Riemannian metrics on surfaces related to Euler-Poinsot rigid motion

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November 19, 2012

Applications

- Attitude control problem-Serret-Andoyer Metric

[Bonnard, Cots, Pomet,Sherbakova-submitted]

- Three spins with unequal coupling

[Yuan, Phd thesis, Harvard, 2000]

1 General Definitions and Motivations

Hamiltonian vector field

$$\boldsymbol{H}(q, p)$$

Exponential mapping (Π : standard projection)

$$exp_{q_0} : (p, t) \to \Pi(expt\boldsymbol{H}(q_0, p))$$

Definition

- Cut point along a given geodesic curve $t \to \gamma(t)$: sup t such that $\gamma(t)$ is of minimal length. Taken all geodesics starting from q_0 this defines the cut locus $C_{cut}(q_0)$.
- Conjugate point along the geodesic : first time it looses the optimality for C^1- closed geodesics. Taking all geodesics this defines the conjugate locus $C(q_0)$.

Extension

- SR-geometry
- Control systems $\frac{dq}{dt} = F(q, u(t)), \to Mint$.

Motivation

- Important geometric problem (See Berger, A panoramic view of Riemannian geometry, Springer, [2003])
- Application to the Monge problem in Optimal Transportation : continuity properties of the optimal transport mapping related to convexity properties of the tangent injectivity domains (Figalli-Rifford-Villani, [2012])

- Hamilton-Jacobi-Bellman equation in optimal control.

2 State of the art

Little is known about the conjugate and cut loci even on surfaces

- Poincaré-Myers result on two-spheres : In the analytic case the cut locus is a finite tree and the extremities of the branches are cusps points of the conjugate locus [1905, 1935]
- Jacobi conjecture on the ellipsoid : the conjugate locus has only 4 cusps [1842]

Remark : Only very recent proofs

- The case of ellipsoids of revolution : Sinclair-Tanaka (2006) : on a two sphere if the Gaussian curvature is monotone from the north pole to the equator the cut locus then the cut locus of a point not a pole is a sub-arc of the antipodal parralel or meridian.
- General case : Itoh-Kiyohara (2004) : general ellipsoids : the cut locus of a non umbilical point is a subarc of the antipodal line

of curvature and the conjugate locus has only four cusps.

Extension to Liouville surfaces [Itoh, Kiyohara-2011]

3 Sketch of the proof

The oblate case for an ellipsoid of revolution [Proof Bonnard, Caillau, Sinclair, Tanaka]

- The metric is given by

$$g = (cos^2\varphi + \epsilon sin^2\varphi)d\varphi^2 + sin^2\varphi d\theta^2$$

and it can be written in Darboux normal form

$$g = d\psi^2 + m(\psi)d\theta^2$$

where $\psi = 0$ is the equator.

- Integrability and Symmetries

The Hamiltonian is $H = \frac{1}{2}(p_\psi^2 + \frac{1}{m(\psi)}p_\theta^2)$

1. p_θ is a constant (Clairaut relation)-Symmetry of the geodesic flow with respect to the meridian

2. $m(\psi) = m(-\psi)$: symmetry of the geodesic flow with respect to the equator.

To integrate we parameterize by arc-length $H = \frac{1}{2}$ one gets the mechanical system

$$(\frac{d\psi}{dt})^2 = 1 - V(\psi, p_\theta)$$

where V is the potential. It has an unique minimum at the equator an for the geodesic flow we have two properties

- Except for meridian and the equator the $\psi-$ variable oscillates periodically between $-\psi_+$ and ψ_+.
- The $\theta-$ variable is monotonic.

Hence we have only one type of generic behaviors for the geodesics.

- To construct the conjugate locus and the cusp locus one must consider The set of geodesics starting from a given point and their intersections because :

Proposition

For a complete Riemannian manifold a cut point can be either a conjugate point or (generic case) a point where two intersecting minimizers are intersecting.

To analyze this property we introduce :

Definition

Fixing the initial point to the equator the first return mapping is the map :

$$R : p_\theta \epsilon (0, \sqrt{m(\psi(0))}) \to \Delta\theta,$$

where $\Delta\theta$ is the rotation angle of θ after the first return mapping to the equator.

To control the intersection of the geodesics on the prolate ellipsoid we use

The first return mapping is decreasing and conjugate and cut point of the geodesics starting from the equator cannot occur before returning to the equator.

To conclude about the cut locus we use the rigidity of the geodesic flow which is a consequence of the integrability :

Lemma

Due to the reflexional symmetry with respect to the equator two geodesics starting from the equator intersect with equal length when returning to the equator.

This gives the cut locus.

To construct the conjugate locus of the equator we proceed as follows.

The Gauss curvature is maximum $= M$ at the equator and the injectivity radius is given by $\pi/\sqrt{M}$. It corresponds to a cusp point for the conjugate locus at the equator.

The four cusps Jacobi conjecture is proved by constructing by continuation the conjugate locus from this cusp point.

To prove that cusp point cannot occur we proved

Lemma

The conjugate locus of the equator is given by

$$p_\theta \to (\psi_{t_c}(p_\theta), \theta_{t_c}(p_\theta))$$

with $\psi'(p_\theta)$non zero for p_θ in $(0, \sqrt{m(0)})$.

Making the computations more precised we get [Bonnard, Caillau, Sinclair, Tanaka 2009]

Theorem

If $R' < 0 < R''$ then the cut locus of a point not a pole is a segment of the antipodal parallel and the conjugate locus has only four cusps.

Conclusion

To conclude one can prove :

The condition on the return mapping is equivalent to the same condition on the period mapping of the $\psi-$ variable

Remark

At this level we don't use the explicit parameterization of this period by Jacobi elliptic functions. But it helps to check the monotonicity conditions.

Work in progress [Bonnard, Caillau, Rifford, CRAS 2011]

Application to the continuity property of the optimal transport mapping in Monge problem on surfaces.

4 Euler-Poinsot rigid body motion and Serret Andoyer metric

Left-invariant metrics on $S0(3)$: geometric optimal control formulation

$$\frac{dR}{dt} = \sum_{i=1,3} u_i R A_i, Min \to \int_0^T \sum_{i=1,3} I_i u_i^2$$

Maximum principle

$$H = \sum u_i H_i - \frac{1}{2} \sum u_i^2, \frac{\partial H}{\partial u_i} = 0.$$

True Hamiltonian

$$H = \frac{1}{2}(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3})$$

Euler equation

$$\frac{dH_i}{dt} = \{H_i, H\}$$

Limit case-SR geometry : $I_1 \to \infty$.

Serret-Andoyer variables

The generic solution envolves on a two-dimensional torus and from Liouville theorem the motion is quasi-periodic and depends upon two frequencies. If action-angles variables can be introduced [Sadov] and intermediate step in the computation is to introduce symplectic coordinates for which the metric is given by

$$g = \frac{2}{f(y)}dx^2 + \frac{2}{(2C - f(y))}dy^2,$$

with $A = 1/I_1 < B = 1/I_2 < C = 1/I_3, f(y) = 2(Asin^2 y + Bcos^2 y)$.

Remark

One additional variables z doesn't appear in the metric.

Geometric integration

It reduces to consider the mechanical system

$$(\frac{dy}{dt})^2 + V(y), V(y) = 2(C - f(y)/2)(p_x^2 f(y) - 1)$$

and the metric can be written in the Darboux normal form on a surface of revolution

$$g = d\varphi^2 + m(\varphi)d\theta^2.$$

Application

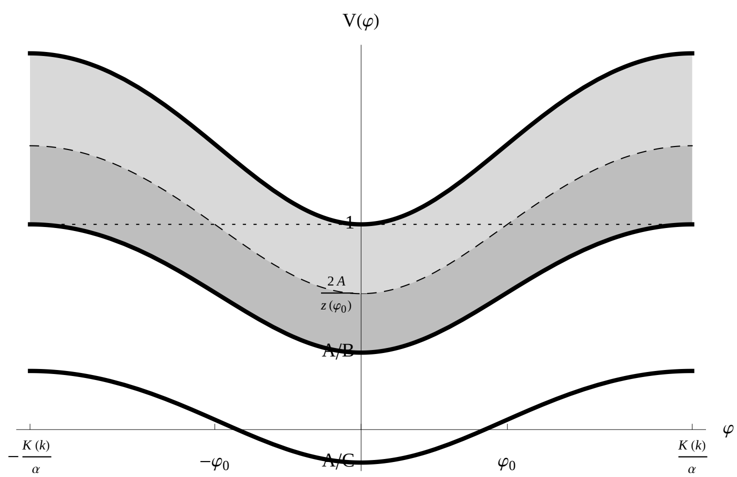
In this interpretation the integration reduces to integrate a standard pendulum with the following transcendance

- φ : Jacobi-elliptic function

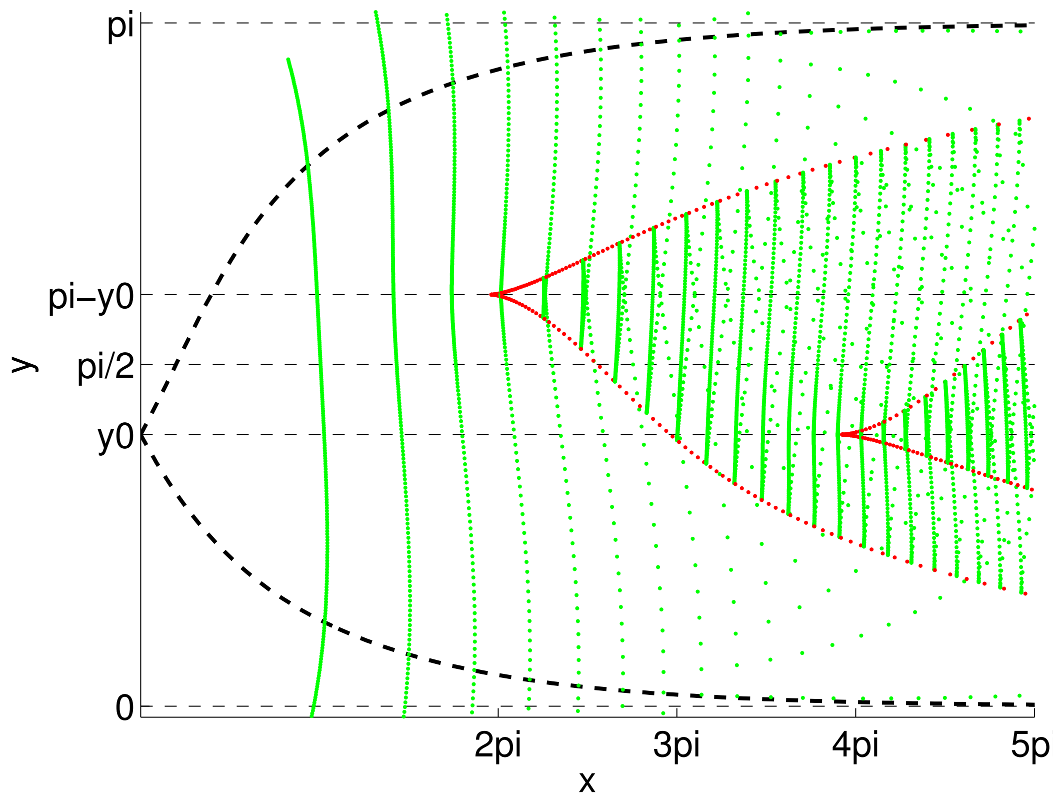
- θ : elliptic integral of the third kind

Below we represent the potential mapping : it has minimum value at the equator but a a maximum ; this gives three types of geodesics (pendulum) and the conjugate locus.

Potential



Conjugate locus



Conclusion

- Analogy for the case of oscillating trajectories with the solution on an ellipsoid of revolution.
- In particular the determination of the conjugate locus using the analogy with the oblate ellipsoid of revolution is a first step towards the determination of the conjugate and cut loci for left-invariant metrics on $S0(3)$.

This is a very important and difficult problem of geometric optimal control with many applications.

RESULT

Only oscillating trajectories can have conjugate points and the conjugate locus is represented above. Contrarly to the case of ellipsoid only two symmetric cusps exists and the asymptotic of the conjugate locus is related to the separatrixes of the pendulum corresponding to separatrixes in Euler equation.

Still the conjugate locus is computed by continuation starting from the cusp point at the equator.

ONE APPLICATION OF THE TECHNIQUES TO OPTIMAL CONTROL

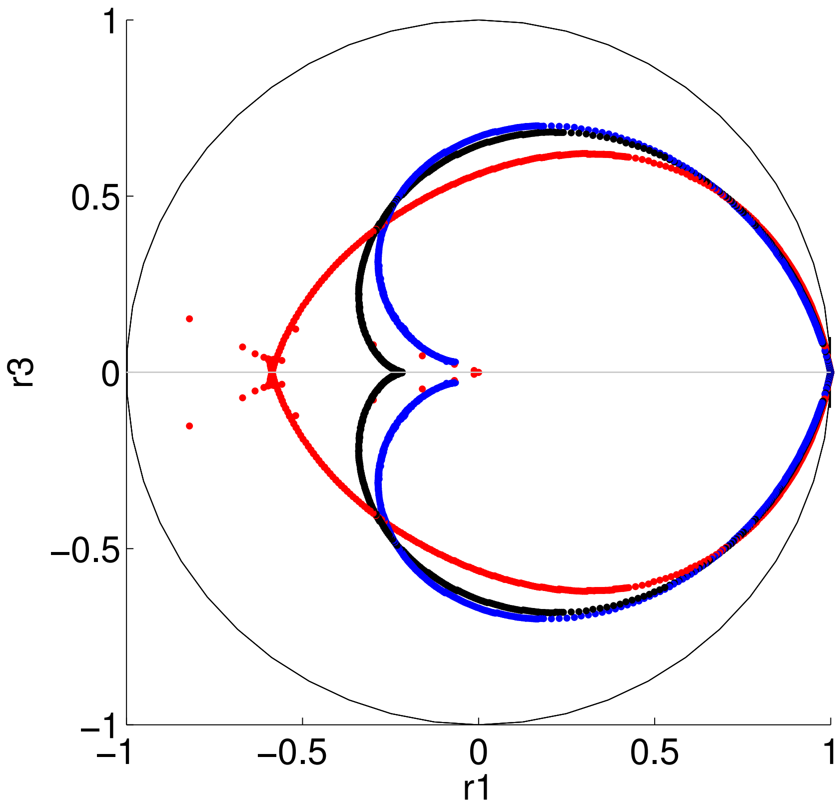
Quantum control of three coupled spins with unequal coupling.

The system is a projection of the SR-case on $S0(3)$ and defines a Riemannian metric on S^2 with polar singularities at the equator and is deformation of the so-called Grusin case-In black on the picture.

The Grusin case is $g = d\varphi^2 + tan^2\varphi d\theta^2$

Due to the singularity the injectivity radius goes to zero (conjugate point of an equatorial point accumulate near this point) and the cusp singularity is replaced by a fold.

Deformation of the Grusin case in the spin dynamics



Work in progress