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(Non)existence of integrals that are polynomial in momenta

Rouen 15.11.2012

- arXiv:1010.4699 (J. Geom. Phys 2011) joint with V. Shevchishin
- arXiv:1111.4690 (Phys. Rev. D 2012) joint with B. Kruglikov

I will show two results:

1. the first result is joint with V. Shevchishin: we constructed all metrics on 2-dimensional surfaces admitting, besides the Hamiltonian, one integral that is linear in momenta and one integral that is cubic in momenta.
   • The result is based on a TRICK that can be applied in many other problems

2. the second result is joint with B. Kruglikov: we proved nonintegrability in the class of integrals that are polynomial of degree $\leqslant 6$ in momenta of a certain metric that was conjectured to be integrable by physicists.
   • The result is based on a METHOD that can be applied in many other situations

Fact (1st year linear algebra course). Let V be a finite-dimensional nontrivial real vector space and $L : V \to V$ be a linear map. Then, there exists a 1- or 2-dimensional linear subspace $\widetilde{V} \subseteq V$ such that $L(\widetilde{V}) = \widetilde{V}$.

Proof. In a certain basis, the matrix of L has one of the following Jordan normal forms

$$\begin{pmatrix} \lambda & 1 & * & * \\ 0 & \lambda & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} \alpha & \beta & * & * \\ -\beta & \alpha & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} \lambda & 0 & * & * \\ 0 & \mu & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

In all the cases the space spanned by the first two basis vectors is invariant. In the 1st and 3rd case the space spanned by the first basis vector is invariant

We consider the following linear system of PDE:

$$\sum_{k,i} c_j^{i,k} \frac{\partial u_i}{\partial x_k} + \sum_i c_j^i u_i = 0 \ , \quad j = 1, \ldots, m. \qquad (1)$$

Here $(u_1, \ldots, u_\ell)$ are the unknown functions to find, the coefficients $c_j^{i,k}$ and c_j^i are functions thought to be known, everything lives in a small neighborhood $W \subset \mathbb{R}^n$ and is at least as smooth as I need in the proofs.

Fact (1st year calculus). Assume the coefficients $c_j^{i,k}$ and c_j^i are independent of x_1. Then, for any solution $(u_1, \ldots, u_\ell)$ of (1), the tuple $\left(\frac{\partial}{\partial x_1} u_1, \ldots, \frac{\partial}{\partial x_1} u_\ell\right)$ is also a solution.

Proof. We differentiate the equations (1) and interchange the partial derivatives to obtain

$$\frac{\partial}{\partial x_1}\left(\sum_{k,i} c_j^{i,k} \frac{\partial u_i}{\partial x_k} + \sum_i c_j^i u_i\right) = \sum_{k,i} c_j^{i,k} \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_1} u_i\right) + \sum_i c_j^i \left(\frac{\partial}{\partial x_1} u_i\right) = 0 \ .$$

Setup. g is a 2D Riemannian (local) metric on $W \subseteq \mathbb{R}^2(x, y)$. We consider its Hamiltonian

$$H : T^*W \to \mathbb{R} \ , \quad H = g^{ij} p_i p_j.$$

A function $F : T^*W \to \mathbb{R}$ is an integral, if $\{H, F\} = 0$. Geometrically, integrals are conservative quantities, i.e. are functions that are constant along the orbits of the Hamiltonian system.

For example, H itself is an integral, since $\{H, H\} = 0$ because of the antisymmetry of $\{\ ,\ \}$.

Functions are functionally independent, if their differentials are linearly independent at almost every point.

Definition 1

The metric g is superintegrable, if there exist two integrals L, F of a certain $\underbrace{\textit{special form}}_{\textit{to be explained}}$ such that L, H, F are functionally independent.

The metric g is superintegrable, if there exist two integrals L, F of a certain $\underbrace{\text{special form}}_{\text{to be explained}}$ such that L, H, F are functionally independent.

I will assume that L is linear in momenta,

$$L = b_1(x,y)p_1 + b_2(x,y)p_2,$$

and F is cubic in momenta,

$$F = a_0(x,y)p_1^3 + a_1(x,y)p_1^2p_2 + a_2(x,y)p_1p_2^2 + a_3(x,y)p_2^3.$$

Why the integrals are homogeneous polynomials in momenta?

- There is no sense to consider integrals that
  - are polynomial in momenta
  - but are not homogeneous polynomial in momenta,

  since every homogeneous term is an integral (Darboux, Whittaker) .
- Moreover, if the integral is analytic in momenta, then there exists an integral that is polynomial in momenta.

- Superintegrable systems possess deep and interesting geometry, are useful, and were actively studied
  - In mathematical physics (Winternitz ...): many physical phenomena can be described with the help of superintegral systems
  - In differential geometry: solution of many natural problems are superintegrable (Koenigs, Darboux, Lie 18xx, Bryant-Manno-M$\sim$ 2006)
  - As source of examples: one can answer many natural questions about superintegrable metrics using algebra and without integrating ODE and PDE (Kalnins, Kress, Miller ...).
- The case Linear + Cubic is the first unsolved case
  - If both integrals are linear, the metric has constant curvature
  - The case when one integral is linear and another is quadratic was solved by Darboux in the XIXth century. He has a complete local classification.
  - The case when both integrals are quadratic was solved by Koenigs in the XIX th century. He has a complete local classification. By Kiyohara 1991, no nontrivial examples on closed surfaces are possible.

Who: Our case, when one integral is linear and another is cubic, was actively attacked recently by Winternitz and his (former) doctoral students Gravel and Marquette, and, independently, by Rañada. They could solve the problem under the additional assumption that the Hamiltonian system has the form

$$\frac{1}{2}(p_x^2 + p_y^2) + V(x, y).$$

In this case no new phenomena compared with the case "two quadratic integrals" appear (in the classical case; in the quantum case there are additional superintegrable systems (Post-Winternitz 2011)).

How: Because of the existence of the linear integral, one can think that the metric is $f(x)(dx^2 + dy^2)$. The existence of the cubic integral is a system of 5 nonlinear PDE on 4 unknown functions of two variables (coefficients of the integral) and the function f, which is intractable by standard methods.

Our advantage: We do essentially the same, but we know the trick

Take the coordinate system such that $g = f(x)(dx^2 + dy^2)$ (the local a.e. existence follows from the existence of the linear integral); in this coordinates the linear integral is p_y.

Then, the existence of the integral

$$F = a_0(x,y)p_1^3 + a_1(x,y)p_1^2 p_2 + a_2(x,y)p_1 p_2^2 + a_3(x,y)p_2^3$$

is equivalent to the solution of a linear system of 5 PDE on the 4 unknown a_i whose coefficients do not depend on y: indeed,

$$\{H,F\} = \sum_{i=1,2} \left[\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x_i}\right] = 0$$

This is a homogeneous polynomial of degree 4 in $p = (p_1, p_2)$ whose coefficients are expressions in $a_i, \frac{\partial a_i}{\partial x_j}, f(x), f'(x)$. Its 5 coefficients are our equations.

Fact (classics). The system is of finite type $\big($its solution space $\mathcal{F}^3$ is a finite-dimensional linear vector space ($dim \leqslant 10$ by Kruglikov 2008)$\big)$

Fact (trivial). Since f does not depend on y, the coefficients of the system of equation do not depend on y.

Assume there exists a 3rd degree integral F that is functionally independent of $L = p_y$ and H.

Consider the space $\mathcal{F}^3$ of all integrals that a polynomial of degree 3 in momenta.

This is a vector space; it is at least 3-dimensional, since the function p_y^3, $p_y \cdot H$ and our 3rd degree integral F are its elements, and at most 10-dimensional by Kruglikov 2008.

On the space, consider the linear endomorphism

$$\mathcal{L} : \mathcal{F}^3 \to \mathcal{F}^3, \ \mathcal{L}(F) = \frac{\partial F}{\partial y}.$$

The formula once more:

$$\begin{gathered}\mathcal{L}(a_0(x,y)p_1^3 + a_1(x,y)p_1^2p_2 + a_2(x,y)p_1p_2^2 + a_3(x,y)p_2^3) \\ = \left(\tfrac{\partial a_0(x,y)}{\partial y}\right)p_1^3 + \left(\tfrac{\partial a_1(x,y)}{\partial y}\right)p_1^2p_2 + \left(\tfrac{\partial a_2(x,y)}{\partial y}\right)p_1p_2^2 + \left(\tfrac{\partial a_3(x,y)}{\partial y}\right)p_2^3.\end{gathered}$$

It is well-defined (the result is again an integral):
indeed, the property that F is an integral is a system of linear PDE on a_i; the coefficients of this system do not depend on y so for every solution $(a_0, ..., a_3)$ we have that $\frac{\partial a_0}{\partial y}, ..., \frac{\partial a_3}{\partial y}$ is also a solution.

Take an eigenvalue μ of $\mathcal{L}$, $\underbrace{\text{assume first that } \mu \text{ is real and} \neq 0}_{\text{additional assumption}}$.

Then, there exists an integral F such that $\{p_y, F\} = \frac{\partial F}{\partial y} = \mu F$.
For

$$F = a_0(x,y)p_1^3 + a_1(x,y)p_1^2 p_2 + a_2(x,y)p_1 p_2^2 + a_3(x,y)p_2^3$$

the equality $\frac{\partial F}{\partial y} = \mu F$ reads

$$\begin{array}{lllll} & \frac{\partial a_0(x,y)}{\partial y} p_1^3 & + \frac{\partial a_1(x,y)}{\partial y} p_1^2 p_2 & + \frac{\partial a_2(x,y)}{\partial y} p_1 p_2^2 & + \frac{\partial a_3(x,y)}{\partial y} p_2^3 \\ = & \mu a_0(x,y)p_1^3 & + \mu a_1(x,y)p_1^2 p_2 & + \mu a_2(x,y)p_1 p_2^2 & + \mu a_3(x,y)p_2^3 \end{array}$$

and implies $a_0(x,y) = e^{\mu y}\tilde{a}_0(x)$, ... , $a_3(x,y) = e^{\mu y}\tilde{a}_3(x)$.

THIS IS AN ANSATZ FOR THE COEFFICIENTS OF THE INTEGRAL; THE FUNCTIONS INSIDE DEPEND ON THE FUNCTIONS OF ONE VARIABLE; SO EVERY PDE ON THE COEFFICIENTS OF THE INTEGRAL IS ACTUALLY AN ODE. WE HAVE 5 ODE ON 5 UNKNOWN FUNCTIONS $\tilde{a}_0, ..., \tilde{a}_3, f$ OF x

The case when μ is complex or the eigenvalue $\mu = 0$ the situation is essentially the same: in the complex case we obtain 10 PDE on 1 + 4 + 4 = 9 unknown functions, which has no solution unless μ is purely imaginary and we again have 5 ODE on 5 unknowns and always have a solution.

The case $\mu = 0$ is also essentially the same; in this case we need to consider the Jordan normal form of $L_{|\mathcal{F}^3}$.

Theorem 1

Let (M,g) be a Riemann surface s.th. $H = \frac{1}{2}g^{ij}p_ip_j$ admits (independent) linear and cubic integrals $L + F$. Then, a.e., $\exists$ coordinates (x,y) s.th. $L = p_y$ and $g = \frac{1}{h_x^2}(dx^2 + dy^2)$ where $h = h(x)$ satisfies one of the eqns (where $h_x := \frac{dh(x)}{dx}$)

$$
\begin{array}{rlcl}
(i) & h_x\left(A_0h_x^2 + \mu^2A_0h(x)^2 - A_1h(x) + A_2\right) & = & \left(A_3\frac{\sin(\mu x)}{\mu} + A_4\cos(\mu x)\right)\\
(ii) & h_x\left(A_0h_x^2 - \mu^2A_0h(x)^2 - A_1h(x) + A_2\right) & = & \left(A_3\frac{\sinh(\mu x)}{\mu} + A_4\cosh(\mu x)\right)\\
(iii) & h_x\left(A_0h_x^2 \qquad\qquad\qquad - A_1h(x) + A_2\right) & = & \left(A_3x \qquad\qquad + A_4\right)
\end{array}
$$

with some real constants $A_0, \ldots, A_4$ and $\mu > 0$ in cases (i,ii). In all three cases $\mathcal{F}^3(g) = \mathrm{span}\langle L^3, L{\cdot}H, F_1, F_2\rangle$ (4-dimensional unless g has constant curvature).

The explicit formulas for F_1, F_2 are of the form:

(i) $F_1(x,y) = \cosh(\mu y)\cdot f_{i,1}$, $\quad F_2(x,y) = \sinh(\mu y)\cdot f_{i,2}$,

(ii) $F_1(x,y) = \cos(\mu y)\cdot f_{ii,1}$, $\quad F_2(x,y) = \sin(\mu y)\cdot f_{ii,2}$,

with some *polynomials* $f_{i,ii;1,2}$ in $h(x), h_x, h_{xx}, h_{xxx}$ (and in p_x, p_y).

Case (iii) is similar.

Message. *Though the formulas in Theorem 1 may look ugly, one can work with them.*

The curvature of $g = \frac{(dx^2+dy^2)}{h_x^2}$ is

$$R = h_{xxx}h_x - h_{xx}^2 \qquad (*)$$

Now, differentiating the ODE for h (for example,
$h_x\left(A_0 h_x^2 + \mu^2 A_0 h(x)^2 - A_1 h(x) + A_2\right) = \left(A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x)\right))$
w.r.t. x and with respect to x, x, we obtain h_{xx} and h_{xxx}, substituting in $(*)$ we obtain the formula for the curvature.

Theorem. $h(x)$ from Theorem 1 corresponds to the metrics of constant curvature iff

- (a) it is a polynomial in x of $\deg \leqslant 2$, or
- (b) $h(x) = c\sin(\mu x + \varphi_0)$, or
- (c) $h(x) = c_+\exp(\mu x) + c_-\exp(-\mu x)$

In particular, for almost all $A_0, \ldots, A_4$ the metric does not have constant curvature.

Theorem. *The metrics from Theorem 1 are Darboux-superintegrable, if and only if* $A_0 = 0$.

Theorem. *For the metrics from Theorem 1, the dimension of the space of 3rd degree integrals is precisely* 4.

Fact (Kruglikov 2008). The maximal dimension of the 3rd degree integrals is 10; the submaximal is $\leqslant 7$; Kruglikov conjectured that the submaximal is actually 4; by our result the Kruglikov conjecture is true under the assumption that there exists a linear integral

Theorem 2 (Global solution on the sphere S^2.)

Assume that parameters of the equation

$$h_x\big(A_0(h_x^2 - h(x)^2) - A_1 h(x) + A_2\big) \;=\; \big(A_3 \sinh(x) + A_4 \cosh(x)\big) \qquad \text{(ii)}$$

satisfy $A_0 > 0$, $A_4 > |A_3|$ (and $\mu = 1$). Let $h(x)$ be a unique local solution of the eqn (ii) such that $h_x(x_0) > 0$. Then the solution $h(x)$ extends to the whole line $x \in \mathbb{R}$ and the metric $g = \frac{(dx^2+dy^2)}{h_x^2}$ extends to a real-analytic metric on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with the complex coordinate $z = e^{x+\mathbf{i}y}$.
Moreover, both cubic integrals F_1, F_2 also extend real-analytically on the whole S^2.

Remark. $\theta := \arctan(e^{-x})$ and $\varphi := y$ are spherical coordinates on S^2.

- Many classical examples (Jacobi, Lagrange, Euler) of integrable metrics on S^2 with linear or quadratic first integral.
- Kovalevskaya top (1889): Metric on S^2 with $\deg F = 4$.
- Goryachev(-Chaplygin) top: (1916) Metric on S^2 with $\deg F = 3$
- Linear or quadratic integrable metrics on closed surfaces are completely understood: Kolokoltsov, Kiyohara, Bobenko-Nehoroshev, M$\sim$,...) (1984-...)
- Selivanova (1999) and Dullin-M$\sim$ (2004) generalized by Valent (2010): new metrics on S^2 admitting cubic F,
- Kiyohara,(2001), metric on S^2 with F of any degree $d \geqslant 3$
- Kiyohara,(1991), If g on S^2 admits quadratic F_1, F_2 $(\dim \mathcal{F}^2(g) \geqslant 3) \Rightarrow R_g \equiv \mathrm{const}$.
- M$\sim$-Shevchishin (2011): The first one which is polynomially superintegrable.

Open Problem 1

Generalize our results for integrals of higher degree in momenta.

Comment: The trick works

Open Problem 2

Generalize our results for the pseudo-Riemannian metrics.

Comment: The trick works

Open Problem 3

Quantize the cubic integral.

Comment 1: For all previously known superintegrable systems, the integrals survive the quantiziation.
Comment 2: The trick still works!!

Open Problem 4

All geodesics of global superintegrable metrics are closed; so the metrics are the so-called Tannery metrics. Describe the metrics in the terms of Tannery. This also could allow to see them as the induced metrics on some surfaces in $\mathbb{R}^3$.

What is ZIPOY-VOORHEES metric? It is solution of the vacuum Einstein equation $Ric \equiv 0$; found by Zipoy and Voorhees in 1966 and 1970:

$$\left(\frac{x+1}{x-1}\right)^{\delta}\left((x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)\,d\phi^2\right)-\left(\frac{x-1}{x+1}\right)^{\delta}dt^2.$$

Here δ is an arbitrary number.

For $\delta=0$ we obtain the flat metric, for $\delta=1$ the Schwarzschild metric.

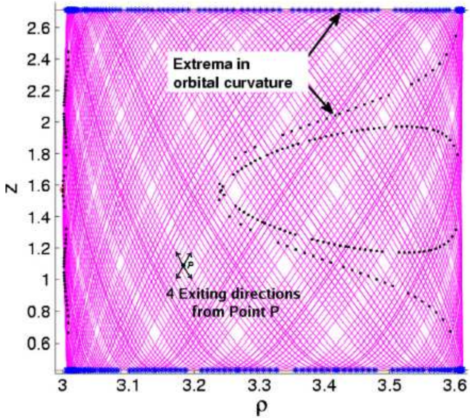
May be the next simplest example is $\delta=2$; in this case the metric looks as follows

$$ds^2=\left(\frac{x+1}{x-1}\right)^2\left[(x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^4\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right]-\left(\frac{x-1}{x+1}\right)^2dt^2.$$

Does the Zipoy-Voorhees metric admits an integral that is polynomial in momenta and that is functionally independent of and is in involution with the linear integrals p_t, p_ϕ?

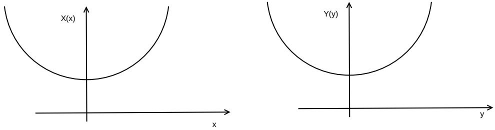
The property that two integrals F_1, F_2 are in involution means $\{F_1, F_2\} = 0$; in our case it means that I does not depend on t and on ϕ. The mathematical background of the latter assumption is clear in view of the first part of my talk: if the function I is an integral, the functions $\frac{\partial I}{\partial t}$ and $\frac{\partial I}{\partial \phi}$ are also integrals. Thus, if $\frac{\partial I}{\partial t} \neq 0$, we obtain too many integrals which is no good.

Numerical observation of J. Brinks (2008): the projection of the orbits of the reduced system to $(x,y)-$plane for Zipoy-Voorhees metric with $\delta = 2$. Similar behavior have some other explicitly given SAV-metrics (Manko-Novikov).

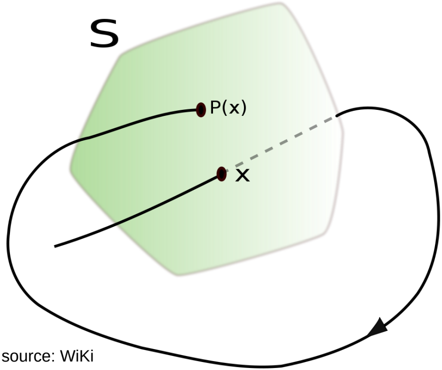


$$ds^2 = \left(\frac{x+1}{x-1}\right)^2\left[(x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^4\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right]-\left(\frac{x-1}{x+1}\right)^2 dt^2.$$

Example: Consider the natural Hamiltonian system with two degrees of freedom with the Lagrangian $H(x, y, p_x, p_y) = p_x^2 + p_y^2 + U(x, y)$ with the potential of the form $U(x, y) = X(x) + Y(y)$, where the functions X and Y have the following diagrams:



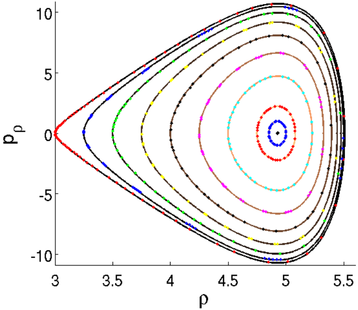
One can check by direct calculations that this system has two independent integrals: $p_x^2 + X(x)$ and $p_y^2 + Y(y)$. Therefore, for any trajectory of the system which we consider as a trajectory in the 4D-phase space $c(t) = (x(t), y(t), p_x(t), p_y(t)) : t \mapsto T^*\mathbb{R}^2$ there exist constants $\mathrm{const}_1, \mathrm{const}_2$ such that $p_x(t)^2 + X(x(t)) = \mathrm{const}_1$, $p_y(t)^2 + Y(y(t)) = \mathrm{const}_2$. We see that that $p_x(t)^2 = \mathrm{const}_1 - X(x(t))$ and similarly $p_y(t)^2 = \mathrm{const}_2 - Y(y(t))$ so the motion exists only the regions such that $\mathrm{const}_1 \geqslant X(x)$ and $\mathrm{const}_2 \geqslant Y(y)$, so the projection of the trajectory lives in a rectangle.



source: WiKi

Def. Poincare section S: hypersurface in the phase space transversal to orbits.
Def. Poincare mapping: sends $x \in S$ to the first intersection of the trajectory starting from x with S.

If the system is integrable, the orbit of the point w.r.t. the Poincare section lies on $n-1$ dimensional (in our case $n-1=1$) surfaces in S.



Such numerical evidences strongly suggest that the system is integrable, or at least is very close to integrable!

Theorem (Kruglikov-Matveev Phys. Rev. D 2012). The Zipoy-Voorhees metric with $\delta = 2$ does not admit nontrivial integral that is polynomial in momenta of degree $\leqslant 7$ and is in involution with p_t and p_ϕ.

Remark. The result for quadratic integrals was known to Brink.

Unbelievable: The numeric tells that their must exist an integral; since everything is analytic the integral is expected to be analytic and the theorem above says that the low order term in the analytic expression is of order 8 at least!!!

$$ds^2 = \left(\frac{x+1}{x-1}\right)^2 \left[(x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^4 \left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right]-\left(\frac{x-1}{x+1}\right)^2 dt^2.$$

Theorem (Kruglikov-Matveev 2011). The Zipoy-Voorhees metric with $\delta = 2$ does not admit nontrivial integrals of degree $\leqslant 7$

We see that the metric is given in elementary functions. The existence of an integral of degree k in momenta is a system of linear PDE of the first order on the unknown entries (functions of x, y) of the tensor whose coefficients are algebraic expressions in the entries of the metrics and their derivatives.

Say, for $m = 6$ we have 120 equations on 64 unknowns.

We differentiate all equations w.r.t. x, y and consider the derivatives of unknowns as new unknowns. In jargon, this is called prolongation. The obtained system is also linear, so can be written in the form $Au = \vec{0}$, where A is a (known) matrix whose coefficients are algebraic expressions in the derivatives of the metric and u is the vector whose components are unknown entries of the Killing tensor and their derivatives. We repeat the procedure 6 times and obtain the following table:

n	0	1	2	3	4	5	6
# of eqn	60+60	180+180	360+360	600+600	900+900	1260 +1260	1680
$\dim(u)$	132 +120	264+240	440+400	660+600	924+840	1232+1440	1584
$rk(A)$	60+60	180+180	360+360	600+ 590	888+ 838	1215+ 1440	1568

This gives us an upper restriction on the dimension of Killing tensors and proves theorem.

- Theory works
- Calculational difficulties in calculating A
- Calculational difficulties in calculating $rk(A)$.
- We did not seriously try
- It is clear though that one can not go too far

I explained two methods in the theory of integrable systems:

- The first one is a simple trick that allows to reduce solving of linear overdetermined PDE-system possessing symmetry to solving of ODE. The trick did solve two classical problems
  - helped to find all (1+3)-superintegrable 2D metrics and
  - found all metrics admitting projective vector fields (the problem was explicitly formulated by Sophus Lie and remained unsolved until 2012.
- The second is the computer-oriented prolongation-projection method:
  - This method can effectively be used in the search for integrals and can rigorously prove nonintegrability in a certain class of integrals – I considered integrals that are polynomial in momenta of degree $\leqslant 7$
- Definitely, there are many more problems to solve with these methods — try. I hope to see your applications of these "my" methods in the next conference.