The Modular class of a differential graded manifold.

Slogan "Algebraization of Physics" (A. Losev)

Manifold M_0 $\rightsquigarrow$ Q-manifold (M, Q)

(space-time, phase space, configuration space)

M: supermanifold; Q: homological vector field

(BRST, BV, BFV)

- DG-manifold : Q-manifold + grading
  - BRST, BV, BFV
  - Poisson structures, Lie Algebroids (Vaintrob '97), Courant algebroids (D. R. '99, '02)
- Invariants of Q-manifolds — characteristic classes (Lyakhovich - Sharapov '04, Lyakhovich - Mosman - Sharapov '09) (earlier for Poisson mflds & Lie algebroids: Crainic, Fernandes)
- Modular class: simplest intrinsic char. class
  - all constructions involve the notion of divergence.
  - Poisson manifolds: Koszul ('85), Weinstein ('96), Huebschmann ('97)
  - Lie algebroids: Evens-Lu-Weinstein ('96),
  - Maps of L.A.'s: Kosmann-Schwarzbach, Laurent-Gengoux, Weinstein ('05, '07) Grabowski - Marmo - Michor

All of the above can be understood as special cases of

- Q-manifolds (Vaintrob '97, Lyakhovich - Sharapov '04, Voronov '07)
- Related to this: BV Laplacian/Antibracket (Schwarz, Khudaverdian, Voronov)
- Related to: holonomy, foliations, integrable hierarchies, quantum anomalies ...

Plan (1) Q-manifolds, dg-manifolds, modular class
(2) Lie algebroids, Poisson mflds, Courant algebroids.
(3) BV anomaly

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(1) **Q-manifold** (M, Q)

M - supermanifold, $Q \in \mathrm{Vect}^1(M)$: $\underline{Q = Q^a \partial_a}$

$$Q^2 = \tfrac{1}{2}[Q,Q] = 0 \qquad \underline{Q^a \partial_a Q^b = 0}$$

$$\Rightarrow H(M,Q) = \mathrm{Ker}\, Q / \mathrm{Im}\, Q = H^0 \oplus H^1$$

DG-supermanifold (M, Q, ε)

$$\underset{\text{degree, ghostnumber}}{\varepsilon = |a|\, x^a \partial_a} \qquad [\varepsilon, Q] = Q \quad (Q \in \mathrm{Vect}^{1,1}(M))$$

DG-manifold : $P = (-1)^\varepsilon$ (parity = degree mod 2)

$\mathbb{N}$-graded DG-manifold: all $|a| \geq 0$

"space with symmetries", $H(M,Q) = \bigoplus_{i \geq 0} H^i(M,Q)$

structure of such manifolds ($\mathbb{N}$-graded)

$$\begin{aligned}
x_0^{i_0} &= x_0^{i_0}(x_0') \\
x_1^{i_1} &= A^{i_1}{}_{i_1'}(x_0')\, x_1^{i_1'} \\
x_2^{i_2} &= A^{i_2}{}_{i_2'}(x_0')\, x_2^{i_2'} + f_2(x_0', x_1') \\
&\cdots \\
x_d^{i_d} &= A^{i_d}{}_{i_d'}(x_0')\, x_d^{i_d'} + f_d(x_0', \dots, x_{d-1}')
\end{aligned}
\Rightarrow J = \begin{pmatrix} J_{00} & & & 0 \\ & J_{11} & & \\ & * & \ddots & \\ & & & J_{dd} \end{pmatrix}$$

$$\begin{tikzcd}
M_1 \arrow[d, shift right, "\pi_1"'] & M_2 \arrow[l] \arrow[dl, shift right, "\pi_2"'] & \cdots \arrow[l] & M_d = M \arrow[l] \arrow[dlll, shift right, "\pi=\pi_d"'] \\
M_0 \arrow[u, shift right, "\iota_1"'] \arrow[ur, shift right, "\iota_2"'] \arrow[urrr, shift right, "\iota=\iota_d"']
\end{tikzcd}$$

vector bundle

The Modular class

Recall $\begin{array}{c} V \\ \downarrow \\ M \end{array}$ vector bundle $\Rightarrow$ $\begin{array}{c} \mathrm{Ber}(V) \\ \downarrow \\ M \end{array}$ line bundle

$\mathrm{Ber}(M) := \mathrm{Ber}(T^*M)$, sections = volume forms

Assume M orientable (for simplicity)

Pick $\nu \in \Gamma(\mathrm{Ber}\, M)$ vol. form

$$\mathrm{div}_\nu Q := L_Q \nu / \nu \in C^\infty(M)^1$$

- $$0 = L_{Q^2}\nu = (L_Q)^2 \nu = L_Q((\mathrm{div}_\nu Q)\nu) =$$
$$= Q(\mathrm{div}_\nu Q)\nu - (\mathrm{div}_\nu Q)^2 \nu = Q(\mathrm{div}_\nu Q)\nu$$
$\therefore$ $\underline{Q(\mathrm{div}_\nu Q) = 0}$

- $\nu' = \rho \nu$
$$\mathrm{div}_{\nu'} Q = L_Q \nu' / \nu' = L_Q(\rho\nu)/_{\rho\nu} =$$
$$= (Q(\rho) + \rho\, \mathrm{div}_\nu Q)/\rho = Q(\log|\rho|) + \mathrm{div}_\nu Q$$
$\therefore$ well-defined class $\mathrm{div}\, Q \in H^1(M, Q)$
called the modular class

- $\nu = \rho(x)\, Dx$
$$\mathrm{div}_\nu Q = \mathrm{str}(\partial_a Q^b) + \rho^{-1} Q^a \partial_a \rho =$$
$$= \underline{\underline{\partial_a Q^a}} + Q^a \partial_a \log|\rho|$$

- For DG-supermanifold, $\mathrm{div}\, Q \in H^{1,1}(M, Q)$
DG-manifold, $\mathrm{div}\, Q \in H^1(M, Q)$

- $\Phi : (M, Q) \longrightarrow (M', Q')$, $\Phi^* Q' = Q \Phi^*$
$$\mathrm{div}\, \Phi := \mathrm{div}\, Q - \Phi^* \mathrm{div}\, Q'$$ <u>Relative Modular class</u>

$\mathbb{N}$-graded case

$Q|_{M_0} = 0 \Rightarrow$ linearize Q at $M_0 \Rightarrow$ tangent/cotangent complex

$$\mathbb{T}M = T_{M_0}M : \quad TM_0 = \mathbb{T}_0 \xleftarrow[\text{anchor}]{\delta} \mathbb{T}_{-1} \xleftarrow{\delta} \mathbb{T}_{-2} \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathbb{T}_{-d}$$

$$\mathbb{T}^*M = T^*_{M_0}M : \quad T^*M_0 = \mathbb{T}_0^* \xrightarrow{d} \mathbb{T}_1^* \xrightarrow{d} \mathbb{T}_2^* \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{T}_d^*$$

bundles over M_0

Splitting: isomorphism $TM \simeq \pi^* \mathbb{T}M$

not canonical, requires extra structure

(and obstructed in holomorphic case)

But: crucial observation

$$\mathrm{Ber}\, J = \prod_i \mathrm{Ber}\, J_{ii}$$

i.e. $\mathrm{Ber}\, M \simeq \pi^*(\mathrm{Ber}\, \mathbb{T}_0^* \otimes \cdots \otimes \mathrm{Ber}\, \mathbb{T}_d^*)$ canonically!

In particular, for a dg manifold, (no $*$)

$$\mathrm{Ber}\, M \simeq \pi^*(\Lambda^{top} \mathbb{T}_0^* \otimes \Lambda^{top} \mathbb{T}_1 \otimes \Lambda^{top} \mathbb{T}_2^* \otimes \Lambda^{top} \mathbb{T}_3 \otimes \cdots)$$

$\Rightarrow$ can use sections of $\nearrow$ to define the modular class,

as is usually done by

"classical" methods

2) - Lie algebroids

$\begin{matrix} A \\ \downarrow \\ M_0 \end{matrix}$ v.b. $\Rightarrow$ $M = M_1 = \Pi A \qquad (x^i, \xi^a)$

$$Q = \xi^a A_a^i(x)\partial_i + \frac{1}{2}\xi^b\xi^a A_{ab}^c(x)\partial_c$$

$$\mathbb{T} = (TM_0 \xleftarrow{a} A)$$

$$\mathrm{Ber}(\Pi A) = \pi^*(\Lambda^{top}T^*M_0 \otimes \Lambda^{top}A) = \pi^*(Q_A)$$

(Evens-Lu-Weinstein)

$\nu \in \Gamma(Q_A)$, $X \in \Gamma(A)$

$\theta_\nu(X) = L_X\nu/\nu$, $\theta_\nu = \mathrm{div}_\nu Q$

$$\mathrm{div}\, Q = [\xi^a(\partial_i A_a^i + A_{ab}^b)] \cancel{+ Q(\ldots)}$$

- $M_0 = *$ $\Rightarrow$ $\mathrm{div}\, Q = [\mathrm{tr}(\mathrm{ad})]$

- Poisson manifolds

M_0, $\{-,-\}$, $\{x^i, x^j\} = \pi^{ij}$, $\pi^{(i\ell}\partial_\ell\pi^{jk)} = 0$ (Jacobi)

$M = \Pi T^*M_0$, $(-,-)$ canonical Schouten bracket (antibracket)

$\pi = \frac{1}{2}\pi^{cb}\theta_b\theta_c \in C^\infty(M)$, $(\pi,\pi) = 0$,

$f \in C^\infty(M_0) \Rightarrow X_f = \{f,-\} = (\pi, f)$; $\{f,g\} = ((\pi,f),g)$

$$Q = (\pi,-) = (\pi, x^i)\partial_i + (\pi,\theta_a)\partial^a = \pi^{ib}\theta_b\partial_i + \frac{1}{2}\theta_a\theta_b\partial_c\pi^{ab}\partial^c$$

let $\mu \in \Gamma(\Lambda^{top}T^*M_0)$, define modular vector field

$\Lambda_\mu(f) := \mathrm{div}_\mu X_f$, $\Lambda = [\Lambda_\mu] \in H^1_\pi(M_0)$

$\Lambda = [(\partial_i\pi^{ij})\partial_j]$, $\mathrm{div}\, Q = 2\Lambda$ (take $\nu = \mu^2$)

(Koszul ('85)
Weinstein ('96))

$$TM = (TM_0 \xleftarrow{\pi^\#} T^*M_0)$$

$$\mathrm{Ber}(M) = \pi^*(\Lambda^{top}T^*M_0 \otimes \Lambda^{top}T^*M_0)$$

Courant algebroids

$$\begin{array}{l} E \\ \downarrow \\ M_0 \end{array} \quad \text{v.b.} + \underset{\text{pseudometric}}{\langle e_a, e_b \rangle = g_{ab}} \Rightarrow M \text{ of dim} = (2n|k) \quad (q^i, p_i, \xi^a).$$

<u>even</u> symplectic $\omega = d(p_i dq^i + \xi^a g_{ab} d\xi^b) \Rightarrow \{-,-\}$

$$H = \xi^a A_a^i(q) p_i + \frac{1}{6} \xi^a \xi^b \xi^c A_{cba}(q)$$

$$\{H, H\} = 0$$

$$a(e_a) = A_a^i \partial_i, \quad \langle [e_a, e_b], e_c \rangle = A_{abc}$$

$\langle -,- \rangle$ ad-invariant

$$Q = \xi^a A_a^i \partial_i + \frac{1}{2} \xi^a \xi^b A_{ba}{}^c \partial_c + A^{ci} p_i \partial_c - (\xi^a \partial_i A_a^j p_j + \frac{1}{6} \xi^a \xi^b \xi^c \partial_i A_{cba}) \partial^i$$

$\nu = D(q, \xi, p)$

$$\operatorname{div}_\nu Q = \partial_i Q^i + \partial_c Q^c + \partial^i Q_i = \xi^a (\partial_i A_a^i + A_{ac}{}^c - \partial_i A_a^i) =$$

$$= \xi^a A_{ac}{}^c = 0$$

by ad-invariance of $\langle -,- \rangle$

<u>Simple explanation</u>: for <u>even</u> symplectic M, there exists a volume form invariant under all Hamiltonian flows.

<u>Tangent ex</u>

$$\mathbb{T}M = (\underset{-2}{T^*M_0} \xrightarrow{a^*} \underset{-1}{E} \xrightarrow{a} \underset{0}{TM_0})$$

$$\operatorname{Ber} M = \pi^* (\Lambda^{top} T^* M_0 \otimes \Lambda^{top} E \otimes \Lambda^{top} TM \simeq \Lambda^{top} E)$$

$\mu \in \Gamma(\Lambda^{top} E)$, $X \in \Gamma(E)$

$\theta_\mu(X) = L_X \mu / \mu = 0$ if μ is ad-invariant

(Stiénon–Xu '08: stop short of proving this)

(3) BV-anomaly

Back to an odd Poisson mfld, basic example $P = \Pi T^* M$ (x^i, x^*_i)

ρ density on P, can take $\rho = \sigma^2$, σ density on M.

$$\Delta_\rho f = \tfrac{1}{2}(-1)^{\tilde f}\, \mathrm{div}_\rho X_f \quad , \quad X_f = (f, -)$$

(Khudaverdian-Schwarz) Now a second-order operator!

(BV-Laplacian) $\quad \Delta = \dfrac{\partial^2}{\partial x^i \partial x^*_i}$

Quantum master action

$S \in C^\infty(P)^0[[\hbar]]$ satisfying Quantum Master equation!

$$\Delta e^{\frac{i}{\hbar}S} = 0 \qquad \left(\Leftrightarrow (S,S) = 2i\hbar\, \Delta S\right)$$

Seek solution perturbatively in $\hbar$

$$S = S_0 + \hbar S_1 + \cdots$$

$(S_0, S_0) = 0$ classical master equation

set $Q = (S_0, -)$ classical BRST

$QS_1 = (S_0, S_1) = i\, \Delta S_0$

$i\Delta S_0 \propto \mathrm{div}_\rho X_{S_0} = \mathrm{div}_\rho Q$

$\therefore$ Solution exists up to 1st order in $\hbar \Leftrightarrow \mathrm{div}\, Q = 0$!

if $\mathrm{div}\, Q \neq 0$ – "One-loop BV anomaly"

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