Non-Abelian Poincaré lemma and Lie algebroids

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A familiar statement

Let $\omega \in \Omega^1(\mathrm{M}, \mathfrak{g})$ be a Lie-algebra-valued 1-form satisfying

$$d\omega + \omega^2 = 0\,. \tag{1}$$

(Here $\omega^2 = \frac{1}{2}[\omega, \omega]$.) Then, if M is simply-connected, ω is a "pure gauge", i.e., there is a function $\mathrm{g} \in \mathrm{C}^\infty(\mathrm{M}, \mathrm{G})$ with values in a Lie group G such that

$$\omega = -d\mathrm{g}\,\mathrm{g}^{-1}\,. \tag{2}$$

Geometric interpretation: $\nabla = d + \omega$ is a connection (for which ω is a connection 1-form); equation (1) is the zero curvature condition; then equation (2) means that

$$d + \omega \,\sim\, d$$

("trivial connection"), i.e., $d + \omega = \mathrm{g} \circ d \circ \mathrm{g}^{-1}$.

Comparison with the Abelian case

If $\mathfrak{g}$ is Abelian (e.g., $\mathfrak{g} = \mathbb{R}$), the Maurer–Cartan equation $\mathrm{d}\omega + \omega^2 = 0$ becomes just $\mathrm{d}\omega = 0$; also, $-\mathrm{d}\mathrm{g}\,\mathrm{g}^{-1} = -\mathrm{d}\ln \mathrm{g}$, so the above statement is nothing but the special case of the Poincaré lemma for 1-forms: every closed 1-form on a simply-connected manifold is exact:

$$\mathrm{d}\omega = 0 \quad \Rightarrow \quad \omega = \mathrm{d}f \quad (\text{for a 1-form } \omega \in \Omega^1(M))\,.$$

Recall that, in general, the Poincaré lemma says that on a contractible manifold, every closed form is exact plus constant:

$$\mathrm{d}\omega = 0 \quad \Rightarrow \quad \omega = \mathrm{d}\sigma + C \quad (\text{for an inhomogeneous form } \omega \in \Omega(M))\,.$$

A "Non-Abelian Poincaré lemma"

What is a non-Abelian analog of the Poincaré lemma for arbitrary forms?

Setup: let M be a (super)manifold and $\mathfrak{g}$ be a Lie superalgebra with a corresponding Lie supergroup G. Let $\omega \in \Omega^{\mathrm{odd}}(M, \mathfrak{g})$ be an odd $\mathfrak{g}$-valued form on M (odd — w.r.t. total parity) satisfying the Maurer–Cartan equation

$$d\omega + \omega^2 = 0\,.$$

Theorem ("Non-Abelian Poincaré lemma")

If M is contractible, then there is an even G-valued form $g \in \Omega^{\mathrm{even}}(M, G)$ such that

$$\omega = gCg^{-1} - dg\, g^{-1}\,,$$

for some odd constant $C \in \mathfrak{g}$.

Remarks to the Non-Abelian Poincaré lemma

Remarks:

- It is not essential that M is a supermanifold, but it is essential that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra.
- For an ordinary manifold M, the form ω may be inhomogeneous, $\omega = \omega_0 + \omega_1 + \omega_2 + \dots$, so that $\omega_{2k+1} \in \Omega^{2k+1}$ take values in $\mathfrak{g}_{\bar{0}}$ while $\omega_{2k} \in \Omega^{2k}$ take values in $\mathfrak{g}_{\bar{1}}$.
- In general, ω is an odd pseudodifferential form, i.e., $\omega \in C^{\infty}(\Pi TM, \Pi\mathfrak{g})$.
- An (even) G-valued form on M is by definition a map $g \in C^{\infty}(\Pi TM, G)$. On ordinary M, $g = g_0 + \dots$, where $g_0 \in C^{\infty}(M, G_0)$.
- An odd constant $C \in \mathfrak{g}_{\bar{1}}$ looks innocent (or unimportant), but it is exactly C that is essential for applications.

Plan of the talk

Multiplicative push-forward

Let M be a supermanifold. Consider $\omega \in \Omega^{\mathrm{odd}}(M \times I, \mathfrak{g})$ where $\mathfrak{g}$ is a Lie superalgebra. (Here $I = [0,1]$.) Consider the projection $p\colon M \times I \to M$. We may write $\omega = \omega_0 + dt\,\omega_1$. We define

$$\mathrm{Texp}\, p_*\colon \Omega^{\mathrm{odd}}(M \times I, \mathfrak{g}) \to \Omega^{\mathrm{even}}(M, G)$$

by the formula

$$\mathrm{Texp}\, p_*\colon \omega \mapsto g = g(1,0) = \mathrm{Texp}\int_0^1 (-\omega)\,.$$

In general, $g(t_1,t_0) = \mathrm{Texp}\int_{t_0}^{t_1}(-\omega)$ is the multiplicative integral along the fibers, which can be defined as the solution at time $t = t_1$ of the differential equation $\frac{dg}{dt} = -\omega_1\, g$ such that $g = 1$ at $t = t_0$.

Multiplicative 'fiberwise Stokes formula'

Let $\Delta g := -dg\, g^{-1}$ be the Darboux derivative (the ordered logarithmic derivative). Denote by $\mathrm{curv}\colon \ \omega \mapsto d\omega + \omega^2$ the 'curvature operator' on odd $\mathfrak{g}$-valued forms.

Theorem

The following commutation formula holds:

$$\Delta \circ \mathrm{Texp}\, p_* + p_* \circ \mathrm{Ad}\, g(1,t) \circ \mathrm{curv} = p'_*$$

where $p'_*\colon \ \omega \mapsto \omega_0|_{t=1} - \mathrm{Ad}\, g\, \omega_0|_{t=0}$ is the (twisted) integral over the fiberwise boundary.

(It is an analog of the formula $d \circ p_* \pm p_* \circ d = p'_*$ for the Abelian case and arbitrary fiber bundles: $\Delta\, \mathrm{Texp} \sim d$ and $\mathrm{curv} \sim d$.)

... and more explicitly

If $g = \mathrm{Texp}\, p_* \omega = g(1,0)$ and $g(t_1, t_0) = \mathrm{Texp}\!\int_{t_0}^{t_1}(-\omega)$ as above, and we denote $\Omega := \mathrm{curv}\, \omega = d\omega + \omega^2$, then the commutation formula reads

$$\boxed{-dg\, g^{-1} + \int_0^1 g(1,t)\, \Omega\, g(1,t)^{-1} = \omega_0|_{t=1} - g\big(\omega_0|_{t=0}\big)g^{-1}\,.}$$

Geometric interpretation: $d + \omega$ is like Quillen's superconnection, and $g(t_1, t_0)$ is the corresponding 'parallel transport' (along the fibers of the projection $M \times I \to M$). (Helps to reconstruct the formula.)

Non-Abelian chain homotopy

Consider a homotopy $F\colon M\times I\to N$, so that $F(x,t)=f_t(x)$.

Theorem

For an arbitrary odd $\mathfrak{g}$-valued form $\omega\in\Omega^{\mathrm{odd}}(N,\mathfrak{g})$, the pull-backs along homotopic maps f_0 and f_1 are related by a (non-linear) formula

$$f_1^*\omega - g\, f_0^*\omega\, g^{-1} = -dg\, g^{-1} + \int_0^1 g(1,t)F^*(d\omega+\omega^2)g(1,t)^{-1},$$

where $g = g(1,0) = \mathrm{Texp}\int_0^1(-F^*\omega)$ and $g(1,t) = \mathrm{Texp}\int_t^1(-F^*\omega)$.

For a proof, apply the commutation formula above to $F^*\omega$.

Flat case

Corollary

If a form $\omega \in \Omega^{\mathrm{odd}}(N, \mathfrak{g})$ satisfies the Maurer–Cartan (or zero curvature) equation:

$$d\omega + \omega^2 = 0\,,$$

then its pull-backs along homotopic maps are gauge-equivalent:

$$f_1^*\omega - g\, f_0^*\omega\, g^{-1} = -dg\, g^{-1}\,,$$

for some $g \in \Omega^{\mathrm{even}}(M, G)$.

Contractible manifolds

On a contractible supermanifold M choose a contraction to a point $x_0 \in M$, i.e., a homotopy h_t between $h_1 = \mathrm{id}$ and $h_0 = i \circ p$ (for the projection $p\colon M \to \{x_0\}$ and inclusion $i\colon \{x_0\} \to M$), e.g., the radial contraction on $\mathbb{R}^{n|m}$. Then if $\omega \in \Omega^{\mathrm{odd}}(M, \mathfrak{g})$ satisfies

$$d\omega + \omega^2 = 0\,,$$

then:

Theorem ("Non-Abelian Poincaré lemma")

$$\omega = gCg^{-1} - dg\, g^{-1}$$

where $C \in \mathfrak{g}_{\bar{1}}$ is the value of ω at x_0, for some $g \in \Omega^{\mathrm{even}}(M, G)$.

So the gauge-equivalence classes of the Maurer–Cartan elements of $\Omega^{\mathrm{odd}}(M, \mathfrak{g})$ correspond uniquely to the $\mathrm{Ad}\, G$-orbits of odd elements of $\mathfrak{g}$.

Remarks

(1) Abstract version: one may replace $\Omega(M, \mathfrak{g})$ by an abstract differential Lie superalgebra (possibly, even an L_∞-algebra).
(2) Parallel results in the literature:

- Schlessinger and Stasheff (the secret work 'Deformation theory and rational homotopy type', unpublished manuscript, 1984 –...) have a statement called the Main Homotopy Theorem: 'homotopic Maurer–Cartan elements are (gauge-)equivalent'.
- Chuang and Lazarev (2008): a proof of S&S's Main Homotopy Theorem for pro-nilpotent differential graded Lie algebras.

(Corresponds to the abstract version of the zero curvature case above.)

(3) Graded version: Suppose there is an extra $\mathbb{Z}$-grading on M and $\mathfrak{g}$ ('weight'), not related with parity. If ω is homogeneous of weight $+1$, then g is homogeneous of weight 0. (Will be used later.)

Lie algebroids: recollection

A Lie algebroid over M is a vector bundle $E \to M$ with a Lie algebra structure on the space of sections $C^\infty(M, E)$ and a bundle map $a\colon E \to TM$ (called the anchor) satisfying

$$[u, fv] = a(u)f\, v + (-1)^{\tilde{u}\tilde{f}} f[u, v]$$

($u \in C^\infty(M, E)$ and $f \in C^\infty(M)$). It implies that on sections $a([u, v]) = [a(u), a(v)]$.

Examples: a Lie (super)algebra $\mathfrak{g}$ where $M = \{*\}$; the tangent bundle $TM \to M$; an integrable distribution $D \subset TM$; the 'action algebroid' $M \times \mathfrak{g}$ of an action of $\mathfrak{g}$ on M; the 'Atiyah algebroid' of a principle bundle (see more below).

Practical description

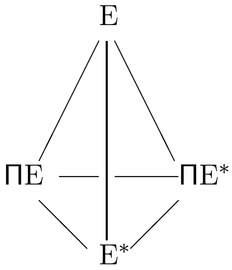
Relative a local frame e_i in E and local coordinates x^a on M, the structure of a Lie algebroid on $E \to M$ is specified by ‘structure functions’ a_i^a and b_{ij}^k:

$$a(e_i) = a_i^a(x)\partial_a\,, \quad \text{and} \quad [e_i, e_j] = b_{ij}^k(x)e_k\,.$$

The structure functions $a_i^a(x)$ and $b_{ij}^k(x)$ satisfy equations expressing together the Jacobi identity for the Lie bracket and the Leibniz formula for the anchor.

Three manifestations

Like a Lie algebra, a Lie algebroid has three other equivalent manifestations in terms of the 'neighbors' of the vector bundle E:



- Poisson bracket on E^* (generalization of the Berezin–Kirillov bracket)
- Schouten bracket on ΠE^*
- Homological vector field on ΠE

Description via Q-manifolds... in coordinates...

A Q-manifold is a supermanifold endowed with an odd vector field $Q \in \mathfrak{X}\,M$ such that $Q^2 = 0$ (a homological vector field).
A Lie algebroid structure on E is equivalent to a Q-manifold structure on ΠE with $w(Q) = +1$. (Weight w of objects on a vector bundle is defined as degree w.r.t. linear coordinates in the fibers. More on graded manifolds later.)
In local coordinates on ΠE,

$$Q = \xi^i Q_i^a(x)\,\frac{\partial}{\partial x^a} + \frac{1}{2}\,\xi^i \xi^j Q_{ji}^k(x)\,\frac{\partial}{\partial \xi^k}\,.$$

where x^a are coordinates on the base and ξ^i coordinates on the fibers. Up to signs, we have $Q_i^a = a_i^a$ and $Q_{ij}^k = b_{ij}^k$.

... and more intrinsically

The anchor and the Lie bracket for E are expressed by

$$a(u)f := \big[[Q, i_u)], f\big]$$

and

$$i_{[u,v]} := (-1)^{\tilde{u}}\big[[Q, i_u], i_v\big].$$

Here the map $i\colon\ C^\infty(M,E) \to \mathfrak{X}(\Pi E)$ is $i_u = (-1)^{\tilde{u}} u^i(x) \frac{\partial}{\partial \xi^i}$. Properties of the bracket and anchor on sections of E are encoded in the identity $Q^2 = 0$ on ΠE.

Transitive Lie algebroids

Recall that a Lie algebroid $E \to M$ is transitive if its anchor $a\colon\ E \to TM$ is onto. Then there is an exact sequence of vector bundles over M,

$$L \rightarrowtail E \twoheadrightarrow TM\,,$$

and $E \to TM$ is itself a fiber bundle (an affine bundle).
Example: the Atiyah algebroid of a principle bundle $P \to M$, defined as $E = TP/G$. A section of the anchor $E \to TM$ is exactly a G-connection on $P \to M$. Here L is the associated 'adjoint' Lie algebra bundle.
For this reason, sections of the anchor for arbitrary transitive Lie algebroids are called (Lie algebroid) connections. The curvature is defined as the $\gamma[X,Y]-[\gamma(X,\gamma(Y))]$ for a connection $\gamma\colon\ TM \to E$.

Strong local triviality

Theorem (K. Mackenzie)

A transitive Lie algebroid always possesses local flat connections $TU \to E$ for $U \subset M$.

(Proof is not easy; it uses cohomology of Lie algebroids.) This theorem is fundamental. It allows to describe transitive Lie algebroids locally as Lie algebroid products $TU \times \mathfrak{g}$ for a fixed Lie algebra $\mathfrak{g}$ ("strong local triviality"), and hence reduce their description to classification of certain "local transition data". In particular, this is used for defining Mackenzie's cohomological obstruction class and further integration theory for transitive Lie algebroids.

Application of the non-Abelian Poincaré lemma...

Reversing parity in the fibers, for a transitive Lie algebroid E, we obtain an affine bundle $\Pi E \to \Pi TM$. In suitable coordinates the algebroid structure is defined by a homological vector field Q on ΠE of the form

$$Q = d + \frac{1}{2}\Big(\xi^i\xi^j Q^k_{ji}(x) + 2\xi^i dx^a Q^k_{ai}(x) + dx^a dx^b Q^k_{ba}(x)\Big)\frac{\partial}{\partial\xi^k}$$

where $d = dx^a\partial/\partial x^a$. Or: $Q = d + \omega_0 + \omega_1 + \omega_2$, with forms taking values in the superalgebra of vector fields on the fiber. Note that Q is homogeneous w.r.t. total weight. Changes of coordinates have the form $\xi^i = \xi^{i'}T^i_{i'}(x) + dx^a T^i_a(x)$ (affine over ΠTM).

... continued

Corollary (From the non-Abelian Poincaré lemma)

There is an affine change of coordinates in each chart such that in new coordinates,

$$Q = d + C \quad \text{where} \quad C = \frac{1}{2}\eta^i\eta^j c_{ji}^k \frac{\partial}{\partial \eta^k}$$

with constant coefficients c_{ij}^k.

This recovers a Lie algebra bundle structure on the kernel of the anchor and local flat connections, i.e., the full "strong local triviality" statement due to Kirill Mackenzie!

Remark: C is exactly the constant term in the statement of the non-Abelian Poincaré lemma.

Graded manifolds...

(Warning about change of notation: what was ΠE before, in the sequel will be just E.)

A graded manifold E is a supermanifold with a privileged class of atlases where the coordinates are assigned weights in $\mathbb{Z}$, so that the coordinate transformations are polynomial in coordinates with nonzero weights and homogeneous with respect to total weight. It is also assumed that the coordinates with nonzero weights run over the whole $\mathbb{R}$ (no restriction on range).

Remark: No relation between weight and parity (in general).

Examples: vector bundles; multiple vector bundles.

Non-negatively graded manifolds

Suppose all coordinates have weights $\geqslant 0$. Such a graded manifold is called non-negatively graded. Non-negatively graded manifolds have nice structure.
For a non-negatively graded manifold E, there is a tower of fibrations,

$$E = E_N \to E_{N-1} \to \ldots E_2 \to E_1 \to E_0 = M\,,$$

where $E_1 \to E_0$ is a vector bundle and each $E_{k+1} \to E_k$ for $k \geqslant 1$ is an affine bundle. Altogether it gives a fiber bundle $E \to M$ with polynomial transition functions (of a special form). So this is a non-linear generalization of vector bundles. The supermanifold M is called the base of E.

Graded Q-manifolds...

Suppose E is a Q-manifold (no grading assumed). Then the odd vector field $Q \in \mathfrak{X}(E)$ tautologically gives a map $Q\colon\; E \to \Pi TM$. If E is non-negatively graded with base M, we can consider the composition $a := Tp^{\Pi} \circ Q\colon\; E \to \Pi TM$. We call it the anchor of a graded Q-manifold E. In coordinates,

$$a^*(x^a) = x^a\,, \quad \text{and} \quad a^*(dx^a) = Q^a(x,\varphi)$$

if

$$Q = Q^a(x,\varphi)\frac{\partial}{\partial x^a} + Q^{\mu}(x,\varphi)\frac{\partial}{\partial \varphi^{\mu}}\,.$$

The anchor is a bundle map over M. If the field Q is homogeneous and we require that the anchor preserves weights, then Q has to be of weight +1.

... and 'non-linear Lie algebroids'

Definition

We call a non-negatively graded manifold $E \to M$ with a homological vector field Q of weight $+1$, a non-linear Lie algebroid over M. The bundle map $a\colon\ E \to \Pi TM$ defined above is called its anchor.

In coordinates,

$$Q = Q^a(x,\varphi)\frac{\partial}{\partial x^a} + Q^\mu(x,\varphi)\frac{\partial}{\partial \varphi^\mu}\,.$$

where $w(Q^a(x,\varphi)) = 1$ and $w(Q^\mu(x,\varphi)) = w(\varphi^\mu) + 1\,.$ The anchor is given by $a^*(dx^a) = Q^a(x,\varphi)\,.$

Remark

$Q^2 = 0 \quad \Longrightarrow \quad a\colon\ E \to \Pi TM$ is a Q-manifold map

The transitive case

The anchor is a bundle map $E \to \Pi TM$ (over M). Mimicking the ordinary ("linear") Lie algebroids we arrive at

Definition

A non-linear Lie algebroid $E \to M$ such that its anchor $a\colon E \to \Pi TM$ is a surjective submersion is called transitive.

A transitive non-linear Lie algebroid can be regarded as a fiber bundle $E \to \Pi TM$ with the projection a. Note that it respects the fields Q and d. The fiber over $x \in M$ splits as $\Pi T_x M \times F$ for some graded manifold F, and in adapted coordinate systems the vector field Q on E takes the form

$$Q = d + Q^i(x, dx, y)\frac{\partial}{\partial y^i}\,,$$

where $d = dx^a \frac{\partial}{\partial x^a}$ and y^i are coordinates on F.

Application of the non-Abelian Poincaré lemma again

Look again at the field Q on E. It has exactly the form

$$Q = d + \omega\,,$$

where

$$\omega = Q^i(x, dx, y)\frac{\partial}{\partial y^i} \in \Omega^{\mathrm{odd}}(M, \mathfrak{X}(F))\,,$$

is a (locally defined) form on M taking values in vector fields on F, odd w.r.t. total parity and of total weight 1.

Corollary (From the non-Abelian Poincaré lemma)

There is a graded change of fiber coordinates $y^i = y^i(x, dx, z)$ such that in new coordinates Q splits:

$$Q = d + Q^i(z)\frac{\partial}{\partial z^i}\,.$$

Interpretation

In 2007, Kotov and Strobl introduced ‘Q-bundles’ as fiber bundles of graded Q-manifolds $E \to B$ satisfying a strong ‘local triviality condition’: locally

$$Q_E = Q_B + Q_F$$

where F is the standard fiber. Of particular interest is the case when $B = \Pi TM$ for some M. As it follows from the above results, one does not have to assume this strong local triviality from the start. Indeed, ‘Q-bundles’ in that sense over ΠTM are nothing but transitive non-linear Lie algebroids described in a special gauge. (Call it “Kotov–Strobl gauge”.) It is parallel to Mackenzie's construction of “local data” for ordinary transitive Lie algebroids. The possibility to choose such a gauge follows from our non-Abelian Poincaré lemma, but one does not have to do that from the start.

Summary

- We introduced and proved a non-Abelian Poincaré lemma.
- Its application to transitive Lie algebroids recovers Mackenzie's result about their "strong" local triviality (which leads to his "local data" description).
- We gave an outline of "non-linear Lie algebroids" as particular graded Q-manifolds.
- For transitive non-linear Lie algebroids, the non-Abelian Poincaré lemma allows to introduce a special gauge (similar to Mackenzie's strong local triviality). This gives a link with the Q-bundles as recently defined by Kotov and Strobl.

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