Integrability conditions for homogeneous potentials of degree −1. Applications in celestial mechanics

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Introduction

We consider here Hamiltonian systems with the following form

$$H=\sum_{i=1}^{n}\frac{p_i^2}{2}+V(q_1,\dots,q_n)$$

with V an homogeneous function of degree -1, meromorphic in $q_1,\dots,q_n$ in $\mathbb{C}^n\setminus\{0\}$. We will note this class of functions $\mathbb{C}\{q_1,\dots,q_n\}$. The question is: does this Hamiltonian possesses n meromorphic first integral which are in involution? To be more precise, we will call a meromorphic first integral a function in $p_1,\dots,p_n,q_1,\dots,q_n$ which is meromorphic everywhere except possibly if $(q_1,\dots,q_n)=0$. To study this problem, we will consider some particular orbit of this system

Definition: We call $c=(c_1,\dots,c_n)\in\mathbb{C}^n$ a Darboux point (or a central configuration in the case of the n-body problem) if it is a solution of the following equation

$$\frac{\partial V}{\partial q_i}(c_1,\dots,c_n)=\alpha c_i \quad \forall i=1..n$$

Then we have that

$$(c.\dot\phi,c.\phi),\ \text{with } \frac{1}{2}\dot\phi^2=-\frac{\alpha}{\phi}+E$$

is an orbit of our system. By changing the normalization, we can choose an arbitrary value for α (if $\alpha\neq 0$), and we will choose here $\alpha=-1$.

The Morales Ramis theorem

Theorem (Morales-Ramis [Morales-Ruiz and Ramis(2001)]): If H admit n meromorphic first integrals in involution, then for all Darboux points c, with $\alpha=-1$, we have

$$Sp(\nabla^2V(c))\subset\{\tfrac{1}{2}(k-1)(k+2),\ k\in\mathbb{N}\}$$

This theorem is very useful, but required often the calculation of all central configurations for a given potential, which can be difficult, for example in celestial mechanics.

Extension of this Theorem in the case of algebraic potentials:
We consider weight-homogeneous polynomials $G\in\mathbb{C}[q,w]^s$ and the surface

$$\mathcal{S}=\{(q,w)\in\mathbb{C}^{n+s},G(q,w)=0\}$$

Let J be the Jacobian matrix of the application $w\mapsto G(q,w)$. We define the derivations of a meromorphic function V on an open set $U\subset\mathcal{S}$

$$\frac{\partial}{\partial q_i}V=\partial_i V+[J^{-1}(\partial_{n+1}V,\dots\partial_{n+s}V)^\intercal]_i$$

This allows to define the dynamical system $\ddot q=\nabla V(q)$ on U outside a singular set

$$\Sigma(V)=\{(q,w)\in U,V(q,w)\notin\mathbb{C}\text{ ou }\det(J)(q,w)=0\}$$

Theorem (in [Combot(2012)], see [Morales-Ruiz et al.(2007)Morales-Ruiz, Ramis, and Simó] for comparaison): Let V be a meromorphic potential on an open set $U\subset\mathcal{S}$ and $\Gamma\subset\mathbb{C}^n\times U$ a non-stationary orbit of V. Suppose $\Gamma\not\subset\mathbb{C}^n\times\Sigma(V)$. If there are n first integrals meromorphic on $\mathbb{C}^n\times(U\setminus\Sigma(V))$ of V that are in involution and functionally independent over an open neighbourhood of Γ, then the identity component of Galois group of the variational equation near Γ is abelian over the base field of meromorphic functions on $\Gamma\setminus(\mathbb{C}^n\times\Sigma(V))$.

Second order variational equations

Theorem (in [Morales-Ruiz et al.(2007)Morales-Ruiz, Ramis, and Simó]): If H admit n meromorphic first integrals in involution, then for all central configurations c, with $\alpha=-1$, the identity component of the Galois group of the variational equation near the orbit $(c.\dot\phi,c.\phi)$ is abelian at any order.

Theorem (in [Combot(2011)]): Let c be a central configuration with $\alpha=-1$. We pose $\lambda_i\ \ i=1\dots n$ the eigenvalues of $\nabla^2V(c)$, the hessian of V on c, and $X_1,\dots,X_m$ its eigenvectors. Assume that $\nabla^2V(c)$ is diagonalizable. If V is meromorphically integrable, then -$\lambda_i=\frac{(p_i-1)(p_i+2)}{2}$ with $p_i\in\mathbb{N}$ (integrability condition at order 1)
-$\forall i,j,k=1...n,\ A_{p_i,p_j,p_k}=0\Rightarrow D^3V(c).(X_i,X_j,X_k)=0$ where A is a 3 index table with values in $\{0,1\}$, invariant by permutation of the indices

$A_{0,i,j}$	0	1	2	3	4	5	$A_{1,i,j}$	0	1	2	3	4	5	$A_{2,i,j}$	0	1	2	3	4	5
0	1	1	1	1	1	1	0	1	0	0	1	1	1	0	1	0	0	0	1	1
1	1	0	0	1	1	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0
2	1	0	0	0	1	1	2	0	0	0	0	0	0	2	0	0	1	0	0	0
3	1	1	0	0	0	1	3	1	0	0	0	0	0	3	0	0	0	1	0	0
4	1	1	1	0	0	0	4	1	0	0	0	0	0	4	1	0	0	0	1	0
5	1	1	1	1	0	0	5	1	1	0	0	0	0	5	1	0	0	0	0	1
$A_{3,i,j}$	0	1	2	3	4	5	$A_{4,i,j}$	0	1	2	3	4	5	$A_{5,i,j}$	0	1	2	3	4	5
0	1	1	0	0	0	1	0	1	1	1	0	0	0	0	1	1	1	1	0	0
1	1	0	0	0	0	0	1	1	0	0	0	0	0	1	1	1	0	0	0	0
2	0	0	0	1	0	0	2	1	0	0	0	1	0	2	1	0	0	0	0	1
3	0	0	1	0	1	0	3	0	0	0	1	0	1	3	1	0	0	0	1	0
4	0	0	0	1	0	1	4	0	0	1	0	1	0	4	0	0	0	1	0	1
5	1	0	0	0	1	0	5	0	0	0	1	0	1	5	0	0	1	0	1	0

For $i,j,k>0$, $A_{i,j,k}=1$ if and only if:

$$\begin{cases} i+j-k\geq 2\\ i-j+k\geq 2\\ -i+j+k\geq 2\\ i+j+k[2]=0\end{cases}\ \text{or}\ \begin{cases}-i+j+k\leq -2\\ i+j+k\ \ [2]=1\end{cases}\ \text{or}$$

$$\begin{cases} i-j+k\leq -2\\ i+j+k\ \ [2]=1\end{cases}\ \text{or}\ \begin{cases} i+j-k\leq -2\\ i+j+k\ \ [2]=1\end{cases}$$

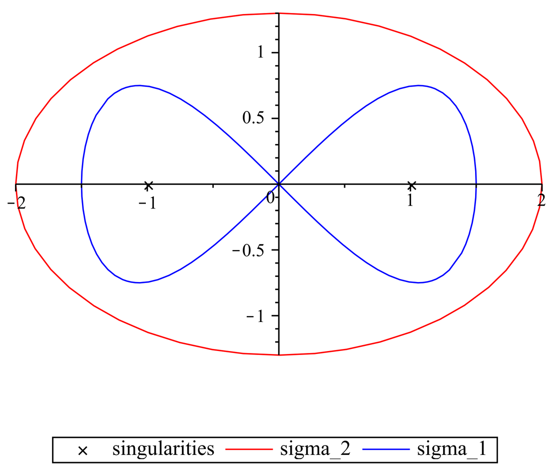
For $j,k>0$, $A_{0,j,k}=1$ if and only if
$\mid j-k\mid\geq 2$ and $\forall k\in\mathbb{N}$, $A_{0,0,k}=1$

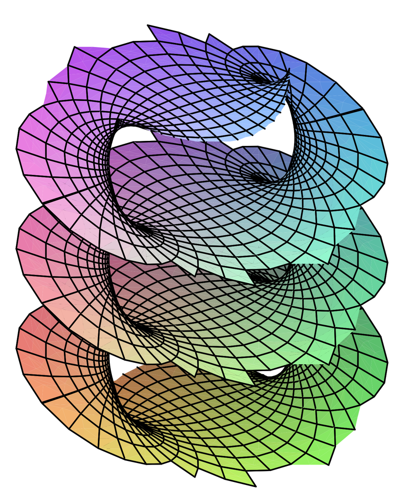
Exhibing a non zero monodromy commutator

The last theorem comes from the study of the monodromy group of the solution of second order variational equations. More precisely, it come from the study of

$$S=\int(Q_i(t)+\alpha P_i(t))(Q_j(t)+\alpha P_j(t))(Q_k(t)+\alpha P_k(t))(t^2-1)^2dt$$

where P_i,Q_i are some type of Legendre polynomials of the first and second kind, and α denote the choice of valuation for the functions Q_i.





Paths corresponding to the monodromy elements σ_1,σ_2, and the Riemann surfaces corresponding to the functions Q_j. The difference between two sheeves is $i\pi\epsilon_jP_j$. As we see, σ_2, corresponding to monodromy at infinity, acts trivially on Q_j.

We search a non zero commutator of the form $\sigma_1^\alpha\sigma_2\sigma_1^{-\alpha}\sigma_2^{-1}$. For the function S, σ_2 is adding to S a polynomial in α of degree at most 3. Eventually, we prove that this polynomial is not constant which imply that $\sigma_1^\alpha\sigma_2\sigma_1^{-\alpha}\sigma_2^{-1}$ does not act trivially on S.

The three body problem on a line

We consider here the Hamiltonian system with the following form

$$H=\sum_{i=1}^{3}\frac{p_i^2}{2}+\frac{m_1m_2}{q_1-q_2}+\frac{m_1m_3}{q_1-q_3}+\frac{m_2m_3}{q_2-q_3}$$

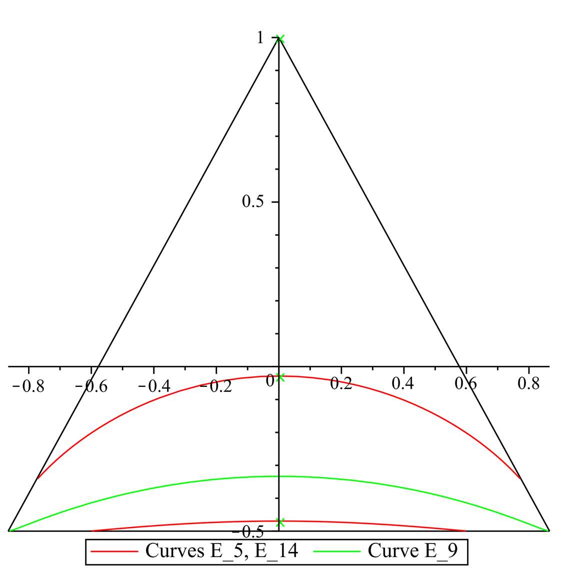
We will suppose that $(m_1,m_2,m_3)\in{\mathbb{R}_+^*}^3$ and $m_1+m_2+m_3=1$. The Hamiltonian H is here a function of $p_1,\dots,p_3,q_1,\dots,q_3$ and is meromorphic everywhere.

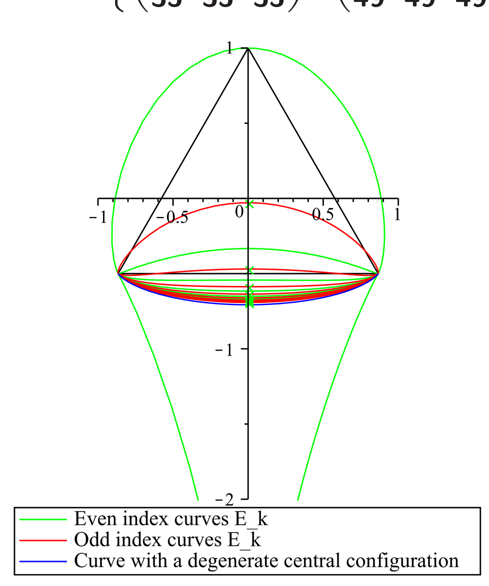
Theorem (see also [Shibayama(2011)], and [Tsygvintsev(2001), Morales-Ruiz and Simon(2009)] for the planar case): If H is meromorphically integrable, then the masses (m_1,m_2,m_3) belong to 3 rationally parametrizable curves $(E_k(\rho)),\ \ k=5,9,14$, which correspond to the case where the central configuration has the eigenvalue k.

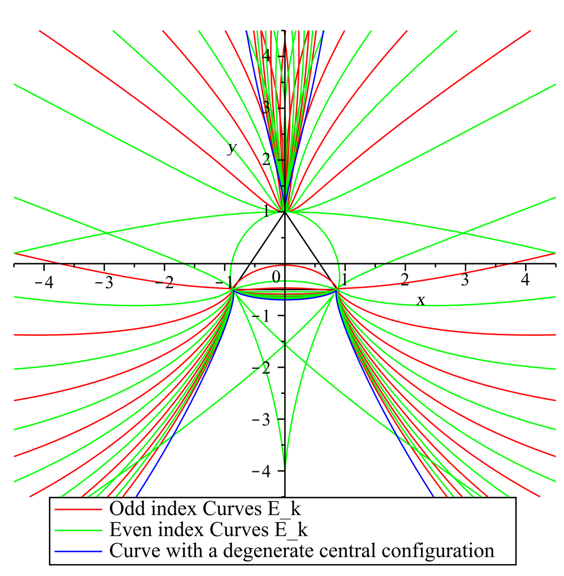
Remark: In the general case, there exists 5 complex central configurations and so we could intersect up to 5 of these curves, but there are degenerate cases which are difficult to study. Instead of using all central configurations, we only use one of them (the real one)

Theorem: If the second order variational equation has a Galois group whose identity component is Abelian, then

$$(m_1,m_2,m_3)\in\left\{\left(\frac{12}{35},\frac{11}{35},\frac{12}{35}\right),\left(\frac{24}{49},\frac{1}{49},\frac{24}{49}\right)\right\}\cup E_9$$







Curves (E_k) represented in barycentric coordinates, with green curves corresponding to integrability at order 2. The cases with positive masses are inside the black triangle. The last plot represent the complexified curves (E_k).

Partial classification of planar homogeneous potentials in the plane

Theorem (in [Combot(2011)]): Let V be a homogeneous potential of degree -1 with two degrees of freedom, such that
$V=r^{-1}U(\theta)$ in polar coordinates, with U meromorphic 2π-periodic.
$\exists c\in\mathbb{C}^2$ such that $c_1^2+c_2^2\neq 0$, $V'(c)=-c$ and $Sp(\nabla^2(V)(c))\subset]-\infty,27[$

If V is meromorphically integrable, then V belongs after possibly rotation to one of the following families

$$V=\frac{a}{q_1}+\frac{b}{q_2}\quad a,b\in\mathbb{C}\qquad V=\frac{a}{r}\quad a\in\mathbb{C}\quad V=\frac{aq_1}{(q_1+\epsilon iq_2)^2}\quad a\in\mathbb{C},\ \epsilon=\pm1$$

This result uses arbitrary order variational equations for $\lambda=-1,0,2$, and variational equations of order 5 or 7 for higher eigenvalues. As the colinear 3 body problem always has a real central configuration, and that the eigenvalue of the Hessian matrix of the potential is bounded by 16, this Theorem applies:

Corollary: The colinear 3 body problem is never meromorphically integrable.

Third order variational equations for planar potentials

Theorem (in [Combot and Koutschan(2012)]): Let V be an homogeneous potential of degree -1 in the plane. Assume that $c=(1,0)$ is a Darboux point of V with multiplier $\alpha=-1$. If the variational equation has a Galois group whose identity component is Abelian up to the order 3, the following conditions are satisfied:

$$Sp(\nabla^2V(c))=\{2,\frac{1}{2}(p-1)(p+2)\}\text{ with } p\in\mathbb{N}$$

$$f_1(p)\,(\partial_{122}V(c))^2+(\partial_{222}V(c))^2\,f_2(p)+(\partial_{2222}V(c))\,f_3(p)=0\text{ if } p \text{ is even,}$$

$$f_1(p)\,(\partial_{122}V(c))^2+(\partial_{2222}V(c))\,f_3(p)=0\quad \partial_{222}V(c)=0\text{ if } p\text{ is odd}$$

and the functions f_1,f_2,f_3 satisfy explicit recurrences with polynomial coefficients.

Applications:
If the potential $V(r,\theta)=r^{-1}\left(a+be^{i\theta}+ce^{2i\theta}+de^{3i\theta}\right)$ in polar coordinates is meromorphically integrable, then V belongs to one of the following families with $a,b\in\mathbb{C}$

$$V=r^{-1}a,\qquad V=r^{-1}\left(a+be^{i\theta}\right),\ V=r^{-1}\left(ae^{i\theta}+be^{3i\theta}\right),$$
$$V=r^{-1}\left(a+be^{2i\theta}\right),V=r^{-1}\left(a+be^{3i\theta}\right),V=r^{-1}\left(a+be^{i\theta}\right)^3,$$

The third order variational equation near the real central configuration of the colinear 3 body problem with positive masses never has a Galois group whose identity component is Abelian.

Integrable germs of homogeneous potentials

We consider a 3-dimensional (look at [Combot(2011)] for dimension 2) homogeneous potential of degree -1, with a series expansion at $q=(1,0,0)$ of the form

$$V(q_1,q_2,q_3)=q_1^{-1}\left(1+\frac{1}{2}\left(\lambda_1\frac{q_2^2}{q_1^2}+\lambda_2\frac{q_3^2}{q_1^2}\right)+\sum_{i=3}^{\infty}\sum_{j=0}^{i}u_{i,j}\frac{q_2^{i-j}q_3^j}{q_1^i}\right)$$

The point $(1,0,0)$ is a Darboux point with multiplier -1. For each couple of eigenvalues λ_1,λ_2, the set of series expansion of order $k+1$ whose variational equations have a Galois group whose identity component is Abelian at order k is an algebraic variety, given by polynomial conditions on $u_{i,j},\ j\leq i\leq k,\ i\geq 3$. Below we have computed this variety and its Hilbert dimension for some eigenvalues

λ_1,λ_2	5	9	14	20	27	35	44	54
5	0	2	0	1	0	1	1	1
9	2	4	2	4	2	3	2	3
14	0	2	0	2	0	2		
20	1	4	2	4	2	4		
27	0	2	0	2	0	2		
35	1	3	2	4	2	4		
44	1	2					0	
54	1	3						0

Hilbert dimension at order 2.

λ_1,λ_2	5	9	14	20	27	35	44	54
5	0	2		1		1	1	1
9	2			4	2	3	2	3
14								
20	1	4						
27		2						
35	1	3						
44	1	2						
54	1	3						

Hilbert dimension at order 3.

λ_1,λ_2	5	9	14	20	27	35	44	54
5		0		0		0	0	1
9	0			0	0	1	1	1
14								
20	0	0						
27		0						
35	0	1						
44	0	1						
54	1	1						

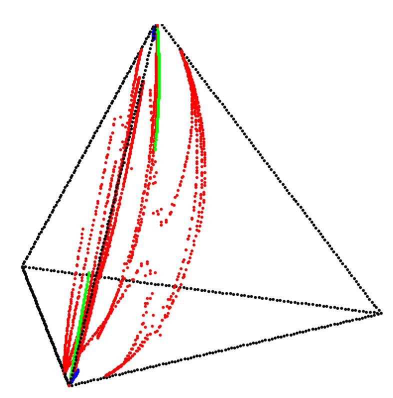
Hilbert dimension at order 4.

The corresponding ideals allow to effectively test Morales-Ramis-Simo integrability condition on higher variational equations, even on families of potentials with parameters. The only condition is that the eigenvalues of the Hessian matrix at the Darboux point should be bounded.

Application: The 4 body problem on a line

Theorem: Let (m_1,m_2,m_3,m_4) be positive masses with $m_1+m_2+m_3+m_4=1$, V the potential of the 4 body problem on a line with Newtonian interaction and a real central configuration c with multiplier -1. Then

$tr(\nabla^2V(c))<70$ $\qquad$ $Sp(\nabla^2V(c)=\{0,2,\lambda_1,\lambda_2\}$ and $\lambda_1,\lambda_2\geq 2$



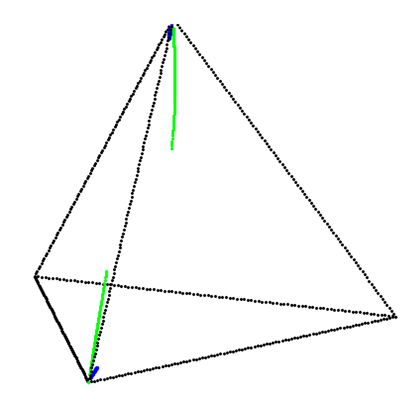


Figure: Set of masses corresponding to allowed eigenvalues of the Hessian matrix. Red curves satisfy integrability conditions at order 1, and the blue one at order 2.

Theorem: The Galois group of the variational equation near the real central configuaration of the colinear 4 body problem has not an Abelian identity component at order 5. Thus the colinear 4 body problem is not meromorphically integrable.

Potentials invariant by rotation

Theorem (in [Combot(2012)]): Let V be a homogeneous potential of degree -1 meromorphic on an algebraic surface $\mathcal{S}$. Assume $c=(1,0,\dots,0)\in\mathcal{S}\setminus\Sigma(V)$ is a Darboux point of V. We assume moreover that there exists a rotation group G, fixing V and a subspace E of an eigenspace of eigenvalue λ of $\nabla^2V(c)$, such that $\{\alpha g.c,\ \alpha\in\mathbb{C}^*,g\in G\}=\{(q_1,q_2,r),\ q_1^2+q_2^2=r^2,\ r\neq 0\}$. Let C be an angular momentum which is a first integral of V such that $C_{|\mathcal{C}}=p_1q_2-p_2q_1$. If V is integrable with meromorphic first integrals on $\mathbb{C}^n\times(\mathcal{S}\setminus\Sigma(V))$, then the equation

$$t(-C^2+2t+2Ht^2)\ddot X+(-t+C^2)\dot X=\lambda X$$

has a Galois group whose identity component is abelian for any H,C. Respectively if V is integrable on a given level of energy and angular momentum H,C, this equation has a Galois group whose identity component is abelian.

(C,H)	λ
$C=0$	$\lambda\in\{\frac{1}{2}(k-1)(k+2),\ k\in\mathbb{N}\}$
$C^2H=-1/2$	$\lambda\in\{-k^2,\ k\in\mathbb{N}\}$
$H=0$	$\lambda\in\{\frac{1}{2}(k-1)(k+2),\ k\in\mathbb{N}\}$
$C^2H\notin\{0,-1/2\}$	$\lambda\in\{0,-1\}$
$(C,H)=(0,0)$	$\lambda\in\mathbb{C}$

Figure: Integrability table at non-zero angular momentum

Corollary: The equal mass n body problem in the plane is not meromorphically integrable even if restricted to a fixed level of energy and angular momentum $(C,H)\neq(0,0)$.

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