UNDERSTANDING BPS BLACK HOLE ENTROPY

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Gauge Theories, Supersymmetry
and Mathematical Physics,
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A intruiging result:

Bardeen, Carter, Hawking, 1973

Black hole mechanics ⟷ Thermodynamics

$$\delta E = T\,\delta S - p\,\delta V$$

$$\delta M = \frac{\kappa_{\rm S}}{2\pi}\,\frac{\delta A}{4} + \phi\,\delta Q + \Omega\,\delta J$$

Planck units!

$$T \leftrightarrow \frac{\kappa_{\rm S}}{2\pi}$$

Hawking temperature
Surface gravity

$$S \leftrightarrow \frac{A}{4}$$

Bekenstein-Hawking area law

Charged Black Holes

(Reissner-Nordstrom)

charged, static, spherically symmetric, solutions to Einstein-Maxwell theory

$P,\ Q$ magnetic/electric charges

$$G_\mathrm{N}\, M^2 > P^2 + Q^2$$ non-extremal, two horizons

$$G_\mathrm{N}\, M^2 = P^2 + Q^2$$ extremal, horizons coalesce

$$G_\mathrm{N}\, M^2 < P^2 + Q^2$$ naked singularity, physically unacceptable

Extremal black holes have zero temperature

BPS ⟷ Partially supersymmetric

Naturally embedded in extended supergravity

In this talk: *N=2 supergravity in D=4 space-time dimensions*

Question: statistical interpretation of black hole entropy ?

microscopic/statistical entropy ⟷ macroscopic/field-theoretic entropy

- microstate counting ⟶ entropy $S_{\rm micro} = \ln d(q,p)$
- supergravity: Noether surface charge *Wald, 1993*
  first law of black hole mechanics

degrees of freedom entropy ⟷ horizon behaviour

In recent years:

new insights and new concepts ✍ amenable to quantitative work ✔

STRING THEORY provides new insights *Strominger, Vafa, 1996*

N=2 SUPERGRAVITY

vector supermultiplets coupled to N=2 supergravity :

scalars: $X^I = X^0,\, X^A$ complex

vectors: $W_\mu{}^I = W_\mu{}^0,\, W_\mu^A$ charges $p^0 \;\; q_0 \;\; p^A \; q_A$

classical solutions :

generalized (extremal) Reissner-Nordstrom black holes

how to calculate ?

N=2 supergravity Lagrangians are encoded in
holomorphic homogeneous functions:

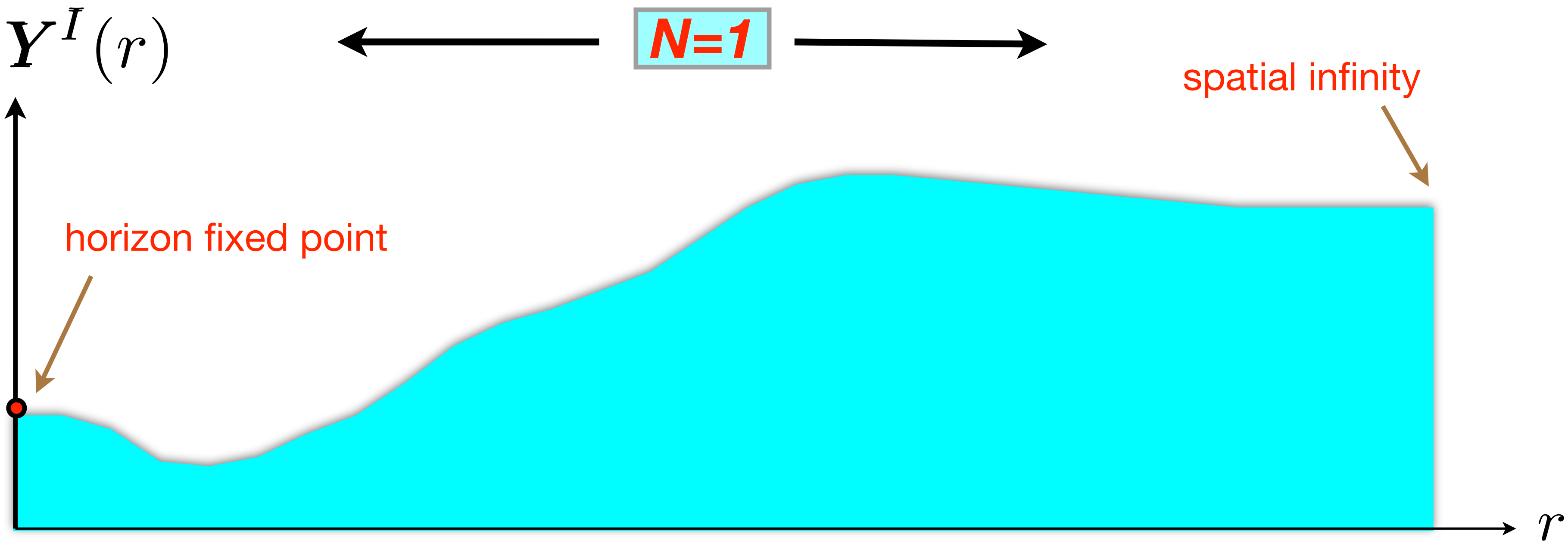
$$F(\lambda\, X) = \lambda^2\, F(X)$$

At the horizon, use rescaled variables $X^I \longrightarrow Y^I$

problem reduces to algebraic equations !

BPS black holes:

solitonic solutions with residual N=1 supersymmetry



N=2	**N=2**
horizon	spatial ∞
Bertotti-Robinson geometry	flat Minkowski space-time
$Y^I\vert_{\text{horizon}}$ determined by $p,\ q$	determines black hole mass
ATTRACTOR MECHANISM !	*Ferrara, Kallosh, Strominger, 1995*

attractor equations

$$Y^I - \bar{Y}^I = \mathrm{i}\, p^I$$
$$F_I(Y) - \bar{F}_I(\bar{Y}) = \mathrm{i}\, q_I$$

$F_I(Y) \equiv \partial_I F(Y)$

(special Kähler geometry)

$$\frac{\mathrm{AREA}}{G_{\mathrm{N}}} = 4\pi\, |Z|^2$$

with

$$|Z|^2 = p^I\, F_I(Y) - q_I\, Y^I$$

e.g. Calabi-Yau compactifications

behaviour at the horizon is determined by the charges

the $Y^I(r)$ *parametrize the* CY_3 manifolds
which change as a function of r.

EXAMPLE

$$F(Y) = -\frac{1}{6}\,\frac{C_{ABC}\,Y^A Y^B Y^C}{Y^0}$$

AREA LAW ⇒ entropy ⇒

$$S_{\mathrm{macro}}(p,q) = \pi\,|Z|^2 = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\,C_{ABC}\,p^A p^B p^C}$$

$$(p^0 = 0)$$

this represents the leading term of the microscopic result : it scales quadratically in the charges!

Subleading corrections:

extend with one ‘extra’ complex field, originating from pure supergravity

Υ

$$F(Y) \longrightarrow F(Y,\Upsilon)$$

↑ Weyl background

homogeneity: $F(\lambda Y, \lambda^2\Upsilon) = \lambda^2\, F(Y,\Upsilon)$

Υ-dependence leads to terms $\propto (R_{\mu\nu\rho}{}^{\sigma})^2$ in effective action

attractor equations remain valid with $\Upsilon = -64$

$Y^I - \bar Y^I = \mathrm{i}p^I$	magnetic charges
$F_I - \bar F_I = \mathrm{i}q_I$	electric charges

Ferrara, Kallosh, Strominger, 1996
Cardoso, dW, Käppeli, Mohaupt, 2000

covariant under dualities!

to ensure the validity of the first law of black hole mechanics, one must modify the definition of black hole entropy.

INSTEAD:
use Wald's prescription based on a Noether surface charge.

Wald, 1993

general N=2 formula:

$$S_{\rm macro} = \pi\,|Z|^2 - 256\,{\rm Im}\,F_\Upsilon(Y,\Upsilon)\Big|_{\Upsilon=-64}$$

$|Z|^2$ → $\dfrac{\rm AREA}{4\,G_{\rm N}}$

${\rm Im}\,F_\Upsilon$ → modification

Cardoso, dW, Mohaupt, 1998

This modifies the Bekenstein-Hawking area law !

The BPS entropy function

entropy function $\Sigma(Y, \bar{Y}, p, q)$

stationary $\partial_Y \Sigma(Y, \bar{Y}, p, q) = 0 \rightarrow Y^*(p, q)$ (attractor equations)

$$S_{\text{macro}} = \pi\, \Sigma\Big|_{Y^*(p,q)}$$

variational principle !

This can also be understood from microscopic arguments!

→ Consider heterotic black holes

Heterotic black holes

on T^6, *dual to type-IIA on* $K3 \times T^2$

also for CHL black holes

Chaudhuri, Hockney, Lykken,1995

N=4 supersymmetry, T- *and* S-*duality*

classical result : $S_{\rm macro} = \dfrac{A}{4\,G_{\rm N}} = \pi\sqrt{q^2p^2 - (q\cdot p)^2}$

$q^2 \quad p^2 \quad p\cdot q$ T-*duality invariant bilinears*

Cvetic, Tseytlin, 1995

Bergshoeff et. al.,1996

two types of BPS states :

1/4 BPS *'dyonic'* $q^2p^2 - (p\cdot q)^2 > 0$

1/2 BPS *'electric'* $q^2p^2 - (p\cdot q)^2 = 0$

↓ zero classical area

microscopic results

1/4 BPS states

dyonic degeneracies

$$d_k(p,q) = \oint_{3-\mathrm{cycle}} \mathrm{d}\Omega \, \frac{\mathrm{e}^{\mathrm{i}\pi[\rho\, p^2 + \sigma\, q^2 + (2\upsilon - 1) p\cdot q]}}{\Phi_k(\Omega)}$$

$k = 10, 6, 4, 2, 1$

$\Omega = \begin{pmatrix} \rho & \upsilon \\ \upsilon & \sigma \end{pmatrix}$ period matrix of g=2 Riemann surface

Dijkgraaf, Verlinde, Verlinde, 1997

Shih, Strominger, Yin, 2005

Jatkar, Sen, 2005

formally S-duality invariant

Asymptotic growth (*1/4 BPS*)

for 'large' dyonic charges $q^2,\, p^2,\, p\cdot q$

dilaton S can remain finite

Cardoso, dW, Käppeli, Mohaupt, 2004 $(k=10)$

Jatkar, Sen, 2005 $(k=6,4,2,1)$

saddle-point approximation reproduces the macroscopic results up to terms inversely proportional to the charges !

Corresponding BPS entropy function:

Cardoso, dW, Mohaupt, 1999

$$\Sigma(S,\bar S,p,q) = -\frac{q^2 - \mathrm{i}p\cdot q(S-\bar S) + p^2|S|^2}{S+\bar S} + 4\,\Omega(S,\bar S,\Upsilon,\bar\Upsilon)$$

$$\Omega = \frac{1}{128\pi}\Big[\Upsilon\,\log\eta^{12}(S) + \bar\Upsilon\,\log\eta^{12}(\bar S) + \tfrac{1}{2}(\Upsilon+\bar\Upsilon)\,\log(S+\bar S)^6\Big]$$

↑ non-holomorphic

this reproduces instanton corrections and non-holomorphic terms!

S-duality invariant!

BPS entropy function

$$\Sigma(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - q_I(Y^I+\bar{Y}^I) + p^I(F_I+\bar{F}_I)$$

$Y^I+\bar{Y}^I$ and $F_I+\bar{F}_I$ play the role of electro- and magnetostatic potentials at the horizon

$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$ *'free energy' to be defined shortly*

$\delta\Sigma = 0$ $\Leftrightarrow$ **attractor equations** (and $\Upsilon=-64$)

$$\pi\Sigma\Big|_* = \mathcal{S}_{\rm macro}(p,q)$$

Non-holomorphic extension incorporated by:

$$F \longrightarrow F^{(0)}(Y) + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$$

Non-holomorphic and homogeneous

R^2-terms in the action yield the subleading corrections in area and entropy

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g}\,\Upsilon^g\, F^{(g)}(t)$$

subleading corrections

$t^A = Y^A/Y^0$

Free energy :

$$\mathcal{F}(Y,\bar Y,\Upsilon,\bar\Upsilon) = -\mathrm{i}\left(\bar Y^I F_I - Y^I \bar F_I\right) - 2\mathrm{i}\left(\Upsilon F_\Upsilon - \bar\Upsilon \bar F_\Upsilon\right)$$

Cardoso, dW, Käppeli, Mohaupt, 2006

- *invariant under dualities!*
- *follows from the requirement that the variation of the entropy function yields the attractor equations*
- *non-holomorphic corrections can again be incorporated*

This free energy equals the Hesse potential $\mathcal{H}(\phi, \chi, \Upsilon, \bar{\Upsilon})$
related to $F(Y, \Upsilon)$ *(with possible non-holomorphic corrections) by a Legendre transformation*

Real special geometry

Freed, 1999
Alekseevsky, Cortés, Devchand, 2002

electro- and magnetostatic potentials

$$\phi^I = Y^I + \bar{Y}^I \qquad \chi^I = F_I + \bar{F}_I$$

attractor equations

$$\frac{\partial \mathcal{H}}{\partial \phi^I} = q_I \qquad \frac{\partial \mathcal{H}}{\partial \chi_I} = -p^I$$

'Hamiltonian form' of the function $F(Y, \Upsilon)$

Thermodynamics and microstates

The role played by the Legendre transform is indicative of a thermodynamic origin (in the spirit of OSV)

Ooguri, Strominger, Vafa, 2004

define a duality invariant canonical partition function

$$Z(\phi,\chi) = \sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]}$$

related to the free energy (naturally formulated as a function of the potentials)

inverse Laplace transform

$$d(p,q) \propto \int \mathrm{d}\chi_I\,\mathrm{d}\phi^I\;Z(\phi,\chi)\,\mathrm{e}^{\pi[-q_I\phi^I + p^I\chi_I]}$$

over periodicity intervals $(\phi-\mathrm{i},\phi+\mathrm{i})$ $(\chi-\mathrm{i},\chi+\mathrm{i})$

identify:

$$\sum_{\{p,q\}} d(p,q)\,\mathrm{e}^{\pi[q_I\phi^I - p^I\hat{\chi}_I]} \sim \sum_{\text{shifts}} \mathrm{e}^{\pi\,\mathcal{H}(\phi,\chi,\Upsilon,\bar{\Upsilon})}$$

Cardoso, dW, Käppeli, Mohaupt, 2006

Integrating out the moduli χ_I, leads to an integral over the ϕ^I.
This must be the original OSV integral with additional terms (measure factor).
This modified OSV integral is consistent with duality.

The integral is similar to the kind of integrals that arise in the asymptotic growth calculation which can be evaluated by saddle-point approximation.

At the semiclassical level there is full agreement, but beyond that not much is known.

Topological string:

Bershadsky, Cecotti, Ooguri, Vafa, 1994

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g}\Upsilon^g\, F^{(g)}(t)$$

$(Y^0)^2 F^{(0)}(t)$

genus-*g* partition function of a twisted non-linear sigma model with CY target space

Y^0 loop-counting parameter

$t^A = Y^A/Y^0$

- Holomorphic anomaly $\partial_{\bar t} F^{(g)} \neq 0$ → $F^{(g)}(t,\bar t)$
- Topological string coupling : $Y^0 = g_{\rm top}^{-1}$
- Duality covariant sections $F^{(g)}$
- $F^{(g)}$ captures certain string amplitudes

Antoniadis, Gava, Narain, Taylor, 1993

A subtle question :

$$F(Y,\Upsilon) = F^{(0)}(Y) + \sum_{g=1} (Y^0)^{2-2g} \Upsilon^g \, F^{(g)}(t)$$

this same expansion is applied to

(a) the topological string and *(b)* the effective action !

But are they identical functions ?

NO! (and still agreement with string amplitudes?)

use duality arguments : Cardoso, dW, Mahapatra, 2008

effective action

- the $F^{(g)}$ are NOT invariant
- the periods transform correctly under monodromies
- the duality transformations are Υ-dependent

topological string

- the $F^{(g)}$ are COVARIANT sections
- the periods refer to $F^{(0)}$
- the duality transformations are Υ-independent

difference has been confirmed:

- $\mathcal{F}^{(1)}$ is still invariant
- for $g \geq 2$ there are differences

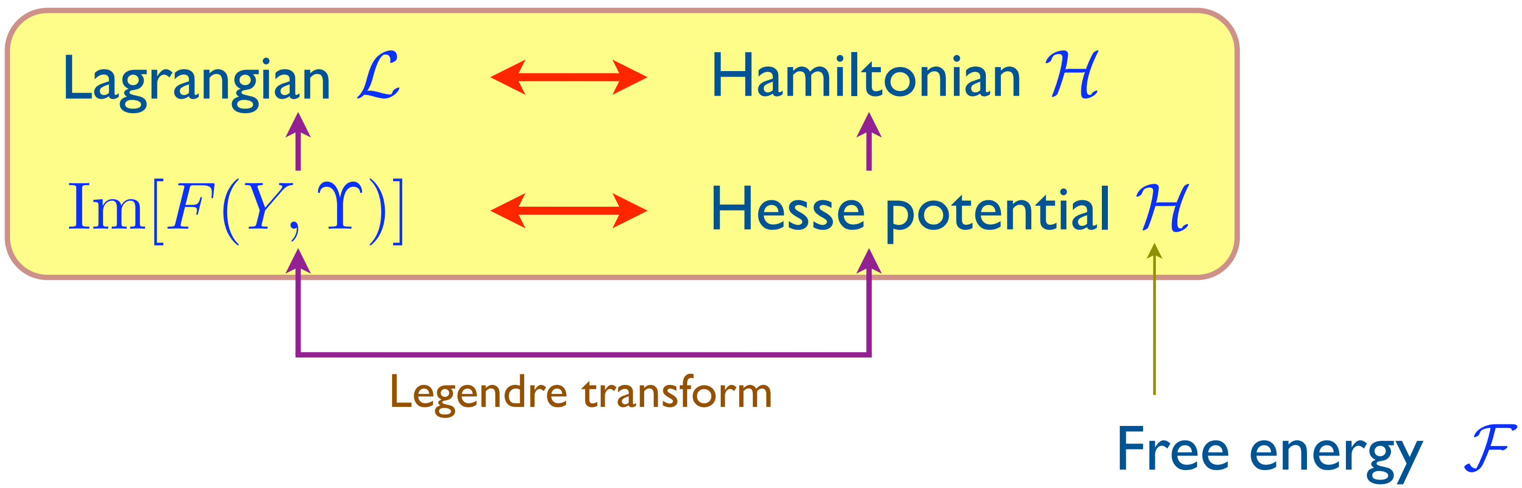
explicit evaluation and comparison of the non-holomorphic corrections for the FHSV model supports this conclusion.

Grimm, Klemm, Marino, Weiss, 2007
Cardoso, dW, Mahapatra, 2008, 2010

What then is the precise relation?

☛ Recall : amplitudes ⇔ connected graphs ⇎ ***1PI*** graphs

☛ Then : compare with E/M duality properties of



Compare : electromagnetism $\mathcal{L}(E, B)$

$(E \leftrightarrow H)$

$(B \leftrightarrow D)$

under E/M duality

transformations depend on the details of the Lagrangian (which is not invariant)

$$D = \frac{\partial \mathcal{L}}{\partial E}, \quad H = \frac{\partial \mathcal{L}}{\partial B}$$

depend on the details of the Lagrangian

$\mathcal{H}(D, B)$ invariant under monodromies

transformations do not depend on the details of the Lagrangian

We close with an example: Born-Infeld Lagrangian

$$\mathcal{L} = -g^{-2}\sqrt{\det[\eta_{\mu\nu} + g\,F_{\mu\nu}]} + g^{-2}$$

spherical symmetry

$$ds^2 = -dt^2 + dr^2 + r^2(\sin^2\theta\,d\varphi^2 + d\theta^2)$$

$$F_{rt} = e \qquad F_{\varphi\theta} = p\,\sin\theta$$

$$\begin{aligned} \mathcal{L}_{\rm red} &= \int d\varphi\,d\theta\,\mathcal{L} \\ &= 4\pi r^2 g^{-2}\left[\sqrt{1-g^2e^2}\sqrt{1+g^2p^2r^{-2}} - 1\right] \end{aligned}$$

symmetry:

$$\begin{cases} \delta e = p\sqrt{\dfrac{1-g^2e^2}{1+g^2p^2}} \\ \delta p = -e\sqrt{\dfrac{1+g^2e^2}{1-g^2p^2}} \end{cases}$$

(suppress: $4\pi, r$)

define electric charge $q = \frac{\partial \mathcal{L}_{\mathrm{red}}}{\partial e}$

Hamiltonian $\mathcal{H} = g^{-2}\left[\sqrt{1 + g^2(p^2 + q^2)} - 1\right]$

symmetry: $\begin{cases} \delta q &= p \\ \delta p &= -q \end{cases}$

Schrödinger, 1935

independent of the coupling constant g !

CONCLUSIONS

- *A lot op progress has been made in the study of BPS black holes. Some of that progress has been reviewed in this talk.*

- *The study of BPS black holes also seems to lead to addressing old questions and to unraveling new structure.*

- *For part of this the BPS black holes play only an ancillary role at the moment*

- *Some of these issues will be addressed by the talk of Gabriel Cardoso.*