Calculus of Variations and Elliptic Equations

–

3rd class

Fenchel-Legendre Transform Let us fix a Banach space X together with its dual X', and denote by $\xi \cdot x$ the duality between an element $\xi \in X'$ and $x \in X$. We say that a function valued in $\mathbb{R} \cup \{+\infty\}$ is proper if it is not identically equal to $+\infty$

Definition - Given a vector space X and its dual X', and a proper function $F = X \to \mathbb{R} \cup \{+\infty\}$ we define its Fenchel-Legendre transform $F^* : X' \to \mathbb{R} \cup \{+\infty\}$ via

$$F^*(\xi) := \sup_x \xi \cdot x - F(x).$$

We note that F^*, as a sup of affine continuous (in the sequel we will just say affine and mean affine and continuous, i.e. of the form $\ell(x) = \xi \cdot x + c$ for $\xi \in X'$ and $c \in \mathbb{R}$) functions, is both convex and l.s.c., as these two notions are stable by sup.

By abuse of notations, when considering functions defined on X' we will see their Fenchel-Legendre transform as a function defined on X (and not on X'': this is possible since $X \subset X''$ and we can restrict it to X, and by the way in most cases we will use only reflexive spaces.

We prove the following restuls.

Proposition 1. *1. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then there exists an affine function ℓ such that $F \geq \ell$.*

2. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then it is a sup of affine functions.

3. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then there exists $G : X' \to \mathbb{R} \cup \{+\infty\}$ such that $F = G^*$.

4. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and l.s.c. then $F^{**} = F$.

Proof. We consider the epigraph $Epi(F) := \{(x,t) \in X \times \mathbb{R} \,:\, t \geq F(x)\}$ of F which is a convex and closed set in $X \times \mathbb{R}$. We take a point x_0 such that $F(x_0) < +\infty$ and consider the singleton $\{(x_0, F(x_0) - 1)\}$ which is a convex and compact set in $X \times \mathbb{R}$. The Hahn-Banach separation theorem provides the existence of a pair $(\xi, a) \in X' \times \mathbb{R}$ such that $\xi \cdot x_0 + a(F(x_0) - 1) < 0$ and $-xi \cdot x + at \geq 0$ for every $(x,t) \in Epi(F)$. Note that this last condition implies $a \geq 0$ since we can take $t \to \infty$. Moreover, we should also have $a > 0$ otherwise taking any point $(x,t)) \in Epi(F)$ with $x = x_0$ we have a contradiction. If we then take $t = F(x)$ for all x such that $F(x) < +\infty$ we obtain $aF(x) \geq -\xi \cdot x + \xi \cdot x_0 + a(F(x_0) - 1)$ and, dividing by $a > 0$, we obtain the first claim.

We now take an arbitrary $x_0 \in X$ and $t_0 < F(x_0)$ and separate again the singleton $\{(x_0, t_0)\}$ from $Epi(F)$, thus getting a pair $(\xi, a) \in X' \times \mathbb{R}$ such that $\xi \cdot x_0 + at_0 < 0$ and $-xi \cdot x + at \geq 0$ for every $(x,t) \in Epi(F)$. Again, we have $a \geq 0$. If $F(x_0) < +\infty$ we obtain as before $a > 0$ and the inequality $F(x) > -\frac{\xi}{a} \cdot (x - x_0) + t_0$. We then have an affine function $\ell$with $F \geq \ell$ and $\ell(x_0) = x_0$. This shows that the sup of all affine functions smaller than F is, at the point x_0, at least t_0. Hence this sup equals F on $\{F < +\infty\}$. The same argument works for $F(x_0) = +\infty$ if for t_0 arbitrary large the corresponding coefficient a is strictly positive. If not, we have $\xi \cdot x_0 < 0$ and $\xi \cdot x \geq 0$ for every x such that $(x,t) \in Epi(F)$ for at least one $t \in \mathbb{R}$, i.e. for $x \in \{F < +\infty\}$. Consider now $\ell_n = \ell - n\xi$ where ℓ is the affine function smaller than F previously found. We have $F \geq \ell \geq \ell - n\xi$ since ξ is non-negative on $\{F < +\infty\}$ and moreover $\lim_n \ell_n(x_0) = +\infty$. This shows that in such a point x_0 the sup of the affine functions smaller than F equals $+\infty$, and hence $F(x_0)$.

Once that we know that F is a sup of affine functions we can write

$$F(x) = \sup_\alpha \xi_\alpha \cdot x + c_\alpha$$

for a family of indexes α. We then set $c(\xi) := \sup\{c_\alpha : \xi_\alpha = \xi\}$. The set in the sup can be empty, which means $c(\xi) = -\infty$. Anyway, the sup is always finite: fix a point x_0 with $F(x_0) < +\infty$ and use since $c_\alpha \leq F(x_0) - \xi \cdot (x_0)$. We then define $G = -c$ and we see $F = G^*$.

Finally, before proving $F = F^*{*}$ we prove that for any function F we have $F \geq F^{**}$ even if F is not convex or lsc. Indeed, we have $F^(\xi) + F(x) \geq \xi \cdot x$ which allows to write $F(x) \geq \xi \cdot x - F^*(\xi)$, an inequality true for every ξ. Taking the sup over ξ we obtain $F \geq F^{**}$. We want now to prove that this inequality is an equality if F is convex and lsc. We write $F = G^*$ and transform this into $F^* = G^{**}$. We then have $F^* \leq G$ and, transforming this inequality (which changes its sign), $F^{**} \geq G^* = F$, which proves $F^{**} = F$. □

Corollary 1. *Given an arbitrary proper function $F : X \to \mathbb{R} \cup \{+\infty\}$ we have $F^{**} = \sup\{G : G \leq F,\ G$ is convex and lsc$\}$.*

Proof. Let us call H the function obtained as a sup on the right hand side. Since F^{**} is convex and lsc and smaller than F, we have $F^{**} \leq H$. Note that H, as a sup of convex and lsc functions, is also convex and lsc, and it is of course smaller than F. We write $F \geq H$ and double transform this inequality, which preserves the signe. We then have $F^{**} \geq H^{**} = H$, and the claim is proven. □