Calculus of Variations and Elliptic Equations
–
8th class

Equations in divergence form with right-hand-side

We consider a point-dependent matrix $a(x)$ (for $x \in \Omega \subset \mathbb{R}^d$) satisfying $\lambda I \leq a \leq \Lambda I$ for two contstants $\Lambda > \lambda > 0$ together with a vector field $F : \Omega \to \mathbb{R}^d$ and the equation $\nabla \cdot (a\nabla u) = \nabla \cdot F$. A function $u \in H_1(\Omega)$ is a solution if for every $\phi \in H^1_0(\Omega)$ we have $\int a\nabla u \cdot \nabla\phi = \int F \cdot \nabla\phi$.

We also call subsolution of the same equation any function $u \in H_1(\Omega)$ such that $\int a\nabla u \cdot \nabla\phi \leq \int F \cdot \nabla\phi$ for every $\phi \in H^1_0(\Omega)$ with $\phi \geq 0$.

We have the following

Proposition 1. *Suppose that $u \in H^1(\Omega)$ is a positive subsolution of weak solution of the equation $\nabla \cdot (a\nabla u) = \nabla \cdot F$ where F is a bounded vector field. Then, for every $x_0 \in \Omega$ with $B(x_0, 2R) \subset \Omega$ we have*

$$||u||_{L^\infty(B(x_0,R))} \leq C \left(\fint_{B(x_0,2R)} |u|^2 \right)^{1/2} + CR||F||_{L^\infty}$$

for a constant $C = C(\lambda, \Lambda, d)$.

Proof. We try to mimick Moser's proof for the case $F = 0$ (see [1]). Since we cannot say that convex functions of subsolutions are subsolutions of the very same equation we will directly use the good test functions, i.e. $\phi = u^{2m-1}\eta^2$ where η is a cut-off fonction with $\eta = 1$ on $B(x_0, r_1)$ and $\eta = 0$ outside $B(x_0, r_2)$ for $r_1 < r_2$. We then obtain

$$(2m-1)\int a\nabla u \cdot \nabla u u^{2m-2}\eta^2 = (2m-1)\int F \cdot \nabla u u^{2m-2}\eta^2 + 2\int a\nabla u \cdot \nabla\eta u^{2m-1}\eta + 2\int F \cdot \nabla\eta u^{2m-1}\eta.$$

Applying suitable young inequalities we do have

$$\int a\nabla u \cdot \nabla\eta u^{2m-1}\eta \leq \frac{1}{4}\int a\nabla u \cdot \nabla u u^{2m-2}\eta^2 + \int a\nabla\eta\nabla\eta u^2$$

as well as

$$\int F \cdot \nabla u u^{2m-2}\eta^2 \leq \int |F|^2 u^{2m-2}\eta^2 + \frac{1}{4}\int |\nabla u|^2 u^{2m-2}\eta^2.$$

and

$$\int F \cdot \nabla\eta u^{2m-1}\eta \leq \frac{1}{2}\int |F|^2 u^{2m-2}\eta^2 + \frac{1}{2}\int |\nabla\eta|^2 u^{2m}.$$

This allows to obtain an inequality of the form

$$\int |\nabla(u^m)|^2\eta^2 = m^2\int u^{2m-2}|\nabla u|^2\eta^2 \leq C(m)\int |F|^2 u^{2m-2}\eta^2 + C(m)\int |\nabla\eta|^2 u^{2m},$$

where $C(m)$ is a constant polynomially depending on m. We then choose η so that $|\nabla\eta| \leq C/(r_2 - r_1)$ and obtain

$$\int_{B(x_0,r_1)} |\nabla(u^m)|^2 \leq \frac{C(m)}{(r_2 - r_1)^2}\int_{B(x_0,r_2)} (r_2 - r_1)^2|F|^2 u^{2m-2} + u^{2m}.$$

The iterations will be the same as in Moser's method, choosing $r_1 = R_{k+1}$ and $r_2 = R_k$ and $R_k := R(1 + 2^{-k})$. This allows to write

$$\int_{B(x_0,R_k)} |\nabla(u^m)|^2 \leq \frac{C(m)4^k}{R^2}\int_{B(x_0,R_k)} R^2|F|^2 u^{2m-2} + u^{2m};$$

note that this inequality is far from being sharp, since we estimated the coefficient $(r_2 - r_1)^2$ in front of $|F|^2$ by R^2, while we could have written $4^{-k}R^2$.

The idea now is to use the L^2 norm of the gradient on the left hand side to estimate an L^p norm of u^m for $p > 2$, but we have to handle the term involving F. The easiest solution is to consider $v := \max\{|u|, R||F||_{L^\infty}\}$, a function which satisfies $|\nabla(v^m)| \leq |\nabla(u^m)|$ and $R|F|, |u| \leq v$. We then have

$$\int_{B(x_0,R_k)} |\nabla(v^m)|^2 \leq \frac{C(m)4^k}{R^2} \int_{B(x_0,R_k)} v^{2m}.$$

We then apply standard Moser iterations, for which we use a sequence m_k of exponents, with $m_k = \beta^k$, $\beta \in (1, \frac{2^*}{2})$, so that the term $C(m)4^k$ can be replaced by another exponential term c^k for a constant $c > 1$ depending on β. We then obtain

$$||v||_{L^\infty(B(x_0,R)} \leq C \left(\fint_{B(x_0,2R)} |v|^2 \right)^{1/2} \leq C \left(\fint_{B(x_0,2R)} |u|^2 \right)^{1/2} + R||F||_{L^\infty},$$

from which the claim follows. □

References

[1] J. Moser A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Communications on Pure and Applied Mathematics* 13.3 (1960): 457-468.