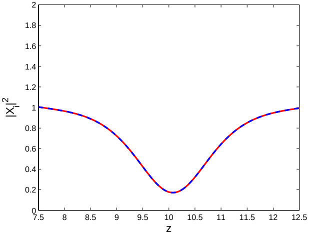
- A non zero shift ($2\pi \neq 0$) $\Leftrightarrow$ Non trivial monodromy.
- Generalized monodromies also exist in physical system (bidromy, fractional monodromy)

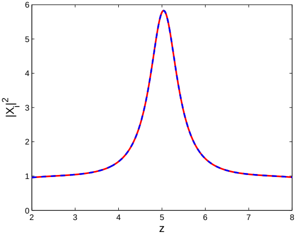
1. E. Assémat, *et al*. Phys. Rev. Lett. **106**, 014101 (2011)

Bragg Model with detuning : Moving solitons

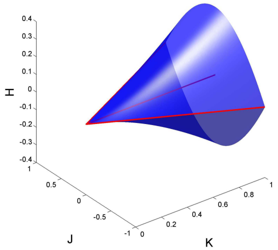


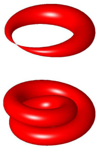


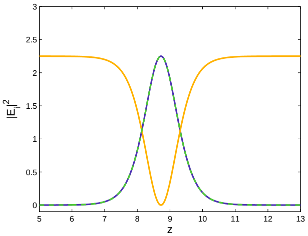




Three waves mixing



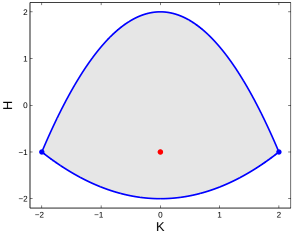


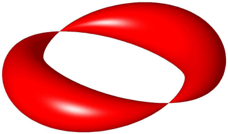


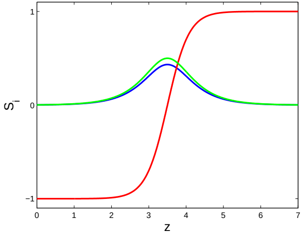
$$\partial_t E_1 + v_1 \partial_z E_1 = iE_3 E_2^*$$
$$\partial_t E_2 + v_2 \partial_z E_2 = iE_3 E_1^*$$
$$\partial_t E_3 + \partial_z E_3 = iE_1 E_2$$

$$H = E_3 E_2^* E_1^* + E_3^* E_2 E_1$$
$$J = v_2 |E_2|^2 + |E_3|^2$$
$$K = v_1 |E_1|^2 - \frac{v_2}{2} |E_2|^2 + \frac{|E_3|^2}{2}$$

Polarization dynamics in Isotropic Fibers







$$\begin{cases} \frac{\partial \vec{S}}{\partial t} + \frac{\partial \vec{S}}{\partial z} = \vec{S} \times \mathcal{I}_s \vec{S} + \vec{S} \times \mathcal{I}_i \vec{J} \\ \frac{\partial \vec{J}}{\partial t} - \frac{\partial \vec{J}}{\partial z} = \vec{J} \times \mathcal{I}_s \vec{J} + \vec{J} \times \mathcal{I}_i \vec{S} \end{cases}$$

$$H = 2(S_x J_x + S_z J_z) - \frac{1}{2}(S_y^2 + J_y^2)$$
$$K = S_y - J_y$$

Outline

Optimal Control

Historical viewpoint

- Calculus of variations (17th century)
- Pontryagin Maximum Principle (1960)
- Applications to quantum physics (1980)

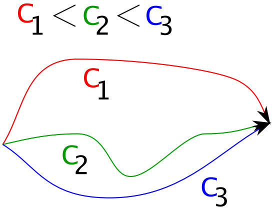
Two main ways

- Direct methods (genetical algorithm, gradient, ...)
- Indirect methods (based on variational principle)

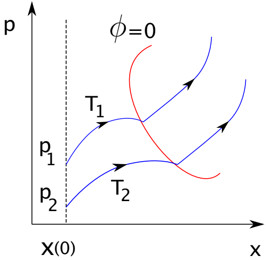
Control problem

Problem :

- Dynamical system : $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$
- Cost : $C = \int_0^T f_0(\mathbf{x}, u)$



Indirect methods
Pontryagin Maximum Principle



Pseudo-Hamiltonian with co-state p :

- $\mathcal{H} = \mathbf{p} \cdot \mathbf{f}(\mathbf{x}, u) + p_0 f_0(\mathbf{x}, u)$

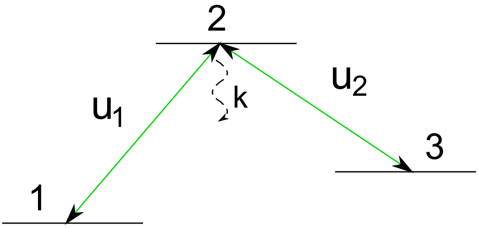
Maximization condition :

- $H = \max_{u \in U} \mathcal{H}(\mathbf{x}, \mathbf{p}, u)$

Hamiltonian dynamical system :

- $\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}$
  $\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}$

Three levels quantum system



- Goal : population transfert from $|1\rangle$ to $|3\rangle$ while avoiding the dissipation on $|2\rangle$
- A solution famous in physics : the STIRAP scheme :
  - Robust but not optimal in time nor in energy
  - Uses counter intuitive controls : u_2 turned on before u_1

Dynamics

- Schrodinger equation : $i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$
- Can be reduced to a 3D real ODE :

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -u_1 & 0 \\ u_1 & -k & -u_2 \\ 0 & u_2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Dissipation $\rightarrow$ dynamics inside a ball.

Energy minimum cost

Quadratic cost :

- $C = \int_0^T [u_1^2(t) + u_2^2(t)]dt$

Pseudo-Hamiltonian :

- $H = -kx_2p_{x_2} + u_1(x_1p_{x_2} - x_2p_{x_1}) + u_2(x_2p_{x_3} - x_3p_{x_2}) -\frac{1}{2}\left(u_1^2 + u_2^2\right)$

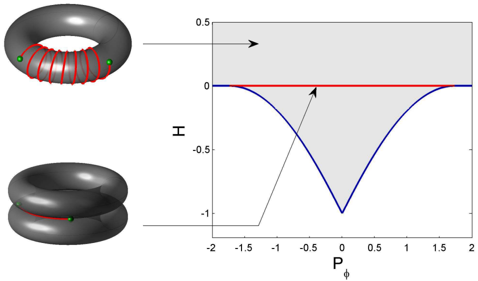
Regular controls (no singular controls here) :

- $u_1 = x_1p_{x_2} - x_2p_{x_1}$ and $u_2 = x_2p_{x_3} - x_3p_{x_2}$

Integrable system with constants of motion :

- $p_\phi = x_1p_{x_3} - x_3p_{x_1}$ and $p_\rho = \frac{1}{|\vec{x}|}(\vec{x} \cdot \vec{p})\exp(|\vec{x}|)$

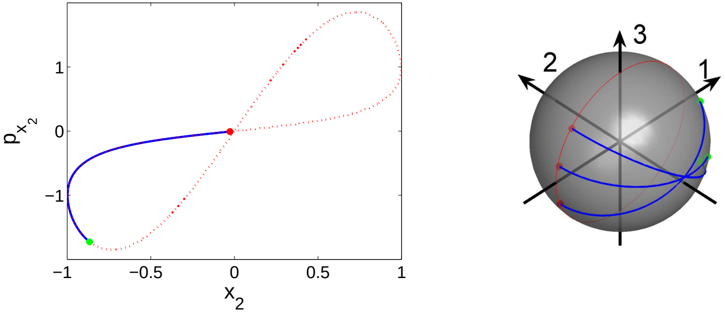
Hamiltonian singularities of the PMP's Hamiltonian [2]



- The phase space structure depends on the cost.
- Energy minimum cost $\rightarrow$ oscillating trajectory
- Non dissipative state $\rightarrow$ singular torus

2. E. Assémat and D. Sugny, Phys. Rev. A, **86**, 023406 (2012)

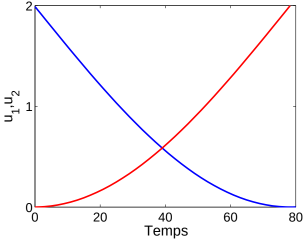
Trajectories on the bitorus

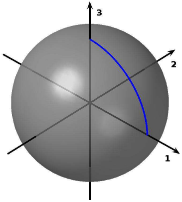


- Physical meaning : family of trajectories that go to a non dissipative state
- Conclusion : impossible to reach the STIRAP with this cost.

STIRAP cost

- Goal : to prevent the oscillations around $x_2 = 0$
- Spherical coordinates : $\begin{cases} x_1 = r \sin\theta \cos\phi \\ x_2 = r \cos\theta \\ x_3 = r \sin\theta \sin\phi \end{cases}$
- STIRAP cost : $C_s = \int \dot{\theta}^2$
- Counter intuitive controls :





Summary

Hamiltonian singularities :

- exist in many physical systems (nonlinear optics, molecular spectra [3]).
- allow to study physical PDE.
- are not yet widely used in physics.

Open questions :

- relation between PDE integrability and ODE integrability.
- relation between singular tori and robustness.
- role of the singular tori in the PMP.

3. E. Assémat, K. Efstathiou, M. Joyeux, D. Sugny, Phys. Rev. Lett. **104**, 113002 (2010)