BPS Black Holes, the Hesse Potential and the Topological String

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Motivation/Recapitulation

Motivation: study the relation of four-dimensional BPS black holes in $N = 2$ theories with topological string theory.

- BPS black holes: charged, supported by complex scalar fields Y^I
- Free energy = Hesse potential. "Hamiltonian" version of effective $N = 2$ Lagrangian, duality invariant.
- Question: relation of Hesse potential $\mathcal{H}$ to topological string theory?

$$e^{\mathcal{H}} = |Z_{\mathrm{top}}|^2 \qquad ? \tag{1}$$

  Microstates of BPS black holes captured by topological string theory? (OSV conjecture)

- To study (1) construct new variables for the Hesse potential.

Type II on a Calabi-Yau threefold

$N = 2$ theory: type II string theory on a Calabi-Yau threefold CY_3. IIB:

- No-where vanishing holomorphic three-form ω.
- ω varies over the space of complex structure deformations $\mathcal{M}$.
- Locally, homogeneous coordinates X^I on $\mathcal{M}$ $(I = 0, \ldots, h^{2,1})$,

$$\text{periods} \quad X^I = \int_{A^I} \omega \quad , \quad F_I^{(0)} = \int_{B_I} \omega \quad , \quad F_I^{(0)} = \partial F^{(0)}(X)/\partial X^I \, ,$$

  where (A^I, B_J) symplectic basis of $H_3(CY_3, \mathbb{Z})$.
- Different choices of symplectic basis differ by $Sp(2 + 2h^{2,1}, \mathbb{Z})$ transformations (symplectic transformations).
- Period vector $(X^I, F_I^{(0)})$ undergoes symplectic transformations.
- Sometimes, change of basis can be undone by picking a different ω. Discrete symmetry group $\Gamma \subset Sp(2 + 2h^{2,1}, \mathbb{Z})$.
- Period matrix $\quad N_{IJ}^{(0)} = \mathrm{Im}\, F_{IJ}^{(0)} \quad , \quad F_{IJ}^{(0)} = \partial F_I^{(0)}/\partial X^J$.
- Special coordinates $\quad t^i = X^i/X^0 \quad , \quad X^0 \qquad (i = 1, \ldots, h^{2,1})$.

Topological String Theory (TST)

Perturbative string theory: CFT on 2d worldsheet Σ.

TST: arises by twisting the internal 2D CFT.

- Cohomological theory: correlation functions independent of worldsheet metric.
- Perturbatively defined in terms of an asymptotic expansion in the topological string coupling g_s (complex), with partition function

$$Z_{\rm top} = \exp \sum_{g=0}^{\infty} g_s^{2g-2} \, F^{(g)}(t) \quad , \quad g_s^{-1} = X^0 \; ,$$

  topological free energies $F^{(g)}$ computed as correlators on orientable Riemann surface Σ_g of genus g .
- Expect non-perturbative corrections of order e^{-1/g_s} to $\ln Z_{\rm top}$.
- Naively, the twisting yields holomorphic $F^{(g)}(t)$. However: Obstruction, holomorphic anomaly. BCOV 1993

Topological String Theory (TST)

- Hence:

$$Z_{\text{top}} = \exp\left[g_s^{-2}\,F^{(0)}(t) + \sum_{g=1}^{\infty} g_s^{2g-2}\,F^{(g)}(t,\bar{t})\right] \quad ,$$

$$F^{(g)}(t,\bar{t}) = F^{(g)}(t) + R^{(g)}[N^{(0)IJ}, F^{(r<g)}(t)] \quad , \quad g \geq 1 \; .$$

- Dependence on $1/g_s = X^0$ remains holomorphic.
- Wave function approach: `Witten 1993, Aganagic+Bouchard+Klemm 2006`
  - $R^{(g)}$ computable via a Feynman graph expansion with
    - $N^{(0)IJ}$ as propagator;
    - vertices constructed out of $F^{(r<g)}(t)$
  - define $F^{(g)}(X,\bar{X}) = g_s^{2g-2}\,F^{(g)}(t,\bar{t}) \quad , \quad g \geq 1$.

Topological Partition Function

Under symplectic transformations $Sp(2 + 2h^{2,1}, \mathbb{Z})$: for $g \geq 1$,

- $\tilde{F}^{(g)}(\tilde{X}, \bar{\tilde{X}}) = F^{(g)}(X, \bar{X})$, functions
- $\Gamma \subset Sp(2 + 2h^{2,1}, \mathbb{Z})$: $F^{(g)}(\tilde{X}, \bar{\tilde{X}}) = F^{(g)}(X, \bar{X})$, invariant.
- $1/g_s = X^0$ transforms under Γ , $F^{(g)}(t, \bar{t})$ modular forms.

Thus, under symplectic transformations $Sp(2 + 2h^{2,1}, \mathbb{Z})$,

$$Z_{\mathrm{top}} = \exp \sum_{g=1}^{\infty} F^{(g)}(X, \bar{X}) \ ,$$

transforms as a function.

Now, let's turn to the $N = 2$ LEEA and the "Hamiltonian" version based on the Hesse potential $\mathcal{H}$ (real):

- $\mathcal{H}$ also transforms as a function under symplectic transformations,
- suggesting the relation

$$\mathrm{e}^{\mathcal{H}} = |Z_{\mathrm{top}}|^2 \quad . \ \text{Subtle.}$$

LEEA

- $N = 2$ Wilsonian action:
  - $N = 2$ vector multiplets, complex scalar fields Y^I (different from X^I)
  - higher-curvature interactions $\propto$ Weyl^2: (Υ, C^2) Weyl background

$$F(Y, \Upsilon) = F^{(0)}(Y) + \sum_{g=1}^{\infty} \Upsilon^g \, F^{(g)}(Y) \, .$$

- Effective action: requires non-holomorphic modifications to make duality symmetries manifest,

G.L.Cardoso, B. de Wit, J. Käppeli, T.Mohaupt, hep-th/0412287

$$F = F^{(0)}(Y) + 2i \, \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \, ,$$

  - Ω is real, homogeneous function of degree 2 .
  - Black hole context, $\Upsilon = \bar{\Upsilon} = -64$ at horizon,
  - $$\Omega = \sum_{g=1}^{\infty} \Upsilon^g \, \Omega^{(g)}(Y, \bar{Y}) \, .$$

  $\Omega^{(g)}$ homogeneous of degree $2 - 2g$.

Hesse Potential

"Hamiltonian" version based on the Hesse potential $\mathcal{H}(\phi, \chi)$, where

$$\phi^I = Y^I + \bar{Y}^{\bar{I}} \quad , \quad \chi_I = F_I + \bar{F}_{\bar{I}} \quad , \quad F_I = \partial F / \partial Y^I \; .$$

In the context of BPS black holes: electro/magneto-static potentials associated to the electric/magnetic charges (q_I, p^I).

Behavior under symplectic transformations:

- Wilsonian/LEEA action based on a complex parametrization $(Y^I, F_I(Y, \bar{Y}, \Upsilon))$. Under symplectic transformations,
  $Y^I \longrightarrow \tilde{Y}^I = \tilde{Y}^I(Y, \bar{Y}, \Upsilon) \quad , \quad \Upsilon \longrightarrow \Upsilon \; .$

  Entanglement with $\Upsilon \Longrightarrow Y^I \neq X^I$!

  $\Omega(Y, \bar{Y}, \Upsilon)$ not a function under symplectic transformations.

- "Hamiltonian" version $\mathcal{H}(\phi, \chi)$ based on a real parametrization. (ϕ^I, χ_I) transform as the charges (p^I, q_I). Thus, they are not subjected to Υ-corrections under symplectic transformations.

Therefore $\quad \phi^I = X^I + \bar{X}^{\bar{I}}$!

New Variables for the Hesse Potential

Proposal:

Construct map between $Y_{\rm sugra}$ and $X_{\rm top}$ by linking them to the Hesse variables (ϕ, χ):

- reexpress the Hesse variables (ϕ, χ) in terms of new variables X,

$$\begin{aligned} Y^I + \bar{Y}^{\bar{I}} &= \phi^I = X^I + \bar{X}^{\bar{I}} \quad , \\ F_I + \bar{F}_{\bar{I}} &= \chi_I = F_I^{(0)}(X) + \bar{F}_{\bar{I}}^{(0)}(\bar{X}) \quad . \end{aligned}$$

- The new variables X transform precisely as the topological string variables. Natural to identify $X = X_{\rm top}$.
- $X^I = Y^I + \Delta Y^I(Y, \bar{Y}, \Upsilon)$, iteratively in Υ, complicated expressions.

  They are non-holomorphic in view of the reality property of the map.

New Variables for the Hesse Potential

In the supergravity variables, the Hesse potential is given by

$$\mathcal{H} = -i\left(\bar{Y}^I F_I - Y^I \bar{F}_{\bar{I}}\right) - 2i\left(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_{\bar{\Upsilon}}\right) \ ,$$

where $\quad F = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}, \Upsilon) \quad , \quad F_\Upsilon = \partial F/\partial\Upsilon$,
with $\Omega = \Upsilon\, \Omega^{(1)}(Y, \bar{Y}) + \Upsilon^2\, \Omega^{(2)}(Y, \bar{Y}) + \ldots$.

To compare with the topological string, need to express $\mathcal{H}$ in terms of the new variables X^I. Compute ΔY^I iteratively $\quad (X^I = Y^I + \Delta Y^I)$.

At order Υ^2 obtain, in the new variables X^I,

$$\begin{aligned} \mathcal{H}(X, \bar{X}, \Upsilon) &= -i\left(\bar{X}^I F_I^{(0)}(X) - X^I \bar{F}_I^{(0)}(\bar{X})\right) + 4\,\Upsilon\,\Omega^{(1)} \\ &\quad + 4\,\Upsilon^2 \left[\Omega^{(2)} - i\,N^{(0)IJ}\left(\Omega^{(1)}_I - \Omega^{(1)}_{\bar{I}}\right)\left(\Omega^{(1)}_J - \Omega^{(1)}_{\bar{J}}\right)\right] \end{aligned}$$

Packaged in terms of symplectic functions, at any order g.

A Generating Function?

Thus, in the new variables X^I,

$$\mathcal{H}(X,\bar{X},\Upsilon) = -i\left(\bar{X}^I F_I^{(0)}(X) - X^I \bar{F}_I^{(0)}(\bar{X})\right) + 4\sum_{g=1}^{\infty}\Upsilon^g\left(\Omega^{(g)} + \ldots\right),$$

where

- $\left(\Omega^{(g)} + \ldots\right)$ symplectic packages at any order g.
- Non-holomorphic extension of the results of `de Wit, hep-th/9602060`.

  Holomorphic set-up: tower of symplectic functions that are modifications of $F_{\Upsilon\ldots\Upsilon}$,

$$\mathcal{D}^{g-1} F_\Upsilon(Y,\Upsilon) \quad , \quad \mathcal{D} = \frac{\partial}{\partial\Upsilon} + iF_{\Upsilon I} N^{IJ} \frac{\partial}{\partial Y^J} .$$

Relation with Topological String Theory

Expressing the Hesse potential in terms of special coordinates
$t^i = X^i/X^0 \quad , \quad X^0 \quad (i = 1, \dots, h^{2,1})$, dependence on

$$1/X^0 = g_s \ , \ \ 1/\bar{X}^0 = \bar{g}_s$$ and mixed powers, such as $1/|X^0|^2$.

This is, in general, not of the topological string type,

$$|Z_{\rm top}|^2 = \exp \sum_{g=1}^{\infty} \left[g_s^{2g-2} \, F^{(g)}(t, \bar{t}) + {\rm c.c.} \right] \ .$$

However, there is no complete knowledge of LEEA Ω for any CY_3 compactification. Thus, various possibilities:

- at this stage, it is consistent to assume

$$\begin{aligned} \left(e^{\mathcal{H}}\right)_{|\Omega} &= |Z_{\rm top}|^2 \quad , \quad \Upsilon = -64 \ , \\ 4 \left(\Omega^{(g)} + \dots \right) &= F^{(g)}(X, \bar{X}) + {\rm c.c} \ . \end{aligned}$$

Determines $\Omega^{(g)}(X, \bar{X})$ in terms of topological string data.

Models with S-, T- Duality Symmetries

- In general,

$$\left(\mathrm{e}^{\mathcal{H}}\right)_{|\Omega} = |Z_{\rm top}|^2\,\mathrm{e}^{P}$$

  with P a symplectic function that
  - depends on mixed powers of X^0, such as $1/|X^0|^2$;
  - determined in terms of the topological string, such as $N^{(0)IJ} F_I^{(1)} F_{\bar J}^{(1)}$.

In models with S-, T- duality symmetries ($\Gamma \subset Sp(2+2h^{2,1},\mathbb{Z})$), partial knowledge about Ω. Examples: FHSV-model, STU-model.

For instance, S-duality restricts $\Omega(Y,\bar Y,\Upsilon)$ in LEEA:

$$\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm S} - \Delta_{\rm S}{}^2\,\frac{\partial\Omega}{\partial S} = \frac{\partial(\Delta_{\rm S}{}^2)}{\partial S}\left[-\tfrac{1}{2}\,Y^0\frac{\partial\Omega}{\partial Y^0} - \frac{\mathrm{i}c}{4\,\Delta_{\rm S}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\frac{\partial\Omega}{\partial T^b}\right]$$

Solve iteratively, $\Omega = \Upsilon\,\Omega^{(1)}(Y,\bar Y) + \Upsilon^2\,\Omega^{(2)}(Y,\bar Y) + \ldots$, up to duality invariant terms.
Adjusting this ambiguity, get agreement with $F^{(1)}(X,\bar X), F^{(2)}(X,\bar X)$.

Outlook

Concluding:

$$\left(e^{\mathcal{H}}\right)_{|\Omega} = |Z_{\rm top}|^2 \, e^{P} \ ,$$

with P a symplectic function that depends on mixed powers of $X^0 = 1/g_s$.

Need to understand:

- the generating function for symplectic packages which build up $\mathcal{H}$,
- non-perturbative corrections e^{-1/g_s} to Ω in models with S-, T-duality symmetries. (LEEA: e^{Y^0} effects).

Thanks!