Calculus of Variations – Homework

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Try to do it in 3h max ; all kind of paper documents (notes, books...) are authorized.

Exercice 1 (7 points)**.** Consider the problem

$$\min\left\{\int_0^L e^{-t}\left(u'(t)^2+5u(t)^2\right)dt \quad : \quad u\in C^1([0,L]),\ u(0)=1\right\}.$$

Prove that it admits a minimizer, that it is unique, find it, compute the value of the minimum, and the limit of the minimizer (in which sense ?) and of the minimal value as $L\to+\infty$.

Exercice 2 (7 points)**.** Consider the functional $F:H^1([0,L])\to\mathbb{R}$ defined through

$$F(u)=\int_0^L\left(u'(t)^2+\arctan(u(t)-t)\right)dt.$$

Prove that

1. the problem $(P):=\min\{F(u)\ :\ u\in H^1([0,L])\}$ has no solution ;
2. the problem $(P_a):=\min\{F(u)\ :\ u\in H^1([0,L]),u(0)=a\}$ admits a solution for every $a\in\mathbb{R}$;
3. we have $F(-|u|)\le F(u)$;
4. the solution of (P_a) is unique as soon as $a\le 0$;
5. there exists $L_0<+\infty$ such that for every $L\le L_0$ the solution of (P_a) is unique for every $a\in\mathbb{R}$
6. the minimizers of (P) and (P_a) are C^∞ functions.

Exercice 3 (7 points)**.** Given a bounded, smooth and connected domain $\Omega\subset\mathbb{R}^d$, and $f\in L^2(\Omega)$, set $X(\Omega)=\{v\in L^2(\Omega;\mathbb{R}^d)\ :\ \nabla\cdot v\in L^2(\Omega)\}$ and consider the minimization problems

$$\begin{aligned}(P) &:= \min\left\{F(u):=\int_\Omega\left(\frac12|\nabla u|^2+\frac12|u|^2+f(x)u\right)dx\ :\ u\in H^1(\Omega)\right\}\\ (D) &:= \min\left\{G(v):=\int_\Omega\left(\frac12|v|^2+\frac12|\nabla\cdot v-f|^2\right)dx\ :\ v\in X(\Omega)\right\},\end{aligned}$$

1. Prove that (P) admits a unique solution ;
2. Prove $\min(P)+\inf(D)\ge 0$;
3. Prove that there exist $v\in X(\Omega)$ and $u\in H^1(\Omega)$ such that $F(u)+G(v)=0$;
4. Deduce that $\min(D)$ is attained and $\min(P)+\inf(D)=0$;
5. Justify by a formal inf-sup exchange the duality $\min F(u)=\sup -G(v)$;
6. Prove that the solution of (D) belongs indeed to $H^1(\Omega;\mathbb{R}^d)$.

Exercice 4 (5 points)**.** Given a bounded, smooth and connected domain $\Omega\subset\mathbb{R}^3$, an exponent p with $2<p<6$, and a function $f\in L^r(\Omega)$ consider the minimization problem

$$\min\left\{\int_\Omega\left(\sqrt{1+|\nabla u|^p}+\frac{|\nabla u|}{1+u^2}+f(x)u\right)dx\ :\ u\in W^{1,q}_0(\Omega)\right\}.$$

Prove that (P) admits a solution for $q\le p/2$ and $r\ge 3p/(4p-6)$ and write the Euler-Lagrange equation satisfied by the minimizer. Prove that this solution also satisfies

$$\int_\Omega\left(\frac p2\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^p}}+\frac{(1-u^2)|\nabla u|}{(1+u^2)^2}+f(x)u\right)dx=0.$$

Exercice 5 (4 points)**.** Let Ω be the $d-$dimensional flat torus (just to avoid boundary conditions, think at a cube), λ a given real number, and $u\in H^1(\Omega)$ a solution of

$$\Delta u=\sqrt{1+|\nabla u|^2}+\lambda u.$$

1. Prove that u is necessarily a C^∞ function.
2. Prove that there is no such a solution for $\lambda=0$.
3. Prove that, if $\lambda>0$, the only solution of this equation is $u=-\lambda^{-1}$.