Calculus of Variations and Elliptic Equations

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2nd class

Convex functionals and functions

Definition - Given a vector space X and a function $F = X \to \mathbb{R} \cup \{+\infty\}$ we say that it is convex if for every $x_0, x_1 \in X$ and every $t \in [0,1]$ we have

$$F((1-t)x_0 + tx_1) \leq (1-t)F(x_0) + tF(x_1).$$

F is said to be strictly convex if the above inequality is strict whenever $x_0 \neq x_1$, $F(x_0), F(x_1) < +\infty$ and $t \in (0,1)$.

We observe for future use a sufficient condition for convexity: if a functional F is such that for every x there exists a linear function $A : X \to \mathbb{R}$ satisfying $F(x+h) \geq F(x) + A(h)$ for every $h \in X$, then F is convex. To see this, we apply it to $x = ((1-t)x_0 + tx_1$ and $h = (1-t)(x_1 - x_0)$, thus obtaining $F(x_1) = F(x+h) \geq F(x) + (1-t)A(x_1 - x_0)$. We then apply the same condition to the same point x but $h = -t(x_1 - x_0)$, thus obtaining $F(x_0) \geq F(x) - tA(x_1 - x_0)$. We multiply the first inequality times t, the second times $(1-t)$, and we sum up, and obtain the desired convexity inequality.

If X is a functional space of functions defined over a domain Ω and F has the integral form $F(u) = \int_\Omega L(x, u(x), \nabla u(x))dx$ then the convexityh of the function $(x, u, v) \mapsto L(x, u, v)$ in the variables (u, v) for a.e. x is sufficient for the convexity of F, but it is not necessary. Consider fonr instance the quadratic form $F(u) = \int_0^T (|u'(t)|^2 - |u(t)|^2)dt$ for $u \in H_0^1([0,T])$. We can easily compute $F(u+h) = F(u) + 2\int_0^T (u'(t) \cdot h'(t) - u(t) \cdot h(t))dt + F(h)$. The second term in the last expression is of the form $L(h)$, thus we just have to see wheterh the last one is non-negative. This is the case, thanks to Poincaré inequality, for small T, and more precisely for $T \leq \pi$. Hence, this functional F is convex despite the integrand being clearly non-convex in u.

Convex functions of one variable satisfy an above-the-tangent inquality that we can write as $j(s) \geq j(0) + sj'(0)$ as soon as j is convex and differentiable at 0. If we apply this to $j(\varepsilon) = F(u + \varepsilon\phi)$ we easily see that the Euler-Lagrange equation, which corresponds to $j'(0) = 0$ for all ϕ, is indeed a sufficient conditions for optimality, and not only necessary. This requires a minor clarification: when writing necessary considitons, we oftn consider a class of smooth perturbations, for instance $\phi \in C_c^\infty(\Omega)$ and impose $j'(0) = 0$ only for those ϕ. This means that we have

$$\int_\Omega (\partial_u L(x, u, \nabla u)\phi + \nabla_v L(x, u, \nabla u) \cdot \nabla\phi)\, dx = 0 \tag{1}$$

for every $\phi \in C_c^\infty(\Omega)$. This is, by definition, what we need to write the Euler-Lagrange equation in the sense of PDE: $\nabla \cdot (\nabla_v L) = \partial_u L$. Yet, if we know information on the summability of $\partial_u L(x, u, \nabla u)$ and $\nabla_v L(x, u, \nabla u)$ we can usually obtain, by density, that if (1) holds for $\phi \in C_c^\infty$ then it also holds for any ϕ. More precisely, if $X = W^{1,p}(\Omega)$ and $\partial_u L(x, u, \nabla u), \nabla_v L(x, u, \nabla u) \in L^{p'}$ (p' being the dual exponent of p, i.e. $p' = p/(p-1)$) then (1) for every $\phi \in C_c^\infty$ implies the validity of the same equality for every $\phi \in W_0^{1,p}$, which is enough to consider the minimization of F in X with prescribed boundary conditions.

Strict convexity and uniqueness

An easy and well-known statement is the following.

Proposition 1. *If $Z \subset X$ is a convex set and $F : X \to \mathbb{R} \cup \{+\infty\}$ is strictly convex and not identically $+\infty$ on Z, then the problem* $\min\{F(x) : x \in Z\}$ *admits at most one minimizer.*

Proof. If two minimizers $x_0 \neq x_1$ existed, then $x = (x_0 + x_1)/2$ would provide a strictly better value for F. ∎

In the boave result, of course strict convexity of F on Z would be enough.

We list some sufficient conditions for strict convexity, in the case $F(u) = \int_\Omega L(x, u(x), \nabla u(x))dx$.

- if for a.e. x the function L is strictly convex in (u, v), then F is strictly convex;
- if for a.e. x the function L is convex in (u, v) and strictly convex in u, then F is strictly convex;
- if for a.e. x the function L is convex in (u, v) and strictly convex in v, then the only possibility for having equality in $F((1-t)u_0 + tu_1) \leq (1-t)F(u_0) + tF(u_1)$ is $\nabla u_0 = \nabla u_1$. If the domain ω is connected, this implies $u_0 - u_1 = const$. Then, if the constraints are such that the minimization is performed over a set which does not contain two functions whose difference is a constant, we have strict convexity of F on the set Z on which we minimize. This happens for instance if the boundary values are fixed, or if the average is fixed.

Convexity and lower-semicontinuity

We first recall few facts abut lower semicontinuity.

Proposition 2. *A function $F : X \to \mathbb{R} \cup \{+\infty\}$ defined on a metric space X is l.s.c. if and only if for every α the set $\{F \leq \alpha\}$ is closed.*

Proof. First suppose that F is lsc. We want to prove that $\{F \leq \alpha\}$ is closed, so we take a sequence $x_n \in \{F \leq \alpha\}$ and we suppose $x_n \to x$. since F is lsc we also have $F(x) \leq \liminf F(x_n) \leq \alpha$, so $x \in \{F \leq \alpha\}$, which proves that the set is closed.

We now suppose that for every α the set $\{F \leq \alpha\}$ is closed and we want to prove that F is lsc. We take a sequence $x_n \to x$ and we want to prove $F(x) \leq \liminf F(x_n)$. If $\liminf F(x_n) = +\infty$ there is nothing to prove, otherwise we take a number $M > \liminf F(x_n)$. By definition of liminf there is a subsequence x_{n_k} with $F(x_{n_k}) \leq M$. Since $x_{n_k} \to x$ and $\{F \leq M\}$ is closed we obtain $F(x) \leq M$. By the arbitrariness of M we obtain $F(x) \leq \liminf F(x_n)$. □

The above characterization of lsc functions can be very useful if combined with a well-known fact in functional analysis, i.e. the fact that closed convex sets in a Banach space are also weakly closed. Tha above proposition is stated for metric spaces, while the weak convergence is in general not metrizable, but it is metrizable on bounded sets of reflexive and separable Banach spaces, which is enough since weakly converging sequences are always bounded. We then have

Corollary 1. *On a reflexive and separable Banach space any functional which is both convex and strongly lsc is also wekly lsc.*

Proposition 3. *Consider the functional $F : L^p(\Omega; \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ given by $F(v) = \int L(v(x))dx$ for a continuous function $L : \mathbb{R}^d \to \mathbb{R}$, $L \geq 0$. Then, F is lsc for the weak L^p convergence if and only if L is convex.*

Proof. In order to that convexity of L is necessary, we will build a sequence $v_n \rightharpoonup v$ on which we will evaluate F. If we take a cube inside Ω and choose $v_n = 0$ outside this cube for every n, we can reduce to the case where Ω is a cube Q. We then fix two vectors $a, b \in \mathbb{R}^d$ and a number $t \in [0, 1]$ and define a sequence v_n in the following way: we divide the cube into n parallel stripes of equal width, and each stripe is further divided into two parallel sub-stripes, one of width $1 - t$ times the width of the original stripe, and on this sub-stripe v_n takes the value a, and one on width t times the width of the original stripe, and on this sub-stripe v_n takes the value b. We easily see that in this case we have $v_n \rightharpoonup v$ where v is the constant vector function equal to $(1-t)a + tb$, and that we have $F(v_n) = (1-t)|Q|L(a) + t|Q|L(b)$, while $F(v) = |Q|L((1-t)a + tb)$. The semicontinuty of F then implies the convexity of L.

For the converse implication, we first observe that F is l.s.c. for the strong convergence in L^p. Indeed, if $v_n \to v$, we can assume that v_n converges a.e. to v, and then Fatou's lemma with the continuity of L

implies $\int L(v) \le \liminf \int L(v_n)$. If L is also convex then F is a convex functional, and convexity together with strong lower-semicontinuity implies the weak lower-semicontinuity (using Corollary 1). $\square$

A note on the Scorza-Dragoni theorem

We provide here a simpler proof of the Scorza-Dragoni theorem (Lemma 4.6 in [?]), based on Lusin's theorem.

A well-known theorem in measure theory states that every measurable function f on a reasonable measure space (X, μ), is actually continuous on a set K with $\mu(X \setminus K)$ small. This set K can be taken compact. Actually, there can be at least two statements: either we want f to be merely continuous on K, or we want f to coincide on K with a continuous function defined on X. This theorem is usually stated for real-valued functions, but we happen to need it for functions valued in more general spaces. Let us be more precise: take a topological space X endowed with a finite regular measure μ (i.e. any Borel set $A \subset X$ satisfies $\mu(A) = \sup\{\mu(K) : K \subset A,\, K \text{ compact}\} = \inf\{\mu(B) : B \supset A,\, B \text{ open}\}$). The arrival space Y will be supposed to be second-countable (i.e. it admits a countable family $(B_i)_i$ of open sets such that any other open set $B \subset Y$ may be expressed as a union of B_i; for instance, separable metric spaces are second-countable).

Theorem 1 (weak Lusin)**.** *Under the above assumptions on X, Y, μ, if $f : X \to Y$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and the restriction of f to K is continuous.*

Proof. For every $i \in \mathbb{N}$, set $A_i^+ = f^{-1}(B_i)$ and $A_i^- = f^{-1}(B_i^c)$. Consider compact sets $K_i^\pm \subset A_i^\pm$ such that $\mu(A_i^\pm \setminus K_i^\pm) < \varepsilon 2^{-i}$. Set $K_i = K_i^+ \cup K_i^-$ and $K = \bigcap_i K_i$. For each i we have $\mu(X \setminus K_i) < \varepsilon 2^{1-i}$. By construction, K is compact and $\mu(X \setminus K) < 4\varepsilon$. To prove that f is continuous on K it is sufficient to check that $f^{-1}(B) \cap K$ is relatively open in K for each open set B, and it is enough to check this for $B = B_i$. Equivalently, it is enough to prove that $f^{-1}(B_i^c) \cap K$ is closed, and this is true since it coincides with $K_i^- \cap K$. $\square$

Theorem 2 (strong Lusin)**.** *Under the same assumptions on X, if $f : X \to \mathbb{R}$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ and a continuous function $g : X \to \mathbb{R}$ such that $\mu(X \setminus K) < \varepsilon$ and $f = g$ on K.*

Proof. First apply weak Lusin's theorem, since $\mathbb{R}$ is second countable. Then we just need to extend $f_{|K}$ to a continuous function g on the whole X. This is possible since $f_{|K}$ is uniformly continuous (as a continuous function on a compact set) and hence has a modulus of continuity ω: $|f(x) - f(x')| \le \omega(d(x, x'))$ (the function ω can be taken sub additive and continuous). Then define $g(x) = \inf\{f(x') + \omega(d(x, x')) : x' \in K\}$. It can be easily checked that g is continuous and coincides with f on K. $\square$

Note that this last proof strongly uses the fact that the arrival space is $\mathbb{R}$. It could be adapted to the case of $\mathbb{R}^d$ just by extending component-wise. On the other hand, it is clear that the strong version of Lusin's Theorem cannot hold for any space Y: just take X connected and Y disconnected. A measurable function $f : X \to Y$ taking values in two different connected components on two sets of positive measure cannot be approximated by continuous functions in the sense of the strong Lusin's Theorem.

Lemma 1. *Let S be a compact metric space, and $f : \Omega \times S \to \mathbb{R}$ be a measurable function, such that $f(x, \cdot)$ is continuous for a.e. x. Then for every $\varepsilon > 0$ there exists a compact subset $K \subset \Omega$ with $|\Omega \setminus K| < \varepsilon$ and f continuous on $K \times S$.*

Proof. The continuity of a function of two variables, when the second lives in a compact space S, is equivalent to the continuity of the map $x \mapsto f(x, \cdot)$ seen as a map from Ω to the Banach space $C(S)$. We can apply the weak version of Lusin's theorem to this map and easily obtain the claim. $\square$

Theorem 3 (Scorza-Dragoni)**.** *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function, such that $f(x, \cdot)$ is continuous for a.e. x. Then for every $\varepsilon > 0$ there exists a compact subset $K \subset \Omega$ with $|\Omega \setminus K| < \varepsilon$ and f continuous on $K \times \mathbb{R}^d$.*

Proof. The statement is more difficult to prove because of the non-compactness of the second space. We then fix a number $R \in \mathbb{N}$ and consider the restriction of f to $\omega \times \overline{B_R}$, where $\overline{B_R}$ is the closed ball of radius R. We consider $\varepsilon_R = \varepsilon 2^{-R-1}$ and apply the previous lemma, thus obtaining a set K_R such that f is continuous on $K_R \times \overline{B_R}$ and $|\Omega \setminus K_R| < \varepsilon_R$. Set $K = \bigcap_R K_R$. We then have $|\Omega \setminus K| < \varepsilon$ and f is continuous on $K \times \overline{B_R}$ for very R. Since is a local property, we deduce that f is continuous on $K \times \mathbb{R}^d$. □

References

[1] E. Giusti *Direct Methods in the Calculus of Variations* World Scientific, 2003