Galoisian Approach to Integrability of Schrödinger Equation

Joint work with
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Motivation: Known shape invariant potentials in Quantum Mechanics

Potential	**Name**
$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2$	Shifted H. O.
$\frac{1}{2}m\omega^2 r^2+\frac{l(l+1)\hbar^2}{2mr^2}-\left(l+\frac{3}{2}\right)\hbar\omega$	3D H.O.
$-\frac{e^2}{r}+\frac{l(l+1)\hbar^2}{2mr^2}-\frac{me^4}{2(l+1)^2\hbar^2}$	Coulomb
$A^2+B^2e^{-2ax}-2B\left(A+\frac{a\hbar}{2\sqrt{2m}}\right)e^{-ax}$	Morse 1
$A^2+\frac{B^2-A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\cosh^2 ax}+\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\sinh ax}{\cosh^2 ax}$	Morse 2
$A^2+\frac{B^2}{A^2}+2B\tanh ax-A\frac{A+\frac{a\hbar}{\sqrt{2m}}}{\cosh^2 ax}$	Rosen-Morse 1
$A^2+\frac{B^2+A^2+\frac{Aa\hbar}{\sqrt{2m}}}{\sinh^2 ar}-\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\cosh ar}{\sinh^2 ar}$	Rosen-Morse 2
$A^2+\frac{B^2}{A^2}-2B\coth ar+A\frac{A-\frac{a\hbar}{\sqrt{2m}}}{\sinh^2 ar}$	Eckart 1
$-A^2+\frac{B^2+A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\sin^2 ax}-\frac{B\left(2A-\frac{a\hbar}{\sqrt{2m}}\right)\cos ax}{\sin^2 ax}$	Eckart 2
$-(A+B)^2+\frac{A\left(A-\frac{a\hbar}{\sqrt{2m}}\right)}{\cos^2 ax}+\frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sin^2 ax}$	Pöschl-Teller 1
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(R. Dutt, A. Khare, U.P. Sukhatme, Am.J.Phys. 56(1988)163-168)

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Notations

$\mathcal{L}_\lambda := H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K.$

$\Lambda \subseteq \mathbb{C}$: set of eigenvalues λ such that $\mathcal{L}_\lambda$ is integrable.

$\Lambda_+ := \{\lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0\}$, $\Lambda_- := \{\lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0\}$.

L_λ: Picard-Vessiot extension of $\mathcal{L}_\lambda$.

$\mathrm{Gal}(L_\lambda/K)$: differential Galois group of $\mathcal{L}_\lambda$.

The set Λ will be called *the algebraic spectrum* (or alternatively *the Liouvillian spectral set*) of H.

Λ can be $\emptyset$, i.e., $\mathrm{Gal}(L_\lambda/K) = \mathrm{SL}(2,\mathbb{C})$ $\forall \lambda \in \mathbb{C}$.

If $\lambda_0 \in \Lambda$ then $(\mathrm{Gal}(L_{\lambda_0}/K))^0 \subseteq \mathbb{B}$.

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Algebraically Solvable and Quasi-Solvable Potentials

We say that the potential $V(x) \in K$ is:

- an *algebraically solvable potential* when Λ is an infinite set, or
- an *algebraically quasi-solvable potential* when Λ is a non-empty finite set, or
- an *algebraically non-solvable potential* when $\Lambda = \emptyset$.

When $\mathrm{Card}(\Lambda) = 1$, we say that $V(x) \in K$ is a *trivial* algebraically quasi-solvable potential.

Examples. Assume $K = \mathbb{C}(x)$.

1. If $V(x) = x$, then $\Lambda = \emptyset$, $V(x)$ is algebraically non-solvable.
2. If $V(x) = 0$, then $\Lambda = \mathbb{C}$, i.e., $V(x)$ is algebraically solvable.
3. If $V(x) = \frac{x^2}{4} + \frac{1}{2}$, then $\Lambda = \{n : n \in \mathbb{Z}\}$, $V(x)$ is algebraically solvable (Weber's equation).
4. If $V(x) = x^4 - 2x$, then $\Lambda = \{0\}$, $V(x)$ is algebraically quasi-solvable (trivial).

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Iso-Galoisian Transformations

Let be $\mathcal{L}$, $\widetilde{\mathcal{L}}$, pairs of linear differential equations defined over differential fields K and $\widetilde{K}$ respectively, with Picard-Vessiot extensions L and $\widetilde{L}$. Let φ be the transformation such that $\mathcal{L} \mapsto \widetilde{\mathcal{L}}$, $K \mapsto \widetilde{K}$ and $L \mapsto \widetilde{L}$, we say that:

1. φ is an *iso-Galoisian transformation* if

$$\mathrm{Gal}(L/K) = \mathrm{Gal}(\widetilde{L}/\widetilde{K}).$$

   If $\widetilde{L} = L$ and $\widetilde{K} = K$, we say that φ is a *strong iso-Galoisian transformation*.

2. φ is a *virtually iso-Galoisian transformation* if

$$(\mathrm{Gal}(L/K))^0 = (\mathrm{Gal}(\widetilde{L}/\widetilde{K}))^0.$$

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Schrödinger Equation with Rational Potentials

Schrödinger Equation with Polynomial Potentials

Theorem (Polynomial potentials and Galois groups)

Let $V(x) \in \mathbb{C}[x]$ a polynomial of degree $k > 0$.
Then

1. $\mathrm{Gal}(L_\lambda/K) = \mathrm{SL}(2, \mathbb{C})$, *or,*
2. $\mathrm{Gal}(L_\lambda/K) = \mathbb{B}$.

Corollary

Assume that $V(x)$ is an algebraically solvable polynomial potential. Then $V(x)$ is of degree 2.

Remark. When a polynomial potential is algebraically solvable or quasi-solvable, then the Galois group of the Schrödinger equation is exactly the Borel group (triangular).

Rational Potentials and Kovacic's Algorithm

Three dimensional harmonic oscillator potential

$$V(r) = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3), \quad \ell \in \mathbb{Z}.$$

The Schrödinger equation is

$$\partial_r^2 \Psi = \left(r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda \right) \Psi.$$

Applying Kovacic's algorithm we obtain $\Lambda = 2\mathbb{Z}$ and the eigenfunctions (considering only $\lambda \in 4\mathbb{Z}$):

$$\begin{aligned}
\Psi_n(r) &= r^{\ell+1} P_{2n}(r) e^{\frac{r^2}{2}}, & \lambda \in \Lambda_-, \\
\Psi_n(r) &= r^{-\ell} P_{2n}(r) e^{\frac{r^2}{2}}, & \lambda \in \Lambda_-, \\
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Bound states.

$$\Psi_n(r) = r^{\ell+1} P_{2n}(r) e^{\frac{-r^2}{2}}, \quad \lambda \in 4\mathbb{N}.$$

Galois groups. For $\lambda \in 4\mathbb{Z}$ we have that $\mathrm{Gal}(L_\lambda/K) = \mathbb{B}$.

Relationship with Whittaker equation. Schrödinger equation with the 3D-harmonic oscillator potential, through the changes $r \mapsto \frac{1}{2}\omega r^2$ and $\Psi \mapsto \sqrt{r}\Psi$, fall in a Whittaker differential equation in where the parameters are given by

$$\kappa = \frac{(2\ell+3)\omega + 2E}{4\omega}, \quad \mu = \frac{1}{2}\ell + \frac{1}{4}.$$

Applying Martinet-Ramis theorem, we can see that for integrability, $\pm\kappa \pm \mu$ must be a half integer.

Darboux Transformation

Theorem (Darboux)

Assume $H_\pm = \partial_x^2 + V_\pm(x)$ *and* $\Lambda \neq \emptyset$. *Let* $\mathcal{L}_\lambda$ *given by* $H_-\Psi^{(-)} = \lambda\Psi^{(-)}$ *with* $V_-(x) \in K$ *and* $\widetilde{\mathcal{L}}_\lambda$ *given by* $H_+\Psi^{(+)} = \lambda\Psi^{(+)}$ *with* $V_+(x) \in \widetilde{K}$. *Let* DT *be the transformation such that* $\mathcal{L}_\lambda \mapsto \widetilde{\mathcal{L}}_\lambda$, $V_- \mapsto V_+$, $\Psi^{(-)} \mapsto \Psi^{(+)}$. *Then*

1. $\mathrm{DT}(V_-) = V_+ = \Psi^{(-)}_{\lambda_1}\partial_x^2\left(\frac{1}{\Psi^{(-)}_{\lambda_1}}\right) + \lambda_1 = V_- - 2\partial_x^2(\ln \Psi^{(-)}_{\lambda_1})$, *where* $\Psi^{(-)}_{\lambda_1}$ *is a particular solution of* $\mathcal{L}_{\lambda_1}$, $\lambda_1 \in \Lambda$.
2. $\mathrm{DT}(\Psi^{(-)}_\lambda) = \Psi^{(+)}_\lambda = \partial_x\Psi^{(-)}_\lambda - \partial_x(\ln \Psi^{(-)}_{\lambda_1})\Psi^{(-)}_\lambda = \frac{W(\Psi^{(-)}_{\lambda_1},\Psi^{(-)}_\lambda)}{W(\Psi^{(-)}_{\lambda_1})}$, $\lambda \neq \lambda_1$, *where* $\Psi^{(-)}_\lambda$ *and* $\Psi^{(+)}_\lambda$ *are the general solutions of* $\mathcal{L}_\lambda$ *and* $\widetilde{\mathcal{L}}_\lambda$ *for* $\lambda \in \Lambda \setminus \{\lambda_1\}$.

Proposition

DT *is isogaloisian and virtually strong isogaloisian. Furthermore, if* $\partial_x(\ln \Psi_{\lambda_1}^{(-)}) \in K$*, then* DT *is strong isogaloisian.*

Proposition

The supersymmetric partner potentials $V_\pm$ *are rational functions if and only if the superpotential* W *is a rational function.*

Corollary

The superpotential $W \in \mathbb{C}(x)$ *if and only if* DT *is strong isogaloisian.*

Rational Shape Invariant Potentials. Assume $V_{\pm}(x;\mu) \in \mathbb{C}(x;\mu)$, where μ is a family of parameters. The potential $V = V_{-} \in \mathbb{C}(x)$ is said to be rational shape invariant potential with respect to μ and $E = E_n$ being $n \in \mathbb{Z}_{+}$, if there exists f such that $E_0 = 0$,

$$V_{+}(x;a_0) = V_{-}(x;a_1) + R(a_1), \quad a_1 = f(a_0), \quad E_n = \sum_{k=2}^{n+1} R(a_k).$$

Theorem

Consider $\mathcal{L}_n := H\Psi^{(-)} = E_n\Psi^{(-)}$ with Picard-Vessiot extension L_n, where $n \in \mathbb{Z}_{+}$. If $V = V_{-} \in \mathbb{C}(x)$ is a shape invariant potential with respect to $E = E_n$, then for $n > 0$,

$$\mathrm{Gal}(L_{n+1}/K) = \mathrm{Gal}(L_n/K).$$

Plan

An example of algebrization

The Morse Potential. $V(x) = e^{-2x} - e^{-x}$.
The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = \left(e^{-2x} - e^{-x} - \lambda\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2\partial_z^2\widehat{\Psi} + z\partial_z\widehat{\Psi} - (z^2 - z - \lambda)\widehat{\Psi} = 0.$$

This equation (normalized) is transformable into a Bessel equation and we can apply Kovacic's algorithm.

Algebraic Spectrum. $\Lambda = \{-n^2 : n \geq 0\} = \mathrm{spec}_p(H)$. Also obtained with Bessel equation.

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The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = \left(e^{-2x} - e^{-x} - \lambda\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2\partial_z^2\widehat{\Psi} + z\partial_z\widehat{\Psi} - (z^2 - z - \lambda)\widehat{\Psi} = 0.$$

This equation (normalized) is transformable into a Bessel equation and we can apply Kovacic's algorithm.

Algebraic Spectrum. $\Lambda = \{-n^2 : n \geq 0\} = \mathrm{spec}_p(H)$. Also obtained with Bessel equation.

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Plan

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