
International workshop on
Integrability in
Dynamical Systems and Control
14-16 November 2012, Rouen, France
Preliminary list of speakers
Alain Albouy (Paris)
Elie Assémat (Dijon)
Boris Bardin (Moscow)
Bernard Bonnard (Dijon)
Yuri Fedorov (Barcelona)
Jean-Pierre Françoise (Paris)
Božidar Jovanović (Belgrade)
Clémence Labrousse (Paris)
Jean-Pierre Marco (Paris)
Vladimir Matveev (Jena)
Eva Miranda (Barcelona)
Juan Morales-Ruiz (Madrid)
Andriy Panasyuk (Olsztyn)
Jean-Pierre Ramis (Toulouse)
Vladimir Roubtsov (Angers)
Dmitry Sinitsyn (Moscow)
Alexei Tsygvintsev (Lyon)
Jacques-Arthur Weil (Limoges)
Nguyen Tien Zung (Toulouse)
The workshop is organized at the National Institute
of Applied Sciences - INSA de Rouen
(V. Salnikov, W. Respondek, A. Maciejewski)
For more information and registration please visit our website:
http://math.univ-lyon1.fr/~salnikov/disco/
INSA ROUEN
MINISTERE DES AFFAIRES ETRANGERES
Normandie-Mathématiques

ISO-IRREGULAR DEFORMATIONS OF LINEAR O.D.E.
and
DYNAMICS OF PAINLEVÉ EQUATIONS

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International workshop on
Integrability in Dynamical Systems and Control
I.N.S.A. de Rouen, 14-16 novembre, 2012

PRESENTATION

We will sketch a *work in progress* in collaboration with Emmanuel Paul and Julio Rebelo, based on some results due to several people, mainly:

Ph. Boalch, S. Cantat, M. A. Inaba, K. Iwasaki, M. Jimbo, F. Loray, B. Malgrange, T. Miwa, M. van der Put, M-H Saito, K. Ueno, E. Witten,...

In the present state it is mainly a *PROGRAM*.

Painlevé differential equations

P_I: $\frac{d^2y}{dt^2} = 6y^2 + t.$

P_{II}: $\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha.$

P_{III}: $\frac{d^2y}{dt^2} = \frac{1}{y}(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.$

P_{IV}: $\frac{d^2y}{dt^2} = \frac{1}{2y}(\frac{dy}{dt})^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}.$

P_V:
$$\frac{d^2y}{dt^2} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t}(\alpha y + \frac{\beta}{y}) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}.$$

P_{VI}:
$$\frac{d^2y}{dt^2} = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}))(\frac{dy}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}).$$

With $\alpha, \beta, \gamma, \delta \in \mathbf{C}$.

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P_V: $\frac{d^2y}{dt^2} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t}(\alpha y + \frac{\beta}{y}) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}.$

P_{VI}: $\frac{d^2y}{dt^2} = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}))(\frac{dy}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}).$

With $\alpha, \beta, \gamma, \delta \in \mathbf{C}$.

Non-linear second order O.D.E. whose all *moving singularities* are *poles* (Painlevé property).

Fixed singularities: ∞ for P_I, P_{II}, plus 0 for P_{III}, P_{IV}, P_V, plus $0, 1$ for P_{VI}.

Figure: Paul Painlevé 1863-1933

Painlevé differential equations

P_I, P_{II}, P_{III}: Painlevé.
P_{IV}, P_V: Gambier.
P_{VI}: Richard Fuchs (son of Lazarus Fuchs), in relation with *isomonodromic deformations* of linear O.D.E..

Painlevé equations are (up to equivalence) all the non-linear second order O.D.E. whose all *moving singularities* are *poles* (Painlevé property) and which are not reducible to "already known cases" (linear equations, Riccati equations, differential equation of elliptic functions).

Property of *"irreducibility"*: *new transcendental functions*, proved by Nishioka (1988) and H. Umemura (1989).

The Painlevé equations except P_{VI} were discovered in relation with the *Painlevé property*.

A misunderstanding

The Painlevé equations, except P_{VI}, were discovered in relation with the *Painlevé property*. At Painlevé time there was some interest in the so-called Painlevé property, in particular (I think...) in relation with the success of S. Kowalevska for the problem of the top. Today Painlevé property remains quite popular, due in particular (I think...) to the fact that there exists an *effective test*.

BUT the fact that so important equations (the O.D.E. of the "special functions of the XX-th century"...) were discovered starting from Painlevé property is a chance (2 is small !). The *really important point* is NOT the Painlevé property but the fact that all Painlevé equations *translate phenomena of isomonodromic (or more generally iso-irregular: Garnier) deformations* of linear O.D.E..

All O.D.E. coming from isomonodromic (or more generally iso-irregular) deformations have the Painlevé property (Malgrange, Miwa...) but the converse can be false for equations of order > 2.

The dynamics of the Painlevé equations

Our project is to use the “true origin” of the Painlevé equations, that is the fact that they translate phenomena of isomonodromic (or more generally iso-irregular) deformations of linear O.D.E., to understand their dynamics.

Our main purpose is to “understand the dynamics” of the *six* Painlevé equations, in particular to be able *to compute their non-linear differential Galois groupoids in Malgrange sense*. We conjecture that, for “generic values” of the parameters, these groupoids are “as big as possible”: *conservation of the area* (as for P_I, a result of G. Casale).

The results are known for P_{VI} (Cantat, Loray, Iwasaki, M.H. Saito...) and our idea is to “imitate” the method for the remaining equations $P_V, \; P_{IV}, \; P_{III}, \; , P_{II}, \; P_I$

In the case of P_{VI} one *translates*, via the Riemann-Hilbert correspondance, the initial *transcendental* problem (the study of the dynamics of the equation, or “equivalently” of the study of the *non-linear* monodromy around $0, 1, \infty$) into a *purely topological problem*: the study of the dynamics induced by a *braid group* on a *character variety*. The character variety is an *affine algebraic surface* (with a complex Poisson structure) and the action is *polynomial* and *explicit*.

In the case of P_{VI} one *translates*, via the Riemann-Hilbert correspondance, the initial *transcendental* problem into a *purely topological problem*: the study of the dynamics induced by a *braid group* on a *character variety*. The character variety is an *affine algebraic surface* (with a complex symplectic structure) and the action is *polynomial* and *explicit*.

In the case of $P_V,\ P_{IV},\ P_{III},\ ,P_{II},\ P_I$, we can *translate*, via the irregular Riemann-Hilbert correspondance (in Martinet-Ramis style), the *transcendental* initial problem into a new one. This new problem is no longer purely topological. It is necessary to replace the four punctured sphere by some *irregular curves*, the braid group by some *wild braid groups* and the character varieties by some *wild character varieties* (replacing *representations* of a *fundamental group* by *representations* of a *wild fundamental groupoid*).

More precisely:

– in the case of P_{VI}, the *Painlevé flow* is *conjugated* to the *isomonodromy flow* on a fibre bundle of *character varieties*;

– in the case of the others Painlevé equations, the *wild Painlevé flow* (i.e. the Painlevé flow “plus” the Stokes actions) is *“conjugated”* to the *wild isomonodromy flow* on a fibre bundle of *wild character varieties*;

We will explain the mechanism in a quite general situation and afterwards we will give an idea of what is happening in the particular case of the irregular Painlevé equations using some recent results of M. van der Put and M.H. Saito.

A dream....

Based on ideas of the Japanese school (Iwasaki...) for the classic case and of Ph. Boalch for the wild case (using Martinet-Ramis approach)

CLAIM (conjectural)

Many ”interesting” *algebraic dynamical systems* “express” *iso-irregular deformations of linear connections*.

Then it is possible to *“compute effectively their dynamics”* (by hand or using computer algebra) as (Poisson) actions of braid goups or wild braid groups on (Poisson) algebraic varieties.

This opens the possibility to compute differential invariants and *the Malgrange differential groupoid* for such systems.

Chaos seems to be more or less the rule and to call “integrable” such systems (it is the “classical” terminology) is not so appropriate !

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CHARACTER VARIETIES

and

THE DYNAMICS OF PAINLEVÉ VI EQUATIONS

Following:

S. Cantat, M. A. Inaba, K. Iwasaki, F. Loray, M-H Saito,...

Character varieties

Let X be a compact Riemann surface X (of genus g) and $a_1, \ldots, a_m$ marked points on it ($m = 0$ is allowed). Let G be a linear complex algebraic group G ($G = GL_n(\mathbf{C})$, $G = SL_n(\mathbf{C})$,...).

We consider the set of representations $\pi_1(X \setminus \{a_1, \ldots, a_m\}, \star) \to G$, $\star \in X \setminus \{a_1, \ldots, a_m\}$ being a fixed base point, modulo the adjoint action of G (equivalence of representations):

$$Hom(\pi_1(X \setminus \{a_1, \ldots, a_m\}), G)/G,$$

$Hom := Hom_{gr}$. It is a *character variety*.

The quotient is in some *algebraic sense* (categorical quotient, Jordan equivalence), I skip the details: for the irreducible case there is no problem...

We suppose now that $X := P^1(\mathbf{C}) \approx S^2$ and $m > 0$ and we denote the character variety $\chi(S^2_m)$. Then the fundamental group $\pi_1(P^1(\mathbf{C}) \setminus \{a_1, \ldots, a_m\})$ is the (non abelian) free group generated by $m - 1$ (homotopy classes of) loops $\gamma_1, \ldots, \gamma_{m-1}$:

We suppose now that $X := P^1(\mathbf{C}) \approx S^2$ and $m > 0$ and we denote the character variety $\chi(S^2_m)$. Then the fundamental group $\pi_1(P^1(\mathbf{C}) \setminus \{a_1, \ldots, a_m\})$ is the (non abelian) free group generated by $m-1$ (homotopy classes of) loops $\gamma_1, \ldots, \gamma_{m-1}$: we can choose $m-1$ points $\tilde{a}_1, \ldots, \tilde{a}_{m-1}$ (respectively) "very near" of $a_1, \ldots, a_{m-1}$, $m-1$ paths $\delta_1, \ldots, \delta_{m-1}$ (from $\star$ to $\tilde{a}_1, \ldots, \tilde{a}_{m-1}$) and $m-1$ loops $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{m-1}$ based at $\tilde{a}_i$ and turning "one time" around a_i, then:

$$\gamma_i := \delta_i \, \tilde{\gamma}_i \, \delta_i^{-1}.$$

We can identify a representation ρ with:

$$(M_1 := \rho(\gamma_1), \ldots, M_{m-1} := \rho(\gamma_{m-1})) \in G^{m-1}$$

and the character variety $\chi(S^2_m)$ with G^{m-1}/G (adjoint action).

There is an alternative description, replacing the fundamental group based at $\star$, by a fundamental groupoid "based at $\star, \tilde{a}_1, \ldots, \tilde{a}_{m-1}$". Then a representation ρ is the data of $\rho(\delta_1) \in G, \ldots, \rho(\delta_{m-1}) \in G$ and $\rho(\tilde{\gamma}_1) \in G, \ldots, \rho(\tilde{\gamma}_{m-1}) \in G$. We can identify the character variety $\chi(S^2_m)$ with G^{2m-2}/G^m.

Variant:

We can "add" to X the *real blow up* of each point $a_1, \ldots, a_{m-1}$, that is $m-1$ *circles* S^1 and choose points $\tilde{a}_1, \ldots, \tilde{a}_{m-1}$ respectively on each circle ("tangential points"). We get a $m-1$ pointed surface $\tilde{X}$ (with a boundary).

In order to parametrize the character variety, we can change the "basis" of the fundamental groupoid, leaving the $\tilde{\gamma}_i$ *fixed* and changing the choice of the (homotopy classes) of the δ_i.

The case of the four punctured sphere

Let $S_4^2 = P^1(\mathbf{C}) \setminus \{a_1, a_2, a_3, a_4\}$ and $G := Sl_2(\mathbf{C})$.

Let $\rho : \pi_1(S_4^2) \to Sl_2(\mathbf{C})$. The knowledge of the matrices $\rho(\gamma_1),\ \rho(\gamma_2),\ \rho(\gamma_3),\ \rho(\gamma_4)$ modulo, *for each one*, the adjoint action of $Sl_2(\mathbf{C})$ is equivalent to the knowledge of the *four parameters*:

$a := \mathrm{Tr}\,(\rho(\gamma_1)),\ b := \mathrm{Tr}\,(\rho(\gamma_2)),\ c := \mathrm{Tr}\,(\rho(\gamma_3)),\ d := \mathrm{Tr}\,(\rho(\gamma_2))$.

We can associate to ρ *three more parameters*:

$x := \mathrm{Tr}\,(\rho(\gamma_1\gamma_2)),\ \ y := \mathrm{Tr}\,(\rho(\gamma_2\gamma_3)), x := \mathrm{Tr}\,(\rho(\gamma_3\gamma_1))$,

invariant under *overall* conjugation.

We have $\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3)\rho(\gamma_4) = I$, the knowledge of the *representation* ρ is equivalent to the knowledge of the *three matrices* $\rho(\gamma_1),\ \rho(\gamma_2),\ \rho(\gamma_3) \in Sl_2(\mathbf{C})$. We can identify $\mathrm{Rep}(S_4^2, Sl_2(\mathbf{C}))$ with the *affine variety* $(Sl_2(\mathbf{C}))^3$ (of dimension 9).

The *polynomial* map: $\chi : \mathrm{Rep}(S_4^2, Sl_2(\mathbf{C})) \to \mathbf{C}^7$ defined by $\rho \mapsto \chi(\rho) := (a, b, c, d, x, y, z)$ is invariant under (overall) conjugation.

The case of the four punctured sphere: the character variety

The *polynomial* map: $\chi : \mathrm{Rep}(S_4^2, Sl_2(\mathbf{C})) \to \mathbf{C}^7$ defined by $\rho \mapsto \chi(\rho) := (a, b, c, d, x, y, z)$ is invariant under conjugation.

The components of χ satisfy the *quartic* equation:

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D,$$

where

$$A := ab + cd,\ B := bc + ad,\ C := ac + bd$$
$$D := 4 - a^2 - b^2 - c^2 - d^2 - abcd.$$

The family $\mathbf{C}[\mathrm{Rep}(S_4^2, Sl_2(\mathbf{C}))]^{Sl_2(\mathbf{C})}$ of polynomial functions on $\mathrm{Rep}(S_4^2, Sl_2(\mathbf{C}))$ invariant par $Sl_2(\mathbf{C})$ is generated by the components of χ, the *algebraic quotient* $\mathrm{Rep}(S_4^2, Sl_2(\mathbf{C}))/Sl_2(\mathbf{C})$, that is the *character variety*, is isomorphic to the hypersurface of $\mathbf{C}^7$ defined by the above equation: a *six-dimensional quartic*.

The *algebraic quotient* $\text{Rep}(S_4^2, Sl_2(\mathbf{C}))/Sl_2(\mathbf{C})$, that is the *character variety*, is isomorphic to the hypersurface of $\mathbf{C}^7$ defined by the equation:

$$F(x,y,z) = x^2 + y^2 + z^2 + xyz - (Ax + By + Cz + D) = 0,$$

a *six-dimensional quartic*.

If we *fix the parameters* a, b, c, d (*"local monodromies"* at the singular points up to conjugation), and therefore A, B, C, D, then x, y, z belongs to a *cubic surface* $S_{(A,B,C,D)}$ of $\mathbf{C}^3$.

There is a *volume form*:

$$\Omega := \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B}$$

on (the smooth part of) this surface ($\Omega \wedge dF = dx \wedge dy \wedge dz$). It defines a *complex symplectic structure*.

There is an *area form*:

$$\Omega := \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B}$$

on (the smooth part of) the cubic surface $S_{(A,B,C,D)}$. It defines a *complex symplectic structure*.

This is a particular case of a very general result: there is a *complex canonical Poisson structure* on the *character varieties* and on the *irregular character varieties* and the corresponding *symplectic foliations* corresponds to the fixation of local data at the singular points: Ph Boalch,...
The *dynamics* induced by *braiding groups* and *wild braiding groups* on the character varieties respects these foliations, they induces *algebraic symplectic dynamics* on algebraic varieties (in the present case cubic surfaces).
One our main projects is *a systematic study of such dynamics*, beginning with the "simplest cases" (!!!) related to the Painlevé équations $P_V,\ P_{IV},\ P_{III},\ ,P_{II},\ P_I$.

Monodromy of a linear differential system

Let (Δ) : $\frac{dY}{dz} = A(z)Y$ be a complex linear differential system of rank n on a Riemann surface X (z local coordinate), with A an *holomorphic* matrix. More generally we can suppose A *meromorphic* and replace X by $X \setminus S$, S being the *singular set* of A.

If $a, b \in X$ and if γ is a continuous path from a to b, we can *extend analytically* along γ any *local solution at a* and we get a *local solution at b*, the result depends only on the *homotopy class* $[\gamma]$ of γ. We get a linear isomorphism $M_\gamma : \mathrm{Sol}_a \to \mathrm{Sol}_b$. If γ is a *loop* ($a = b$), then M_γ is a linear automorphism of the complex vector space Sol_a, the monodromy of γ. We get a map:

$$\rho : \pi_1(X, a) \to GL(\mathrm{Sol}_a)$$

It is an homomorphism of groups (opposite structure on π_1), the *monodromy representation* associated to the system (Δ).

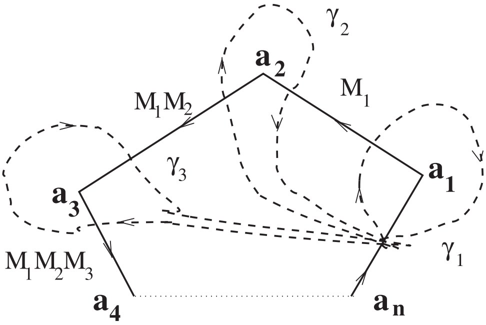

Figure 2. Fuchsian Riemann-Hilbert problem.

Figure: Monodromy

Fuchsian systems on the Riemann sphere, Riemann-Hilbert map

We consider a logarithmic connection ∇ on a trivial rank n complex vector bundle on the Riemann sphere with singurarities at $a_1, \dots, a_m$ $(m \geq 2)$. Choosing a coordinate z on the sphere, with $a_m = \infty$, this amounts to giving a differential system:

$$(\Delta) : \frac{dY}{dz} = A(z)Y, \quad \text{with} \quad A(z) := \sum_{i=1,\dots,m-1} \frac{A_i}{z - a_i},$$

the matrices A_i (residues) being constant. The residue at $a_m = \infty$ is $A_m := -A_1 - \dots - A_{m-1}$.

The Riemann Hilbert map associates to the Fuchsian system (Δ) its *monodromy representation*

$$\rho : \pi_1(P^1(\mathbf{C}) \setminus \{a_1, \dots, a_m\}, \star) \rightarrow GL_n(\mathbf{C}).$$

Riemann-Hilbert map, Riemann-Hilbert problem

We choose simple loops γ_i around a_i, based at $\star$, such that $\gamma_1 \dots \gamma_m$ is contractile.
We set $\mathbf{a} := (a_1, \dots, a_m)$, $\mathbf{A} := (A_1, \dots, A_m)$,
then $RH_{\mathbf{a}}$ amounts to the map:
$\{\mathbf{A} \,|A_1 + \cdots + A_m = 0\} \to \{(M_1, \dots, M_m)|M_m \cdots M_1 = I\}$,
where $M_i := \rho(\gamma_i)$.
If an *irreducible* representation ρ is given, by
$\mathbf{M} := (M_1, \dots, M_m)$, with $M_m \cdots M_1 = I$, then there exists
$\mathbf{A}$, with $A_1 + \cdots + A_m = 0$, such that $RH_{\mathbf{a}}(\mathbf{A}) = \mathbf{M}$. The
(strong) *Riemann-Hilbert problem* admits a solution.

It is important to notice that the Riemann-Hilbert map is in general *transcendental* (cf. the case of $m = 3$: the *hypergeometric functions*). Therefore it is not so surprising that, as we will see, *RH* can transform a *transcendental dynamics* into an *algebraic dynamics*.

Isomonodromic deformations

If we move $\mathbf{a} \in P^1(\mathbf{C})$, the points $a_1, \dots, a_m$ remaining *distinct*, the *topology* of $P^1(\mathbf{C}) \setminus \{a_1, \dots, a_m\}$ does not change, but, if $m \geq 4$, the *complex structure* changes, there are *moduli*. The basis of the deformation is $\mathcal{B} := \mathbf{C}^m \setminus \bigcup_{i \neq j} \Delta_{ij}$, where $\Delta_{ij} := \{x_i = x_j\}$.

Schlesinger studied the following problem, the problem of *isomonodromic deformations*: is it possible to vary the matrices $\mathbf{A}$ as one move $\mathbf{a} \in \mathcal{B}$ in order to realise "the same monodromy data" $\mathbf{M}$ (up to overall conjugation).

Locally this makes sense, starting from $\mathbf{a}^0 := (a_1^0, \dots, a_m^0)$ one can use *the same loops* to generate:

$$\pi_1(P^1(\mathbf{C}) \setminus \{a_1^0, \dots, a_m^0\}, \star) \text{ and } \pi_1(P^1(\mathbf{C}) \setminus \{a_1, \dots, a_m\}, \star),$$

if $\mathbf{a}$ is "sufficiently near" of $\mathbf{a}^0$.

Isomonodromic deformations and Schlesinger equations

Schlesinger discovered that if the matrices A_i satisfy the following differential equations, then the monodromy data is localy preserved (up to overall conjugation):

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j}, \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}$$

Conversely, in the generic case, an isomonodromic deformation satisfies the Schlesinger equations.

Figure: Ludwig Schlesinger 1864-1933, follower and son-in-law of Lazarus Fuchs

Schlesinger equations and P_{VI}

We suppose that $m = 4$ and that the A_i are *trace free* rank two matrices: $A_i \in sl_2(\mathbf{C})$.

Using the action of the *Möbius group* on $P^1(\mathbf{C})$, we can suppose that $a_1 = 0, a_2 = t, a_3 = 1, a_4 = \infty$. Then the Schlesinger equations are:

$$\frac{\partial A_1}{\partial t} = \frac{[A_2, A_1]}{t} \quad \frac{\partial A_2}{\partial t} = \frac{[A_1, A_2]}{t} + \frac{[A_3, A_2]}{t-1} \quad \frac{\partial A_3}{\partial t} = \frac{[A_2, A_3]}{t-1}.$$

This is a differential system with the unknown function (A_1, A_2, A_3) (9 scalar unknown functions).

The Schlesinger system preserves $A_4 = -A_1 - A_2 - A_3$. We fix the eigenvalues of A_i: $\pm\theta_i/2$ $(i = 1, 2, 3)$ and we suppose that $A_4 = -A_1 - A_2 - A_3$ is *diagonalizable*. Then we can conjugate the system such that

$$A_4 = \mathrm{Diag}\,(\theta_4/2, -\theta_4/2).$$

Schlesinger equations and P_{VI}

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$$A_4 = \text{Diag}\,(\theta_4/2, -\theta_4/2).$$

It is possible to choose a pair (y, x) of conjugate coordinates on the space of the entries of A_1, A_2, A_3 such that the Schlesinger system is equivalent to a differential system on (y, x). There is a good choice such that, eliminating x, we get an equivalent second order differential equation which is a Painlevé VI equation with a convenient choice of parameters.

Explicitly, using the fact that the $(1, 2)$ entry of A_4 is 0, we see that the $(1, 2)$ entry of $z(z-1)(z-t)\sum_{i=1}^{3}\frac{A_i}{z-a_i}$ is a *degree one* polynomial in z. We define $y(t)$ as the unique zero of this polynomial (Jimbo-Miwa).

Another parametrization of P_{VI}

$P_{VI}(\theta)$:

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$

$$+\frac{y(y-1)(q-t)}{t^2(t-1)^2}\left(\frac{(\theta_4-1)^2}{2} - \frac{\theta_1^2}{2}\frac{t}{y^2} + \frac{\theta_3^2}{2}\frac{t-1}{(y-1)^2} + \frac{1-\theta_2^2}{2}\frac{t(t-1)}{(y-t)^2}\right),$$

$$\theta := (\theta_1, \theta_2, \theta_3, \theta_4).$$

Each solution $t \mapsto y(t)$ extends analytically as a *meromorphic function* on the universal covering of $P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$: the Painlevé property.

The space of initial conditions of P_{VI}

The naive phase space of $P_{VI}(\theta)$ is $(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}) \times \mathbf{C}^2$:

$$(t, q(t), q'(t)) \in P^1(\mathbf{C}) \setminus \{0, 1, \infty\}) \times \mathbf{C}^2$$

The "good" phase space is a convenient semi-compactification, a fiber space

$$\mathcal{M}(\theta) \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\},$$

whose fiber at any point $t_0 \in P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ is the *Okamoto space of initial conditions*: the Hirzebruch surface F_2 blown-up at 8 points minus some divisor, a union of 5 rational curves (Kazuo Okamoto 1979). The Painlevé foliation gives a local *analytic* trivialisation of the bundle (which is not algebraically locally trivial).

Then the *non-linear monodromy* of $P_{VI}(\theta)$ is given by a representation:

The Painlevé foliation gives a local *analytic* trivialisation of the bundle:

$$\mathcal{M}(\theta) \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\},$$

Then the *non-linear monodromy* of $P_{VI}(\theta)$ is given by a representation:

$$\pi_1(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}, t_0) \to Diff(\mathcal{M}(\theta), t_0)$$

We get a dynamics on the space of initial conditions. The main idea is, using the *Riemann-Hilbert correspondence*, to replace this *transcendental dynamics* by a simple *algebraic dynamics*.

Character varieties and P_{VI}: comparison of dynamics

The Okamoto space of initial conditions $\mathcal{M}(\theta)$ can be interpreted as the moduli space of rank 2, trace free, meromorphic connections having simple poles at $0, t, 1, \infty$ with prescribed residual eigenvalues

$$\pm\frac{\theta_1}{2}, \pm\frac{\theta_2}{2}, \pm\frac{\theta_3}{2}, \pm\frac{\theta_4}{2}.$$

The *Riemann-Hilbert correspondence* provides an analytic diffeomorphism $\mathcal{M}(\theta) \to \hat{S}_{(A,B,C,D)}$, where $\hat{S}_{(A,B,C,D)}$ is *the minimal desingularization* of $S_{(A,B,C,D)}$. The Painlevé foliation corresponds to the Schlesinger foliation and the Schlesinger foliation corresponds to the *natural isomonodromic connection* on the bundle of character varieties $S_{(A,B,C,D)} \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$. Therefore the monodromy of $P_{VI}(\theta)$ corresponds to the before described representation :

$$\pi_1(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}, \star) \to \mathrm{Aut}(S_{(A,B,C,D)}).$$

INVARIANTS OF MEROMORPHIC O.D.E.

Transformations

In order to study a differential system we can use two types of transformations:

1. Transformations of the independant variable z, that is analytic automorphisms ϕ of U: $z := \phi(u)$. We replace the infinitesimal automorphism $\frac{d}{dz}$ by $\frac{d}{du} = \frac{dz}{du}\frac{d}{dz} = \phi'(u)\frac{d}{dz}$.
2. Linear transformations of the unknown vector function Y.

In the two cases we can use only "known" transformations. In the first case we will use *Möbius transformations* (homographies), in the second we will use "rational" transformations $Y = PZ$, with $P \in GL_n(K)$, $K \subset \mathcal{M}(U)$ being a field of "known" functions, containing the entries of the matrix A of the given system, the "rationality field".

Rationality fields: differential fields

We can work with various rationality fields, we need on K (or $\hat{K}$) a structure of *differential field*.

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  $K := \mathbf{C}(z)$, $K := \mathcal{M}(\mathbf{C})$, $K := \mathcal{M}(U),\ U \subset P^1(\mathbf{C})$; d/dz;

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  $K := \mathbf{C}(\{z\})$ (the field of fractions of convergent power series $\mathbf{C}\{z\}$) $K := \mathbf{C}(\{(z-a)\})$;

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- Local cases:

  $K := \mathbf{C}(\{z\})$ (the field of fractions of convergent power series $\mathbf{C}\{z\}$) $K := \mathbf{C}(\{(z-a)\})$;
- Formal cases:

  $\hat{K} := \mathbf{C}((z))$ (the field of fractions of $\mathbf{C}[[z]]$);
  $\hat{K} := \mathbf{C}((1/z))$ (the field of fractions of $\mathbf{C}[[1/z]]$)...

According to the cases we need on K (or $\hat{K}$) a structure of *differential field*.

More generally we introduce the notions of *differential ring*.

A commutative ring (with an unity 1) with an operator (R, ∂), $\partial : R \to R$ (resp. (R, ψ), $\psi : R \to R$) is a differential (resp. difference) ring if:

- ∂ is additive: if $f, g \in R$, then $\partial(f + g) = \partial f + \partial g$;
- ∂ is a derivation: if $f, g \in R$, then $\partial(fg) = (\partial f)g + f\partial g$ (resp. ∂ is a homomorphism: if $f, g \in R$, then $\partial(fg) = (\partial f)(\partial g)$).

Constants: $C_R := \{f \in R | \partial f = 0\}$. Here in all the cases the constants field is $\mathbf{C}$.

Gauge transformations

We consider a differential field (K, ∂) and a differential system:

$$\partial Y = AY, \quad A \in M_n(K).$$

If we introduce a linear transformation $Y = PZ$ on the unknown vector Y: $P \in Gl_n(K)$, whose entries belong to the "rationality field" K, we get:

$$AY = \partial Y = \partial(PZ) = (\partial P)Z + P\partial Z$$

$$APZ = \partial Y = \partial(PZ) = (\partial P)Z + P\partial Z$$

$$\partial Z = (P^{-1}AP - P^{-1}\partial P)Z$$

or

$$\partial Z = BZ, \quad \text{with} \quad B := P^{-1}AP - P^{-1}\partial P.$$

We can replace the gauge group $Gl_n(\mathbf{C})$ by a *linear algebraic group G*: $P \in G(K)$.

Following Birkhoff (1913) we consider some problems of *classification* of differential systems that is the study of the quotients of $Gl_n(K)$ by the corresponding equivalence relations. More precisely the purpose of Birkhoff was to identify each equivalence class by a set of invariants (the *analysis*) such that conversely to a set of invariants corresponds an unique equivalence class (the *synthesis*).

There are two classes of invariants: *algebraic invariants* (computable by hand or using computer algebra) and *transcendental invariants*.

If the matrix A of the system is *constant* (i.e. if its entries belong to C_K) then we can choose the matrix P constant and we get:

$$B := P^{-1}AP - P^{-1}\partial P = P^{-1}AP$$

and the corresponding equivalence class is the *similitude class* of the matrix A.

Riemann-Hilbert Problems

The *direct problem* of the classification is to get a *complete set of invariants*, characterizing the class. The *inverse problem* is to start from a set of “invariants” and to build an equation (up to equivalence).

The main tool of Birkhoff for the solution of the inverse problem is a theorem of factorization of matrix functions (Birkhoff factorization theorem 1911). The problem was initially stated by Riemann (posthumous note, 1876), anterior results are due to Hilbert (1905) and Plemelj (1906, 1908). Birkhoff result was later improved by Garnier (1951).

A weaker version is equivalent to a theorem of classification of fibre bundles on the Riemann sphere (Grothendieck 1957).

Fundamental solutions

Let $Y' = AY$ be a meromorphic differential system on $P^1(\mathbf{C})$ (or more generally on a Riemann surface X).

Let S be its singular set (a finite subset of $P^1(\mathbf{C})$). Let $\star$ be a *regular* point, due to Cauchy theorem there exists a *holomorphic fundamental solution* F in a neighborhood of $\star$ (we can choose $F(\star) = I$), extending it by analytic continuation, we get an holomorphic matrix F on the universal covering of $(P^1(\mathbf{C}) - S; \star)$ (or $(X - S; \star)$) which is a solution (in the evident sense).

If F_1 is another fundamental solution, then $F_1 = FC$, where $C \in Gl_n(\mathbf{C})$ is a constant matrix and conversely.

Let G be a *connected* linear algebraic group and let $\mathfrak{g}$ be its Lie algebra. We suppose $A \in \mathfrak{g}(K)$, if $F(z_0) = I$, then F takes its values in G. If F_1 is another fundamental solution taking its values in G, then $F_1 = FC$, where $C \in G(\mathbf{C})$.

An example: the hypergeometric differential equations

The first solution of the classification problem is due to B. Riemann in the case of the *hypergeometric* differential equations:

$$(E_{\alpha,\beta,\gamma}) \quad z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta = 0,$$

$\alpha,\ \beta,\ \gamma \in$ **C**. Hypergeometric series:

$${}_2F_1(\alpha,\beta;\gamma;z) := \sum_{n=0}^{+\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!},\ (a)_n = \alpha(\alpha-1)\cdots(\alpha-n+1).$$

Fundamental system of solutions at the origin:

$${}_2F_1(\alpha,\beta;\gamma;z), \quad z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z).$$

Fundamental system of solutions at the origin (generic case):

$$ {}_2F_1(\alpha,\beta;\gamma;z), \quad z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z). $$

Monodromy around the origin: $z \to e^{2i\pi}z$ (analytic continuation along a simple loop around 0):

$$ M_0 := \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi\gamma} \end{pmatrix}. $$

Monodromy exponents:

- at 0: $0, 1-\gamma$

We can put the hypergeometric equation in system form, then starting from a fundamental solution F in a small neighborhood of the point $\star$: $z = 1/2$, we get, by analytic continuation along a loop ℓ at $1/2$ a *linear permutation* of the solutions (monodromy transformation):

$\mathcal{M}_\ell \in Gl(Sol_{1/2})$, and, using F, a monodromy matrix M_ℓ:

$$ F \mapsto FM_\ell. $$

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- at 1: α, β

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Fundamental system of solutions at the origin (generic case):

$$ {}_2F_1(\alpha, \beta; \gamma; z), \quad z^{1-\gamma}{}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z). $$

Monodromy around the origin: $z \to e^{2i\pi} z$ (analytic continuation along a simple loop around 0):

$$ M_0 := \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi\gamma} \end{pmatrix}. $$

Monodromy exponents:

- at 0: $0, 1 - \gamma$
- at 1: α, β
- at ∞: $0, \gamma - \alpha - \beta$.

We can put the hypergeometric equation in system form, then starting from a fundamental solution F in a small neighborhood of the point $\star$: $z = 1/2$, we get, by analytic continuation along a loop ℓ at $1/2$ a *linear permutation* of the solutions (monodromy transformation): $\mathcal{M}_\ell \in Gl(Sol_{1/2})$, and, using F, a monodromy matrix M_ℓ:

$$ F \mapsto F M_\ell. $$

The map $\ell \mapsto \mathcal{M}_\ell$ induces a map (the monodromy representation):

$$\pi_1(P^1(\mathbf{C}) - \{0, 1, \infty\}; 1/2) \to Gl(Sol_{1/2})$$

and, using F (equivalently a basis of $Sol_{1/2}$) a map:

$$\pi_1(P^1(\mathbf{C}) - \{0, 1, \infty\}; 1/2) \to Gl_2(\mathbf{C}),$$

defined *up to conjugation*.

Using a *"good choice"* for F, B. Riemann computed this representation for the hypergeometric differential equations using only *trigonometric functions*. More generally we can compute it from "natural choices" for F using trigonometric functions and Γ function. The *monodromy* is (generically) a *transcendental function* of α, β, γ.

If $F = PF_1$, with $P \in Gl_2(\mathbf{C}(z))$ (gauge transformation), then P is *invariant by the monodromy*. As $FM_\ell = PF_1 M_\ell$, F_1 is transformed by the monodromy into $F_1 M_\ell$, therefore the monodromy is *invariant by equivalence* (local monodromy or global monodromy).

This remains true in the general case: $Y' = AY$ with $A \in Gl_n(\mathbf{C}(z))$. If $S \subset P^1(\mathbf{C})$ is the set of poles of A, then we have a monodromy representation:

$$\pi_1(P^1(\mathbf{C}) - S; z_0) \to Gl(Sol_{z_0}),$$

z_0 being a regular point ($z_0 \in P^1(\mathbf{C}) - S$), and using a fundamental solution F at z_0 (Cauchy existence theorem) we get a map:

$$\pi_1(P^1(\mathbf{C} - S; z_0)) \to Gl_n(\mathbf{C})$$

This *monodromy representation* is *invariant by equivalence*.
There is also a local version and a formal version (using the formal monodromy).

In the global, local, formal cases, the monodromy representation gives some invariants. It is natural to ask the following questions:

- Is it possible to deduce “all the invariants” from the knowledge of the monodromy (is the monodromy a “complete set of invariants”) ?
- Is it possible from the knowledge of a finite dimensional representation of the fundamental group π_1 to get a differential system up to equivalence ?

For the hypergeometric differential equations and the complex linear two-dimensional representations of the free group $\mathbf{Z} * \mathbf{Z}$ the answers are yes. It is due to Riemann.

In the general case it can be no. The simplest example is $y' - y = 0$ on $P^1(\mathbf{C})$: the monodromy is trivial, it is the same than the monodromy of $zy' - y = 0$ and the two equations are not equivalent.

There is a nice class of linear differential systems such that the answers are yes, the *regular singular systems*.

The condition is local at the singularities.
A system $Y' = AY$ is *Fuchsian* at z_1 if A admits a simple pole at z_1, it is *regular singular* at z_1 if it is locally meromorphically equivalent to a Fuchsian system at z_1.
If $z_1 = 0$, if $\frac{A_0}{z}$ is the polar part of A at 0 and if $A_0 \in M_n(\mathbf{C})$ is non resonant (i.e. the difference of the eigenvalues are never positive integers), then the system $Y' = AY$ is *locally* equivalent to the system:

$$zY' = A_0 Y.$$

For differential equations there is no difference between Fuchsian and regular singular, the condition (at z_1) is:
it is possible to write the equation:

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0 y = 0$$

with $b_0, \ldots, b_{n-1}$ holomorphic at z_1.

In the non regular singular case there are other invariants than the monodromy. A non regular singular singularity is said *irregular*. The question is essentially local at the irregular singularities and it is related to the *divergence of the fundamental solutions*.

If 0 is an *irregular singularity* of the system $Y' = AY$, then this system admits a *formal* fundamental solution (Hukuhara-Turrittin), x local coordinate:

$$\hat{F} = \hat{H}(t)x^L e^{Q(1/t)},$$

with $x = t^\nu, \nu \in \mathbf{N}^*$, $\hat{H} \in Gl_n(\mathbf{C}((t)))$, $L \in M_n(\mathbf{C})$, $Q = \mathrm{Diag}(q_1, \ldots, q_n)$, $q_1, \ldots, q_n \in \frac{1}{t}\mathbf{C}[\frac{1}{t}]$.

"In general" $\hat{H}$ is *divergent*.

For simplicity we will describe only the *unramified case*: $\nu := 1$.

The archetypal example is the Euler equation:

$$x^2y' + y = x$$

or its “homogeneous version”: $(x^{-1}(x^2y' + y))' = 0$.

Euler equation admits a *divergent* formal power series solution:

$$\sum_{n=0}^{+\infty} (-1)^n n! x^{n+1}.$$

We return to a formal fundamental solution:

$$\hat{F} = \hat{H}(x)x^L e^{Q(1/x)},$$

Formal monodromy $\hat{M} := e^{2i\pi L}$ $(\nu = 1)$, we have

$$\hat{F}(e^{2i\pi}x) = \hat{F}(x)\hat{M}.$$

The *formal invariants* (in particular the formal monodromy) are obtained from L, Q (and ν).

The *meromorphic invariants* for a fixed normal formal form stems from the *divergence* of $\hat{H}$ (*Stokes phenomena*). Intuitively they correspond to a “branching” of the “sum” of $\hat{H}$, a (purely unipotent) “monodromy” around “infinitely near singularities” defined by Q.

Summability and Stokes phenomena

"Generically" the entries of the matrix $\hat{H}$ are *k-summable* (Ramis), where k is the Katz rank of the system (Δ) (the biggest slope of the Newton polygon of (Δ)). More generally they are *multisummable* (Martinet-Ramis...).

Therefore, for all $d \in S^1$, it is possible to define two *summation operators* $\mathbf{S}_d^+$ and $\mathbf{S}_d^-$ associating to $\hat{H}$ two matrices H_d^+ and H_d^- which are holomorphic on a "small" sector bisected by d.

In general $H_d^+ = H_d^-$, they differ only for a finite number of directions: the *singular directions* (or Stokes-lines, or anti-Stokes lines ...). Choosing a branch of the Logarithm, that is $\mathbf{d} \in \mathbf{R}$ above $d \in S^1$, we get two actual fundamental solutions $F_{\mathbf{d}}^+ := \mathbf{S}_{\mathbf{d}}^+ \hat{F}$ and $F_{\mathbf{d}}^- := \mathbf{S}_{\mathbf{d}}^- \hat{F}$.

There exists a unique *constant* matrix $St_{\mathbf{d}}$ such that:

$$F_{\mathbf{d}}^+ = F_{\mathbf{d}}^- \, St_{\mathbf{d}},$$

$St_{\mathbf{d}} \neq I$ if and only if d is a singular direction.

There exists a unique *constant* matrix $St_{\mathbf{d}}$ such that:

$$F_{\mathbf{d}}^{+} = F_{\mathbf{d}}^{-}\, St_{\mathbf{d}},$$

$St_{\mathbf{d}} \neq I$ if and only if d is a singular direction. In that case $St_{\mathbf{d}}$ is the *Stokes matrix* associated to $\mathbf{d}$. It is *unipotent*.

There is a relation between the *formal monodromy* $\hat{M}$, the Stokes matrices and the *actual monodromy* $\hat{M}$:

$$M = \hat{M}\, St_{\mathbf{d_m}} \cdots St_{\mathbf{d_1}}$$

$(\mathbf{d_1} < \cdots < \mathbf{d_m} < \mathbf{d_1} + 2\pi)$.

For the Stokes matrices, a change into the choice of a branch of the logarithm corresponds to a conjugation by a power of $\hat{M}$.

Intuitively a Stokes matrix corresponds to a loop around a singularity (or a pack of singularities) “infinitely near” of 0 on d.

Generalized Riemann-Hilbert correspondance, local version

To a germ of irregular meromorphic system, in the unramified case, we can associate:

$$Q \in \frac{1}{z}[\frac{1}{z}], \quad \hat{M}, \quad \text{and the} \quad \mathrm{St}_{\mathbf{d}}.$$

We will say that these data defines a *representation* of the *wild local fundamental group* $\pi_{1,w}(\mathbf{C}, 0)$. (It is possible to give precise definitions at the price of some abstraction...)

Conversely to such data (satisfying some compatibility conditions), it is possible to associate a germ of meromorphic system. It is the *generalized (or wild) Riemann-Hilbert correspondance* (Birkhoff, Balser-Jurkat-Lutz, Sibuya, Malgrange...) in the local case.

Generalized Riemann-Hilbert correspondance, global version

Let X be a Riemann surface.
In the *regular singular-case* we associate to a meromorphic differential system (Δ) on X its *monodromy representation*:

$$\rho : \pi_1(X \setminus S; \star) \to Gl(\mathrm{Sol}_\star(\Delta))$$
$$RH : \quad (\Delta) \to \rho.$$

In the *irregular case* we can imitate this process, replacing the fundamental group by a *wild fundamental group* (Martinet-Ramis, Bolibruch-Malek-Mitschi), however the correspondance it not "perfect", it is in fact necessary to use a *wild fundamental groupoid*. What are missing are some *"links"* between the base point $\star$ and each irregular singular point a_i, more precisely a continuous path between $\star$ and a (generic) direction $\tilde{a}_i$ at a_i, followed by an "antisummation path" along $\tilde{a}_i$ (Jimbo-Miwa-Ueno, Boalch, Witten, van der Put-Saito).

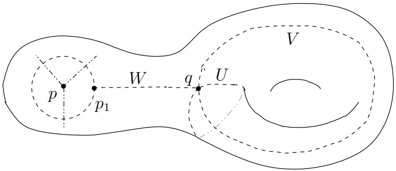

FIGURE 2. A Riemann surface C, here taken to be of genus $g_C = 1$, with an irregular singularity at a point p. A basepoint is taken at q. Show are the Stokes rays near p and the important paths in defining the generalized monodromy data.

Figure: Wild fundamental groupoid: g=1, m=1

Representations of the wild fundamental groupoid, global version
(The idea)

Links: $L_i : \mathrm{Sol}_\star \to \mathrm{Sol}_{a_i},\ i = 1, \ldots m.$

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(The idea)

Links: $L_i : \mathrm{Sol}_\star \to \mathrm{Sol}_{a_i}, \; i = 1, \ldots m.$

- $X := P^1(\mathbf{C})$, genus $g := 0$:

$$I = L_m^{-1} M_m L_m \ldots L_1^{-1} M_1 L_1,$$
$$M_i = \hat{M}_i \, \mathrm{St}_{\mathbf{d_{i,r_i}}} \ldots \mathrm{St}_{\mathbf{d_{i,r_1}}} \quad i = 1, \ldots m.$$

Representations of the wild fundamental groupoid, global version
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- $X := X_g$, genus g:

$$I = U_1 V_1 U_1^{-1} V_1^{-1} \ldots U_g V_g U_g^{-1} V_g^{-1} L_m^{-1} M_m L_m \ldots L_1^{-1} M_1 L_1,$$
$$M_i = \hat{M}_i \mathrm{St}_{\mathbf{d_{i,r_i}}} \ldots \mathrm{St}_{\mathbf{d_{i,r_1}}}, \quad i = 1, \ldots m.$$

Gauge group

Let G be a *connected* linear algebraic group and let $\mathfrak{g}$ be its Lie algebra.

Let $(\Delta) : \frac{dY}{dz} = A(z)Y$, we suppose $A \in \mathfrak{g}(K)$, where $K := \mathbf{C}(\{z\})$ (A is a meromorphic matrix taking its values in the vector space $\mathfrak{g}$), then:

$$Q \in \frac{1}{z}\mathfrak{g}[\frac{1}{z}], \quad \hat{M} \in G, \quad \text{and the} \quad St_{\mathbf{d}} \in G$$

(Kolchin, Babbitt-Varadarajan, Martinet-Ramis, Boalch).

We say that G is a *gauge group* for the system (Δ).

BRAID GROUPS AND MAPPING CLASS GROUPS

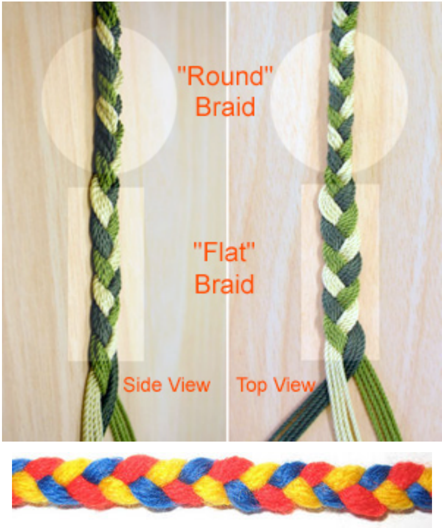

Figure: Braids

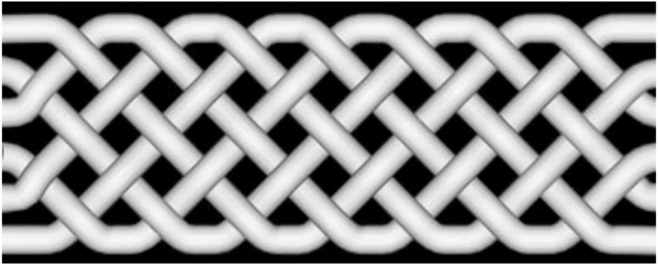

Figure: Braid

E. Brieskorn (1988):

The beauty of braids is that they make ties between so many different parts of mathematics: combinatorial theory, number theory, group theory, algebra, topology, geometry and analysis, and, last but not least, singularities.

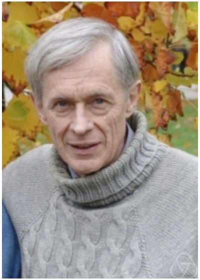

Figure: E. Brieskorn

Description using braiding of strands
The group law

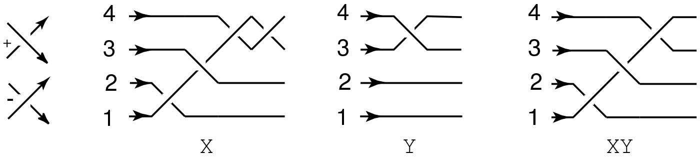

Figure 1: Examples of 4-braids X, Y and their product XY

Figure: Braids

Braid groups, following E. Artin (1925)

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Figure: Emil Artin 1898-1962

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- $\sigma_i \sigma_j = \sigma_j \sigma_i$, if $|j-i| \geq 2$;
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, \ldots, n-2$, if $n \geq 3$.

Figure: Emil Artin 1898-1962

Generators of braid groups: geometric interpretation with strands

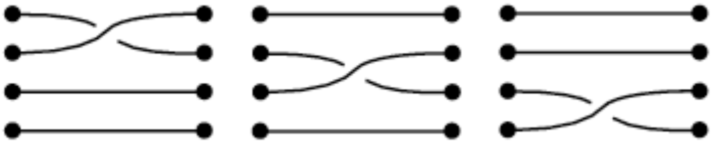

Figure: Generators $\sigma_1, \sigma_2, \sigma_3$ of B_4

The braid group B_1 is trivial and the braid group B_2 is isomorphic to $\mathbf{Z}$.

For $n = 3$, the Artin braid group B_3 is generated by σ_1 and σ_2 with only one relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

The *center* of B_3 is the subgroup generated by

$$(\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2.$$

Let $S_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S_2 := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\sigma_1 \mapsto S_1$, $\sigma_2 \mapsto S_2$ induces a group homomorphism:

$$B_3 \to PSL_2(\mathbf{Z}).$$

There is an exact sequence:

$$I \to ((\sigma_1\sigma_2)^3) \to B_3 \to PSL_2(\mathbf{Z}) \to 1$$

Permutation groups and pure braid groups

Let $\mathfrak{S}_n$ be the permutation group of n elements.

Taking account only of the *origin* and the *extremity* of the strands, we get a group homomorphism: $B_n \mapsto \mathfrak{S}_n$. This homomorphism is onto and, by definition, its kernel is the:

pure braid group P_n.

Braid groups and configuration spaces

The *configuration space* of n points on the complex plane $\mathbf{C}$ is, by definition,

$$\mathrm{Conf}_n := \{z := (z_1, \dots, z_n) \in \mathbf{C}^n \mid z_i \neq z_i,\ \text{if } i \neq j\}.$$

The permutation group $\mathfrak{S}_n$ acts on Conf_n by permutation of the coordinates of z.

There is an interpretation of the groups B_n and P_n as *fundamental groups*:

$$P_n = \pi_1(\mathrm{Conf}_n, \star), \quad B_n = \pi_1(\mathrm{Conf}_n/\mathfrak{S}_n, \star).$$

Let $\gamma := (\gamma_1, \dots, \gamma_n) : [0, 1] \to \mathrm{Conf}_n$ be a continuous loop. The graphs of $\gamma_1, \dots, \gamma_n$ form a subset of $[0, 1] \times \mathbf{C}^n$ which can be interpreted as a *geometric (pure) braid*.

Variants. We can replace the complex plane $\mathbf{C}$ by the disk D. We can also interpret B_n as a *mapping class group*.

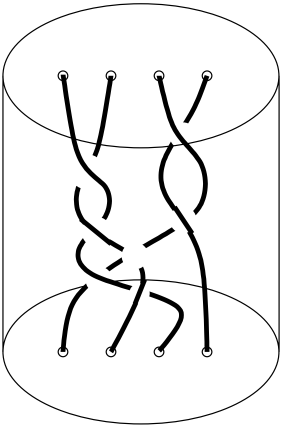

Figure: Braids and punctured disks

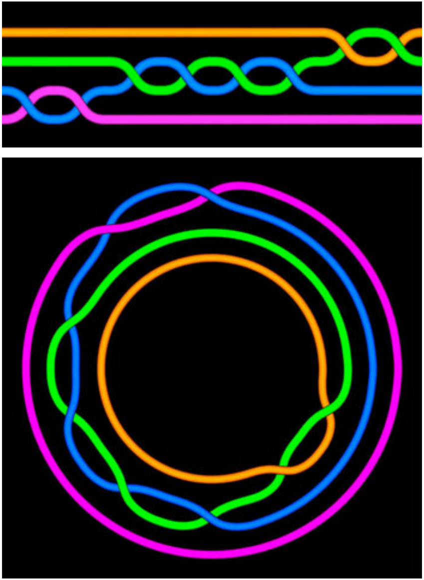

Figure: Pure braid and loop on the configuration space

Braid groups and mapping class groups

Let $\overline{D}_n$ be the closed unit disc with n marked points $a_1, \ldots, a_n$. The braid group B_n is isomorphic to the group $\mathfrak{M}(\overline{D}_n)$ of *diffeotopies* of $\overline{D}_n$ that is the subgroup of the group of diffeotopies of the closed disc leaving fixed the boundary $\partial\overline{D} = S^1$ and the *set* of marked points.

Idea of a proof

Let $f \in \mathfrak{M}(\overline{D}_n)$, we represent it as a diffeomorphism ϕ preserving the orientation. Interpreting ϕ as a diffeomorphism of $\overline{D}$, we know that it is isotopic to $\mathrm{id}_{\overline{D}_n}$. If $t \in [0,1] \mapsto \phi_t$ is an isotopy, the n maps $(t, a_i) \mapsto (t, \phi_t(a_1)) \in [0,1] \times \overline{D}$ $(i = 1, \ldots, n)$ define a geometric braid and we get a map $\mathfrak{M}(\overline{D}_n) \to B_n$. It is clearly an homomorphism of group, exhibiting the inverse one can prove that it is an isomorphism.

The group $\mathfrak{M}(\overline{D}_n)$ induces an *automorphism* of the fundamental group $\pi_1(\overline{D}_n)$. It acts on the "natural" systems of generators of $\pi_1(\overline{D}_n)$ (such a system identifies the fundamental group with a free group $F_n := \mathbf{Z} * \cdots * \mathbf{Z}$).

We are interested in the subgroup of $\mathfrak{M}(\overline{D}_n)$ fixing *each* a_i, $i = 1, \ldots, n$. It is isomorphic to the *pure* braid group P_n and also to the mapping class group of the $n+1$ punctured sphere S^2_{n+1}.

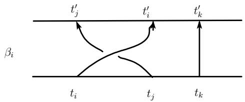

Figure 10: Basic braid β_i, where (i, j, k) is a cyclic of $(1, 2, 3)$

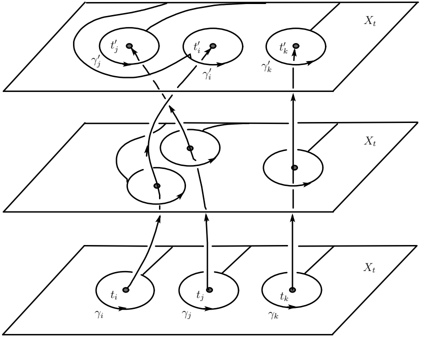

Figure 11: The braid action $\beta_i : (\gamma_i, \gamma_j, \gamma_k) \mapsto (\gamma'_i, \gamma'_j, \gamma'_k)$

Figure: Half-monodromy and braiding (Iwasaki...)

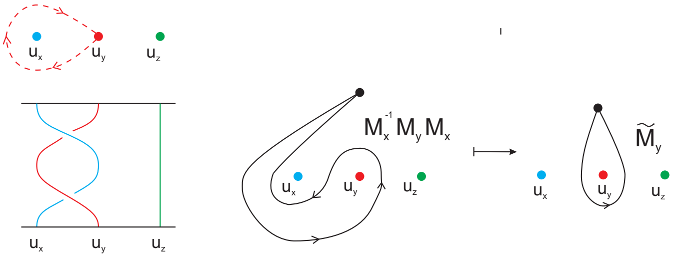

Figure: Monodromy and braiding (O. Lisovyy)

The pure braid group P_3 and the dynamics of the Painlevé equation P_{VI}

Using the Riemann-Hilbert map, it is possible to interpret the (analytic) dynamics of P_{VI} as a dynamics on the character variety $\chi(S_4^2, Sl_2(\mathbf{C}))$ corresponding to the action of the mapping class group of S_4^2, or equivalently to the pure braid group P_3. Hence we get a *polynomial* action of $PSL_2(\mathbf{Z})$ on the cubic surfaces $S_{(A,B,C,D)}$.

Let $\Gamma_2^{\pm} \subset PSL_2(\mathbf{Z})$ be the subgroup whose elements coincide with identity modulo 2. Then (El'-Huti):

the morphism $\Gamma_2^{\pm} \to \mathrm{Aut}(S_{(A,B,C,D)})$ is injective, the index of its image is bounded by 24 and it is generically an *isomorphism*.

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Using this result it is possible to prove that the

non-commutative Galois differential groupoid,

of P_{VI}, in Malgrange sense, is

the groupoid of transformations conserving the area,

except in the Picard-Painlevé case (Cantat-Loray).

The irreducibility of P_{VI} in Nishioka-Umemura sense follows (Casale).

Let, as above, $S_4^2 = P^1(\mathbf{C}) \setminus \{a_1, a_2, a_3, a_4\}$ and $G := Sl_2(\mathbf{C})$.

Let $\rho : \pi_1(S_4^2) \to Sl_2(\mathbf{C})$ be a representation.
We set:

$$M_1 := \rho(\gamma_1),\ M_2 := \rho(\gamma_2),\ M_3 = \rho(\gamma_3).$$

We consider

$\Lambda := \{(s_x, s_y, s_z) |\ s_x^2 = s_y^2 = s_z^2 = 1\} = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2.$

$$s_x = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, s_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, s_z = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$$

The group $\Gamma_2^{\pm}$ can be identified with Λ and the standard modular group Γ_2 corresponds the subgroup (of index 2) of Λ containing the words of even length in the *involutions* s_x, s_y, s_z.

$$
\begin{aligned}
s_x &: (M_1, M_2, M_3) \to (M_1^{-1}, M_2^{-1}, M_1 M_3^{-1} M_1^{-1}) \\
s_y &: (M_1, M_2, M_3) \to (M_2 M_1^{-1} M_2^{-1}, M_2^{-1}, M_3^{-1}) \\
s_z &: (M_1, M_2, M_3) \to (M_1^{-1}, M_3 M_2^{-1} M_3^{-1}, M_3^{-1})
\end{aligned}
$$

The traces a, b, c, d (and therefore A, B, C, D) are fixed by $\Lambda = \Gamma_2^{\pm}$.

The action of $\Lambda = \Gamma_2^{\pm}$ on the cubic surface $S_{(A,B,C,D)}$ is (Loray-Cantat, Lisovyy):

$$
\begin{aligned}
s_x(x, y, z) &= (A - x - yz, y, z) \\
s_y(x, y, z) &= (x, B - y - zx, z) \\
s_z(x, y, z) &= (x, y, -z - xy)
\end{aligned}
$$

The standard modular group $\Gamma_2 \subset PSL_2(\mathbf{Z})$ is generated by the three elements:

$$g_x = s_z s_y = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = S_1^2$$

$$g_y = s_x s_z = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = S_2^2$$

$$g_z = s_y s_x = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} = S_1^{-2} S_2^{-2}$$

This corresponds, modulo the Riemann-Hilbert map, to the *Painlevé VI non-linear monodromy* (t turning around, $0, 1, \infty$). We have $g_x g_y g_z = 1$.

From P_{VI} to the other Painlevé equations

Our aim is to find a similar mechanism for the other Painlevé equations.

It is necessary (and, we hope, sufficient...) to replace, mutatis mutandis,

tame objects (regular singularities)

by

wild objects (irregular singularities).

REDUCTIVE GROUPS

and

GENERALIZED BRAID GROUPS

Motivation: Infinitely near points and how to use them

As we have said it is possible to interpret an *irregular singular point* as a *"pack of infinitely near singularities"* in some "transcendental sense" (Garnier 1919, and later Ramis, Deligne, Martinet-Ramis...).

It is possible to give rigourous definitions, using in particular *roots* on sub-tori of algebraic groups (Martinet-Ramis)[1] then one can generalize the notion of *configuration space* for such packs of infinitely near singularities and the corresponding fundamental groups are:

generalized braid groups,

more precisely G-braid groups, G being a *reductive algebraic group* (cf. E. Brieskorn).

[1] An algebraic version of Ecalle pointed alien derivations.

Reductive groups

Let G be a complex linear algebraic group, by definition it is *reductive* if it does not contain an invariant algebraic subgroup isomorphic to G_a (i. e. to the additive group $(\mathbf{C},+)$). It is equivalent to say that the *unipotent radical* $R_u(G)$ of G is *trivial*, or that the *radical* $R(G)$ of G is an algebraic *torus*.

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The following conditions are equivalent:

- G is reductive;
- G contains a compact group (in the sense of the usual topology) which is Zariski dense;
- every rational representation of G (morphism of algebraic groups $G \to GL(V)$) is semi-simple.

Let G be a *connected complex reductive group*, We fix a *maximal torus* $T \subset G$ (Cartan sub-group), we denote the corresponding Lie algebras $\mathfrak{t} \subset \mathfrak{g}$. Let $\mathcal{R} \subset \mathfrak{t}^*$ be the set of *roots* of G relative to T (non zero weights). To $\alpha \in \mathcal{R}$, we associate the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$:

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x, \forall h \in \mathfrak{t}\}.$$

Then $\dim_{\mathbf{C}} \mathfrak{g}_\alpha = 1$. We have $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$.

Let $N(T)$: be the normalizer of T:
$N(T) := \{g \in G \,|\, gT = Tg\}$, then $W := N(T)/T$ is the *Weyl group* of G. It can be interpreted as a finite group of complex reflections on the complex space $\mathfrak{t}$. To each root α is associated a reflection hyperplane Δ_α.

If $G := Sl_n(\mathbf{C})$, then we can choose for T the group of invertible diagonal matrices. Then $\mathfrak{t} \approx \mathbf{C}^n$, the roots are $x \mapsto x_i - x_j$, the roots hyperplanes are $\Delta_{ij} := \{x_i = x_j\}$ and the Weyl group is the permutation group $\mathfrak{S}_n$.

Regular elements of a Cartan sub-algebra, generalized pure braid groups

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By definition, the *regular subset* of the Cartan Lie-algebra $\mathfrak{t}$ is:

$$\mathfrak{t}_{\mathrm{reg}} := \mathfrak{t} \setminus \bigcup_{\alpha \in \mathcal{R}} \Delta_\alpha.$$

We have $A \in \mathfrak{t}_{\mathrm{reg}}$ if and only if $\alpha(A) \neq 0$ for all $\alpha \in \mathcal{R}$.

If $G := Sl_n(\mathbf{C})$, then $\mathfrak{t}_{\mathrm{reg}} = \mathbf{C}^n \setminus \bigcup_{i \neq j} \Delta_{ij}$ (configuration space).

By definition the *generalized pure braid group* associated to the complex algebraic Lie-algebra $\mathfrak{g}$ is $\pi_1(\mathfrak{t}_{\mathrm{reg}})$. If $G := Sl_n(\mathbf{C})$, then we get the *classical pure braid group*.

Reductive groups and deformations of O.D.E.

In order to have a "good theory" of *iso-irregular deformations*, we will suppose in the following that the *local gauge groups* at the *irregular singularities* of our O.D.E. are connected *reductive groups*.

Reductive groups, positive roots, Borel subgroups

Let G a *connected complex reductive group*, $T \subset G$ a maximal torus and $\mathcal{R} \subset \mathfrak{t}^*$ the corresponding set of roots. We say that $\mathcal{R}^+ \subset \mathcal{R}$ is a subset of *positive roots* if:

- for all $\alpha \in \mathcal{R}$, $\alpha \in \mathcal{R}$ or $-\alpha \in \mathcal{R}$
- for all $\alpha, \beta \in \mathcal{R}^+$, $\alpha \neq \beta$, if $\alpha + \beta$ is a root, then $\alpha + \beta \in \mathcal{R}^+$.

  If $\mathcal{R}^+$ is a subset of positive roots, then $\mathcal{R}^- := -\mathcal{R}^+$ is also a subset of positive roots.

  The one-parameter subgroups $U_\alpha := \exp \mathfrak{g}_\alpha$, $\alpha \in \mathcal{R}^+$, generate a *unipotent* subgroup U^+ of G, the *unipotent radical* of a *Borel subgroup* B^+ of G.
  Replacing $\mathcal{R}^+$ by $\mathcal{R}^-$, we get U^- and the opposite Borel subgroup B^-.
  Using convenient coordinates, this corresponds to *triangular subgroups*.

WILD CHARACTER VARIETIES

BRAIDING OF STOKES DATA

and

ISO-IRREGULAR DEFORMATIONS

Irregular points

Let D be a germ of complex disc and a local coordinate z vanishing at the center. We denote $\hat{K} := \mathbf{C}((z))$ and $\hat{O} := \mathbf{C}[[z]]$ (intrinsic differential algebras).

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- a positive integer $r \in \mathbf{N}$, $r > 0$ (Katz rank),
- an *irregular type*, that is:

$$Q := \frac{A_r}{z^r} + \ldots + \frac{A_1}{z},$$

where $A_j \in \mathfrak{t} \quad (j = 1, \ldots, r)$.
Intrinsically $Q \in \mathfrak{t}(\hat{K})/\mathfrak{t}(\hat{O})$.
We denote by $H \subset G$ the *normalizer* of Q.

If X is a Riemann surface, we will denote an irregular point “at $a \in X$” by (a, Q) (omitting G and T). If $Q = 0$, $(a, Q) = (a, 0)$ is, by definition, *a regular singular point*.

Irregular curves or wild Riemann surfaces

We consider a Riemann surface X with $m \geq 1$ marked points $a_1, \ldots, a_m$. For each a_i, $i = 1, \ldots, m$, we fix:

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- an *irregular type* at a_i, that is:

$$Q_i := \frac{A_{i,r_i}}{z^{r_i}} + \ldots + \frac{A_{i,1}}{z},$$

where z is a local coordinate vanishing at a_i and

$$A_{i,j} \in \mathfrak{t}_i, \quad (j = 1, \ldots, r_j).$$

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These data define an *irregular curve* or *wild Riemann surface*. Equivalently such a curve is the data of a Riemann surface with m marked *irregular points* (a_i, Q_i) $(i = 1, \ldots, m)$.

We denote by $H_i \subset G_i$ the *normalizer* of Q_i.

An irregular curve or wild Riemann surface

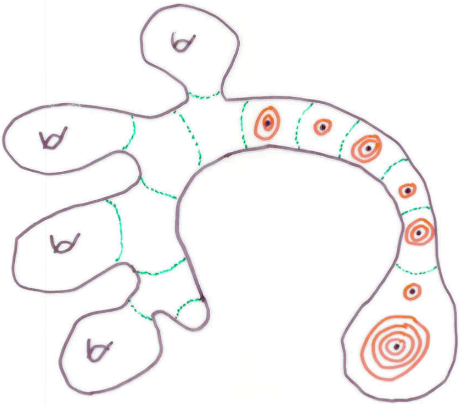

Figure: From Ph. Boalch

We will associate to an irregular curve a moduli of representations. In order to have a *good notion of deformation* (some kind of flatness) we will suppose that the coefficient A_{i,r_i} of the *most polar part* $\frac{A_{i,r_i}}{z^{r_i}}$ of Q_i is *regular*:

$$A_{i,r_i} \in (\mathfrak{t}_i)_{\text{reg}} \quad (i = 1, \dots, m).$$

Intuitively an irregular point is *a pack of infinitely near points* and the regularity hypothesis allows the deformation of this pack “without crossing”. The homotopy class of a deformation loop of the pack corresponds to an element of a *generalized braid group*. This element will acts naturally on the classes of wild representations.

According to E. Witten, we can interpret the positions of the a_i and the Q_i as the *non topological data* of a *wild representation* and, on the contrary, the *Stokes matrices* and the *actual* (or equivalently the *formal*) *monodromy* as the *generalized topological data*. In an iso-irregular deformation the *non topological data* *move* but the *generalized topological data* *remain fixed*.

We have several (pack of) *"times"*. One is controlling the deformations of the set of singularities (a_i) (modulo the action of the Möbius group), the others control the deformations of the most polar part $\frac{A_{i,r_i}}{z^{r_i}}$ of each Q_i.

Stokes data and monodromy associated to an irregular point

Following Martinet-Ramis, Ramis, Boalch. Cf. van der Put-Singer book

Let (a, Q) be an (unramified) irregular point. We choose a coordinate z vanishing at a and set $a = 0$. We will define the Stokes data associated to $(0, Q)$.

For each root $\alpha \in \mathcal{R} \subset \mathfrak{t}^*$, we may define $q_\alpha := \alpha \circ Q$, using z we can interpret q_α as a *polynomial* without constant term $q_\alpha \in \frac{1}{z}\mathbf{C}[\frac{1}{z}]$ of degree $\deg q_\alpha = r_\alpha$.

We interpret the real blow up of the origin as the circle S^1, a direction $d \in S^1$ will be said to be a *singular direction* (or Stokes line) supported by α if the holomorphic function e^{q_α} has a *maximal decay* as $z \to 0$ in the direction d.

We denote by $\mathcal{R}(d) \subset \mathcal{R}$ the non empty finite subset of the roots supporting $d \in S^1$. It is a subset of some subset of *positive roots*.

Stokes data and monodromy associated to an irregular point

We denote by $\mathcal{R}(d) \subset \mathcal{R}$ the non empty finite subset of the roots supporting $d \in S^1$. It is a subset of some subset of *positive roots*. By definition, the Stokes group associated to a singular direction $d \in S^1$ is the algebraic subgroup St_d of G generated by the one parameter subgroups $\exp \mathfrak{g}_\alpha$, $\alpha \in \mathcal{R}(d)$. The Stokes groups are *unipotent*. The Lie algebra st_d of St_d is:

$$\bigoplus_{\alpha \in \mathcal{R}(d)} \mathfrak{g}_\alpha \subset \mathfrak{g},$$

it is *nilpotent*.

The Stokes groups St_d are *normalized* by H.

Stokes data and monodromy associated to an irregular point

Let $(0, Q)$ be an irregular point, we denote by H the *normalizer* of Q in G. We can define:

- a formal monodromy $\hat{M} \in H$
- a set of Stokes multipliers $S_d \in \mathrm{St}_d$, $d \in S^1$ being a *singular direction*; we call this set: *Stokes data*..

These data form "by definition" a *representation* ρ of the (local) *wild-fundamental group*:

$$\hat{M} = \rho(\hat{\mu}), \qquad S_d = \rho(\sigma_d)$$

Let $\Lambda \in \mathfrak{h}$ such that $\hat{M} = \exp 2i\pi\Lambda$. The preceding representation is associated to a meromorphic connection formally equivalent (modulo the gauge action of $G(\hat{K})$) to the connection $d - (dQ + \Lambda\frac{dz}{z})$.

Starting from a base point on S^1 (a non singular direction) and indexing the singular directions counterclockwise, we get the actual monodromy M:

$$M = \hat{M} S_{d_1} \dots S_{d_n}.$$

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Therefore it is natural to define an element μ of the wild fundamental group by

$$\mu = \hat{\mu} \sigma_{d_1} \ldots \sigma_{d_n},$$

and to identify it with a generator of $\pi_1(S^1)$. Then $\pi_1(S^1)$ is identified to a subgroup of the wild fundamental group $\pi_{1,w}(0, Q)$.

Be careful, there are subtle *constraints* on the generators $\hat{\mu},\ \sigma_d$ of $\pi_{1,w}(0, Q)$. It is possible to build "free generators" (in alien derivations style): Martinet-Ramis.

There are also *constraints* on the representations, they disappear in Martinet-Ramis construction.

Deformations leaving the Stokes data and monodromy "constant"

We want to move an irregular point with Stokes data and monodromy in such a way that Stokes data and monodromy remain "constant" in a reasonable sense.

We start from an irregular point $(0, Q)$ and we "move it a little bit": (a, Q_a), with $Q_a := \frac{A_r(a)}{(z-a)^r} + \ldots + \frac{A_1(a)}{z-a}$, such that $A_r(a) \in \mathfrak{t}_{reg}$ (we allow only deformations such that the order of the pole of Q_a dos not change).

We work with matrices: $G \subset GL_n(\mathbf{C})$, T diagonal. Starting from a system of Stokes data at $(0, Q)$, we can associate to it a set of *effective singular directions*: the singular directions d such that $S_d \neq I$. When we move, it can happen that an effective direction *"splits"* in two or more effective directions (two different roots can support the same direction...).

Deformations leaving the Stokes and monodromy data "constant"

We work with matrices: $G \subset GL_n(\mathbf{C})$, T diagonal. Starting from a system of Stokes data at $(0, Q)$, we can associate to it a set of *effective singular directions*: the singular directions d such that $S_d \neq I$. When we move, it can happen that an effective direction *"splits"* in two or more effective directions, therefore the good definition is *to keep the local products of Stokes matrices constant* (Ueno-Miwa-Jimbo, Boalch).

In terms of data associated to a family of germs at 0 of irregular connection, this corresponds to fix the action of $\mathbf{S}_{d''}^{-1} \circ \mathbf{S}_{d'}$, $d', \ d''$ being two *fixed* regular directions, $d' < d < d''$ "sufficiently near" d.

Supposing moreover that the *formal monodromy remains constant*, we define a local version of iso-irregular deformation.

Wild character varieties: the local case

We start from an *irregular point* $(0, Q)$. The group $H \subset G$ (normalizing Q) acts by adjoint action on the Stokes data and the monodromy (formal or actual), by definition the set of classes is the *wild character variety* associated to $(0, Q)$, equivalently it is the set of equivalence classes of representations of the wild fundamental group associated to $(0, Q)$.

Wild fundamental groupoid: the global case

We start from an *irregular curve* $(X, (a_1, Q_1), \dots, (a_m, Q_m))$.
We suppose that all the G_i are equal: $G_i = G$ and, for simplicity, that G is an algebraic subgroup of $GL_n(\mathbf{C})$, T being *diagonal*.

We define an associated *wild fundamental groupoid*, "adding" some *irregular data* to the fundamental group $\pi_1(X \setminus \{a_1, \dots, a_m\}, \star)$, $\star \in X \setminus \{a_1, \dots, a_m\}$.

As explained before, it is possible to define a wild fundamental *group* but in order to have a "good RH correspondence" it is necessary to define a *groupoid* (Jimbo-Miwa, Boalch, Witten, van der Put-Saito).

In X we replace each a_i by its *real blow* up ∂_i ($\partial_i \approx S^1$), we get a "partial compactification" $\tilde{X}$ of $X \setminus \{a_1, \dots, a_m\}$:

$$\tilde{X} := X \setminus \{a_1, \dots, a_m\} \cup \bigcup_{i=1,\dots,m} \partial_i.$$

For each i, we choose $\tilde{a}_i \in \partial_i$ (a direction at a_i). We consider the space $\tilde{X}$ with $m+1$ (resp. m) base points: $\star \in X \setminus \{a_1, \dots, a_m\}$, $\tilde{a}_1, \dots, \tilde{a}_m$ (resp. $\tilde{a}_1, \dots, \tilde{a}_m$). We will define "the" *wild fundamental groupoid*

$$\pi_{1,w}((X, (a_1, Q_1), \dots, (a_m, Q_m)); \star, \tilde{a}_1, \dots, \tilde{a}_m)$$

(resp. $\pi_{1,w}((X, (a_1, Q_1), \dots, (a_m, Q_m)); \tilde{a}_1, \dots, \tilde{a}_m)$) of the irregular curve.

The idea is to "add" *irregular data* to the classical fundamental groupoid, with $m+1$ base points:

$$\pi_1(\tilde{X}; \star, \tilde{a}_1, \dots, \tilde{a}_m) \qquad ,$$

more precisely at each point $\tilde{a}_i$ we "add" the wild fundamental group $\pi_{1,w}(a_i, Q_i)$:

The idea is to “add” *irregular data* to the classical fundamental group, with $m+1$ base points:

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more precisely at each point $\tilde{a}_i$ we “add” the wild fundamental group $\pi_{1,w}(a_i,Q_i)$:

we glue $\pi_{1,w}(a_i,Q_i)$ with $\pi_1(\tilde{X};\star,\tilde{a}_1,\ldots,\tilde{a}_m)$ “in van Kampen style”:

$$\pi_1(\tilde{X};\star,\tilde{a}_1,\ldots,\tilde{a}_m)*_{\pi_1(\partial_i,\tilde{a}_i)}\pi_{1,w}(a_i,Q_i).$$

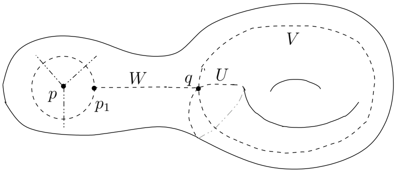

FIGURE 2. A Riemann surface C, here taken to be of genus $g_C = 1$, with an irregular singularity at a point p. A basepoint is taken at q. Show are the Stokes rays near p and the important paths in defining the generalized monodromy data.

Figure: Wild fundamental groupoid: g=1, m=1

Wild character varieties: the global case

Let G be a complex *reductive* linear algebraic group.

Let $(X, (a_1, Q_1), \ldots, (a_m, Q_m))$ be an *irregular curve*, with $G_i = G$, forall $i = 1, \ldots, m$.

The space of "group homomorphisms":

$Hom_{gr}(\pi_{1,w}((X, (a_1, Q_1), \ldots, (a_m, Q_m)); \star, \tilde{a}_1, \ldots, \tilde{a}_m), G)$

is a smooth affine variety, it has an action of G^{m+1}, the quotient (in an algebraic sense) is a *wild character variety*.

Wild character varieties and deformations

We interpret the wild character variety associated to an irregular curve as a fiber above a “point” corresponding to this irregular curve, then we get a fiber space above the moduli of irregular curves (X is fixed and we move the regular and irregular singularities respecting the “non crossing” restrictions). Iso-irregular deformation defines a *connection* on this bundle, that is a way to identify canonically the fibres above a small “simply-connected” open subset of the basis. Hence the “fundamental group of the basis” acts on the fibres, that is on each wild character variety.

It remains to understand the “fundamental group of the basis”, that is the *global wild braid group*, and its action.

Wild braids: the global case

Let G be a complex *reductive* linear algebraic group.

Let $(X,(a_1,Q_1),\dots,(a_m,Q_m))$ be an *irregular curve*, with $G_i = G$, forall $i = 1,\dots,m$.

There are some actions on the corresponding *wild character variety* :

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There are some actions on the corresponding *wild character variety* :

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  It remains to "put the things together". I *conjecture* that there is a:

  "braiding of braids",

  a G-braid is "hidden" in an "infinitesimal neighborhood" of each classical strand (of P_{m-1}).

Figure: Braiding of braids

Braiding of braids and wild mapping class groups

The *global wild braid group* is described as a (pure) braiding of (generalized) braids group.

Using “natural generators” of braids of braids groups, it is “easy” to compute the action of these groups on “natural generators” of the wild fundamental groups (the action of “wild mapping class groups”). Then we can see that the corresponding actions on the wild character varieties are *polynomial* (they are computed using matrix multiplications).

Braiding of braiding of... braiding of braids

For more *degenerate* cases of irregular points, there are *multi-levelled Stokes phenomena* (several slopes), I conjecture that in such cases we get a *multi-scale braiding*, in Russian dolls style (cf. Boalch).

Figure: Russian dolls: *a multiscale metaphor*...

Generalized isomonodromic systems

Following Y. Sibuya

Let U be an open set in $\mathbf{C}$ and V a polydisc in $\mathbf{C}^n$. We consider an holomorphic *parametrized* differential system:

$$(\Delta) : \frac{dY}{dz} = A(z,t)Y \text{ on } U \times V \; ((z,t) \in U \times V).$$

We will say that the system (Δ) is *isomonodromic* if there exists a covering $(U_i)_{i \in I}$ of U by simply connected open sets and fundamental systems of solutions Y_i on $U_i \times V$ such that the *connection matrices* $C_{jk} := Y_j^{-1} Y_k$ are *independant* of $t \in V$.

The system (Δ) is *isomonodromic* if and only if there exists n holomorphic invertible matrices $B_1 \ldots, B_n$ on $U \times V$ such that the Pfaffian system:

$$\frac{\partial Y}{\partial z} = A(z,t)Y, \quad \frac{\partial Y}{\partial t_h} = B_h(z,t)Y, \quad h = 1, \ldots, n$$

is *completely integrable*.

The system $(\Delta): \frac{dY}{dz} = A(z,t)Y$ is *isomonodromic* if and only if there exists n holomorphic invertible matrices $B_1 \dots, B_n$ on $U \times V$ such that the Pfaffian system:

$$\frac{\partial Y}{\partial z} = A(z,t)Y, \quad \frac{\partial Y}{\partial t_h} = B_h(z,t)Y, \quad h = 1, \dots, n$$

is *completely integrable*, that is if we have $(*)$:

$$\frac{\partial B_h}{\partial z} - \frac{\partial A}{\partial t_h} = [A, B_h], \quad h = 1, \dots, n$$
$$\frac{\partial B_h}{\partial t_k} - \frac{\partial B_k}{\partial t_h} = [B_k, B_h], \quad h, k = 1, \dots, n$$

Equivalently, if $\Omega := Adz + \sum_{h=1}^{n} B_h\, dt_h$, $d\Omega - \Omega \wedge \Omega = 0$.

Using *summation in sectors* at each irregular singular point, we can define a notion of *iso-irregular* deformation. Using *wild RH* it can be translated in terms of *natural connections on wild character varieties*. Then:

RH_w: *transcendental dynamics* of $(*) \rightarrow$ *algebraic dynamics*

APPLICATION TO THE DYNAMICS

OF THE PAINLEVÉ EQUATIONS

(Conjectures...)

The rules of the game
following van der Put and Saito

We consider local families of rational linear systems of rank two on $P^1(\mathbf{C})$, with structure group $SL_2(\mathbf{C})$ (i. e. trace-free matrices):

$$\frac{dY}{dz} = A(z,t)Y$$

(regular or irregular singularities, ramified or not),
$t \in U \subset \mathbf{C}^m$ is a *parameter*. We denote S the singular set and $|S| \in \mathbf{N}^*$ its cardinal. We denote $r(a)$ the Katz rank (slope of the Newton polygon) at a singular point a, $r(a) \in \mathbf{N}$ or $r(a) \in \frac{1}{2} + \mathbf{N}$.
We ask that the dimension of the fibers of RH (or RH_w is one (working up the action of the Mobius group or a subgroup). Then $|S| > 4$ is excluded and there are only the following possibilities:

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1. $|S| = 4$, then $S = \{0, 1, \infty, t\}$, only regular singular points;
2. $|S| = 3$, then $S = \{0, 1, \infty\}$, only one irregular point ∞ with $r(\infty) \in \{1, \frac{1}{2}\}$;

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2. $|S| = 3$, then $S = \{0, 1, \infty\}$, only one irregular point ∞ with $r(\infty) \in \{1, \frac{1}{2}\}$;
3. $|S| = 2$, then $S = \{0, \infty\}$, one regular point and one irregular point *or* two irregular points;

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4. $|S| = 1$, then $S = \{\infty\}$, one irregular point.

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2. $|S| = 3$, then $S = \{0, 1, \infty\}$, only one irregular point ∞ with $r(\infty) \in \{1, \frac{1}{2}\}$;
3. $|S| = 2$, then $S = \{0, \infty\}$, one regular point and one irregular point *or* two irregular points;
4. $|S| = 1$, then $S = \{\infty\}$, one irregular point.

   There are relations with *Heun equations* (ordinary, confluent, biconfluent, doubly-confluent and triconfluent).

The ten families of “Painlevé systems”

Dynkin	Painlevé equation	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{P}$
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	$\mathrm{PV}_{\mathrm{deg}} = \mathrm{PIII(D6)}$	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

TABLE 1. Classification of Families

Figure: Following van der Put and Saito, $r(.)$ is the Katz rank at the singular points. There are ramified cases

Table of the systems and of the (wild) characteristic varieties

(0,0,0,0). PVI. $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}$, all $tr(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+x_3^2-s_1x_1-s_2x_2-s_3x_3+s_4=0$, with
$s_i=a_ia_4+a_ja_k$, $(i,j,k)=$ a cyclic permutation of $(1,2,3)$,
$s_4=a_1a_2a_3a_4+a_1^2+a_2^2+a_3^2+a_4^2-4$ with $a_1,a_2,a_3,a_4\in\mathbb{C}$.

(0,0,1). PV. $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+t/2\cdot\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, all $tr(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2-(s_1+s_2s_3)x_1-(s_2+s_1s_3)x_2-s_3x_3+s_3^2+s_1s_2s_3+1=0$ with $s_1,s_2\in\mathbb{C},\ s_3\in\mathbb{C}^*$.

(0,0,1/2). $\mathrm{PV}_{\mathrm{deg}}$. $\frac{d}{dz}+\frac{A_0}{z}+\frac{A_1}{z-1}+\begin{pmatrix}0&t^2\\0&0\end{pmatrix}$, all $tr(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+s_0x_1+s_1x_2+1=0$ with $s_0,s_1\in\mathbb{C}$.

(1,-,1). PIII(D6). $z\frac{d}{dz}+A_0z^{-1}+A_1+\begin{pmatrix}\frac{t}{2}&0\\0&-\frac{t}{2}\end{pmatrix}z$, all $tr(A_*)=0$.

$x_1x_2x_3+x_1^2+x_2^2+(1+\alpha\beta)x_1+(\alpha+\beta)x_2+\alpha\beta=0$ with $\alpha,\beta\in\mathbb{C}^*$.

Figure: Following van der Put and Saito

(1/2,-,1). PIII(D7). $z\frac{d}{dz} + A_0 z^{-1} + A_1 + \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{t}{2} \end{pmatrix} z,$ all $tr(A_*) = 0.$

$x_1x_2x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0$ with $\alpha \in \mathbb{C}^*$.

(1/2,-,1/2). PIII(D8). $z\frac{d}{dz} + \begin{pmatrix} 0 & 0 \\ -q & 0 \end{pmatrix} z^{-1} + \begin{pmatrix} \frac{p}{q} & -\frac{t}{q} \\ 1 & -\frac{p}{q} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z.$

$x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0.$

(0,-,2). PIV. $z\frac{d}{dz} + A_0 + A_1 z + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2.$

$x_1x_2x_3 + x_1^2 - (s_2^2 + s_1s_2)x_1 - s_2^2x_2 - s_2^2x_3 + s_2^2 + s_1s_2^3 = 0$ with $s_1 \in \mathbb{C},\ s_2 \in \mathbb{C}^*$.

(0,-,3/2). PIIFN. $z\frac{d}{dz} + A_0 + \begin{pmatrix} 0 & t+q \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2$

$x_1x_2x_3 + x_1 - x_2 + x_3 + s = 0$, with $s \in \mathbb{C}$.

(-,-,3). PII. $\frac{d}{dz} + A_0 + A_1 z + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^2,$ all $tr(A_*) = 0.$

$x_1x_2x_3 - x_1 - \alpha x_2 - x_3 + \alpha + 1 = 0$ with $\alpha \in \mathbb{C}^*$.

(-,-,5/2). PI. $\frac{d}{dz} + \begin{pmatrix} p & t+q^2 \\ -q & -p \end{pmatrix} + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2.$

$x_1x_2x_3 + x_1 + x_2 + 1 = 0.$

Table of the equations of the monodromy spaces for the 10 families.

The case of P_{III} $(D6)$
(following van der Put-Saito)

Two irregular singularities: $0, \ \infty$, with Katz rank 1 $(r(0) = r(\infty) = 1)$, no regular singularity.

We have a family of differential systems: $z\frac{d}{dz} + A_0 z^{-1} + A_1 + A_\infty z$, up to a scaling $z \mapsto \lambda z$, we can suppose that:

$$Q_0 = (\tfrac{t}{2} z^{-1}, -\tfrac{t}{2} z^{-1}) \quad \text{and} \quad Q_\infty = (\tfrac{t}{2} z, -\tfrac{t}{2} z).$$

Choosing adapted basis for the spaces of formal solutions at 0 and ∞, say (e_1, e_2) for $\hat{Sol}_0$ and (f_1, f_2) for $\hat{Sol}_\infty$, we get:

$$\hat{M}_0 := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \ S_{0,2} := \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}, \ S_{0,1} := \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix},$$

$$\hat{M}_\infty := \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \ S_{\infty,2} := \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}, \ S_{\infty,1} := \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}.$$

We have:

$$M_0 = \hat{M}_0 S_{0,2} S_{0,1} = \begin{pmatrix} \alpha & \alpha a_2 \\ \alpha^{-1} a_1 & \alpha^{-1}(1 + a_1 a_2) \end{pmatrix}$$

$$M_\infty = \hat{M}_\infty S_{\infty,2} S_{\infty,1} = \begin{pmatrix} \beta & \beta b_2 \\ \beta^{-1} b_1 & \beta^{-1}(1 + b_1 b_2) \end{pmatrix}.$$

The matrice of the link $L : \hat{\text{Sol}}_0 \to \hat{\text{Sol}}_\infty$ in the choosen basis is $\begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix} \in Sl_2(\mathbf{C})$. We set $M_0 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$. Using $M_\infty L = L M_0$, we eliminate the data at ∞. Then the coordinate ring for the variety of representations is the localization of:

$$\mathbf{C}[m_1, \dots, m_4, \ell_1, \dots, \ell_4]/(m_1 m_4 - m_2 m_3 - 1, \ell_1 \ell_4 - \ell_2 \ell_3 - 1)$$

given by:

$$0 \neq \alpha = m_1 \text{ and } 0 \neq \beta = \ell_1 \ell_4 m_1 + \ell_2 \ell_4 m_3 - \ell_1 \ell_3 m_2 - \ell_2 \ell_3 m_4.$$

We quotient by the action of $\mathbf{C}^* \times \mathbf{C}^*$ on the basis (algebraically):

$$e_1, e_2, f_1, f_2 \mapsto \lambda e_1, \lambda^{-1} e_2, \mu f_1, \mu^{-1} f_2.$$

The corresponding ring of invariants is a localization of a quotient of $\mathbf{C}[m_1, m_4, \ell_1\ell_4, m_2\ell_1\ell_3, m_3\ell_2\ell_4]$. It is the wild character variety.

Setting $y_1 := \ell_1\ell_4,\ y_2 := m_2\ell_1\ell_3,\ y_3 := m_4$, we get, for fixed $\alpha,\ \beta$, a *cubic surface*:

$$y_2(\beta - \alpha y_1 + y_2 + (y_1 - 1)y_3) + y_1(y_1 - 1)(1 - \alpha y_3) = 0,$$

and, after some simple transformations the *cubic surface*:

$$S_{\alpha,\beta} := \{x_1x_2x_3 + x_1^2 + x_2^2 + (1+\alpha\beta)x_1 + (\alpha+\beta)x_2 + \alpha\beta = 0\}.$$

The wild braid groups at 0 and ∞ are isomorphic to $\pi_1(\mathbf{C}^*) \approx \mathbf{Z}$, they acts *polynomially* on the *wild character variety $S_{\alpha,\beta}$*.

As B_1 is trivial, following a conjecture above (braiding of braids) the wild group must be isomorphic to $\mathbf{Z} * \mathbf{Z}$.

INTEGRABILITY AND NON-LINEAR GALOIS THEORY

Non-linear differential Galois theory

At the end of XIX-th century and at the beginning of XX-th century, J. Drach and afterwards E. Vessiot tried to create a non-linear differential Galois theory (which seduced P. Painlevé who used it to “prove that the solutions of P_I are *“new transcendental functions”*). Unfortunately their definitions were quite imprecise and there were important gaps in some of their proofs.

Recently H. Umemura (1996) and B. Malgrange (2001) returned to the problem. Their approaches are a priori quite different (they are equivalent in some sense in the algebraic case). Malgrange approach is similar to one of the approaches proposed by Vessiot.

We must replace the notion of group by the notion of *groupoid* (small category whose all the arrows are isomorphisms).

Non-linear differential Galois theory

B. Malgrange:

La théorie de Galois différentielle est ce que l'algèbre voit de la dynamique.

Differential Galois theory is what algebra sees from the dynamics.

More formally, the *differential Galois groupoid* is *the Zariski closure of the dynamics*.

It remains to precise this definition...

Ajoutons que la méthode de Jules DRACH, qui est *algébrique* et conduit directement aux invariants différentiels des *types* de sous-groupes du groupe ponctuel général, en *partant des équations différentielles*, est, au fond, indépendant de la théorie des groupes de Lie et permet même de retrouver cette théorie.

Ernest VESSIOT a discuté et précisé les principes de la théorie précédente, et donné une autre théorie rationnelle d'intégration qui repose sur la considération du plus petit groupe de transformations en $x, y_1, \dots, y_n$, laissant le système S invariant, et contenant la transformation infinitésimale

$$\frac{\partial f}{\partial x} + \sum_{h=1}^{n} \xi_h (x, y_1, \dots, y_n) \frac{\partial f}{\partial y_h},$$

dont les équations de définition soient rationnelles. Ce *groupe spécifique* est isomorphe au groupe de rationalité : celui-ci exprime la manière dont les transformations du groupe spécifique échangent les courbes intégrales $\mathfrak{J}$ de S.

C'est la théorie de l'intégration logique qui a permis à Paul PAINLEVÉ d'affirmer que les équations qui portent son nom définissent des transcendantes irréductibles aux transcendantes précédemment introduites dans l'analyse mathématique. On le constate en montrant que les équations dont il s'agit ont pour groupe de rationalité le groupe (simple) formé des transformations ponctuelles du plan qui laissent les aires invariantes.

Ernest VESSIOT,
Directeur honoraire de l'École normale supérieure.

Figure: E. Vessiot. Groupe de Galois différentiel: groupe de rationalité et groupe spécifique, irréductibilité de PI

Galois-Malgrange groupoid

For sake of simplicity M is a smooth complex affine algebraic variety.

Let $\mathcal{G}(M)$ be the groupoid of formal diffeomorphisms $(\hat{M}, a) \rightarrow (\hat{M}, b)$. roughly speaking a $\mathcal{D}$-groupoid is a sub-variety of $\mathcal{G}(M)$ defined by PDEs (differential ideal) whose projections on the jet spaces of finite order form an algebraic groupoid on $M \setminus Z$ (Z being an hypersurface). The Galois-Malgrange groupoid of an *autonomous system* $\dot{x} = X(x)$ is, by définition (Malgrange), the smallest $\mathcal{D}$-groupoid "containing" the flow of X.

This makes sense (Malgrange). Il y a version for *foliations* and a *discrete*: the smallest $\mathcal{D}$-groupoid "containing" an automorphism[2] or, more generally, a subgroup of automorphisms.

[2] Similarly a dominant morphism

Galois-Malgrange groupoid

A $\mathcal{D}$-groupoïde admits a *Lie algebra*.

The Galois-Malgrange groupoid of an *autonomous system* $\dot{x} = X(x)$ is the smallest $\mathcal{D}$-groupoid whose Lie algebra contains X.

In the algebraic case, one can define a $\mathcal{D}$-groupoid by the *conservation of some differential invariants* (Drach ?, P. Gabriel, J.F. Pommaret).

Intégrability and non-linear differential Galois theory

There is a *non-linear* variant of Morales-Ramis theorem.

Theorem (Ramis 2002)

If an Hamiltonian system is integrable in Liouville sense by meromorphic first integrals, then the Lie algebra of its Galois-Malgrange groupoid $\mathfrak{G}$ *is abelian.*

This result follows easily from a symplectic lemma.

In the algebraic case one can deduce from the above result Morales-Ramis-Simó theorem (and a fortiori Morales-Ramis theorem) using Artin theorem: Casale 2009.

Integrability: a change of point of view

In the spirit of Drach and Vessiot, it is possible to *define* some notions of integrability using the Galois-Malgrange groupoid $\mathfrak{G}$.

Some possible definitions ?

More and more restrictive definitions of integrability.

1. A dynamical system (continuous or discrete) on a complex analytic manifold M is said *integrable* if its Galois-Malgrange groupoid is *strictly smaller* than $\mathfrak{G}(M)$.

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4. A dynamical system (continuous or discrete) on a complex analytic manifold M is said *integrable* if its Lie Galois-Malgrange algebra $\mathfrak{G}$ is *abelian*.

More precisely there is a whole hierarchy of notions of integrability according to the list of $\mathcal{D}$-subgroupoids of $\mathcal{G}(M)$. It is an old idea of Drach and Vessiot.

Integrability in the sense 3 is strongly related to the notion of integrability by quadratures (Casale, Malgrange).

Non-integrability of Painlevé équations

To a Painlevé equation $y'' = \frac{d^2y}{dt^2} = f(t, y)$, we associate the vector field

$$X := \frac{\partial}{\partial t} + y' \frac{\partial}{\partial y} + f(t, y) \frac{\partial}{\partial y'}$$

on the extended phase space $(t, y, y') \in \mathbf{C}^3$. We consider the 2-form $\gamma := i_X\, dt \wedge dy \wedge dy'$.

The Galois groupoid of a Painlevé équation is a sub-groupoid of the invariance groupoid $\mathcal{G}_\gamma$ of the form γ, whose solutions are $\{\Gamma | \Gamma^* \gamma = \gamma\}$.

In the case of P_I, the Galois groupoid is $\mathcal{G}_\gamma$ (Casale). It is the same for P_{VI}, except for the Picard-Painlevé case (Loray-Cantat). We conjecture that it is also the same for all the Painlevé équations for generic values of the parameters.

A problem

The Riemann-Hilbert map is *analytic* but *not algebraic*, therefore the algebra is not the same on the two sides and, even if the two dynamics are conjugated, “what algebra sees from the dynamics” (therefore the Malgrange-Galois groupoid) *could be* different.

We conjecture that they “coincide” in the generic case (it is true for P_{VI}, according to Loray-Cantat).

OPEN PROBLEMS

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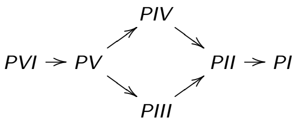
4. Understand the *"confluence"* of character varieties towards *irregular* character varieties (Cayley cubic surfaces ?). Understand the dynamics of Painlevé equations by *"confluence"* of the dynamics of P_{VI}.



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6. Search for possible analogies with the *Langlands program in number theory* via some analogies between *wild phenomena in complex foliations* and *wild phenomena in number theory*: *two-scaled infinitesimals* (A. Weil, Deligne, Katz, André, Ramis...).

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7. Develop similar *discrete* theories: q-analog theories

...

THE END

THANK YOU FOR YOUR ATTENTION