Calculus of Variations – Homework

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Try to do it in 3h max ; all kind of paper documents (notes, books. . .) are authorized.

Exercice 1 (6 points)**.** Given $L > 0$, consider the problem

$$\min \left\{ \int_0^L \left(u'(t)^2 + 2e^t u(t) + 4u(t)^2 \right) dt \quad : \quad u \in C^1([0, L]),\ u(0) = 1 \right\}.$$

Prove that it admits a minimizer, that it is unique, find it, and compute the value of the minimum as a function of L.

Exercice 2 (6 points)**.** Given a bounded, smooth and connected domain $\Omega \subset \mathbb{R}^d$, consider the minimization problem

$$\min \left\{ \int_\Omega (1 + e^{-u^2})(1 + e^{|\nabla u|^2}) dx \ : \ u \in H_0^1(\Omega) \right\}.$$

Prove that a minimizer exists, and that it is a continuous function. Also prove that, if $\lambda_1(\Omega) < 1$, the function $u = 0$ cannot be a minimizer. In this same case, prove that the minimizer is not unique.

Exercice 3 (5 points)**.** Let Ω denote the d-dimensional flat torus, and $f \in (H^1(\Omega))'$. Consider the minimization problem

$$\min \left\{ G(v) := \int_\Omega \left(\frac{1}{2}|v|^2 + |v_1| \right) dx \ : \ v \in L^2(\Omega; \mathbb{R}^d),\ \nabla \cdot v = f \right\},$$

where v_1 stands for the first component of v.

Find the dual problem to the above one and prove that the optimal vector field v is H^1 if $f \in H^1$.

Exercice 4 (7 points)**.** Given an exponent $\beta \in]0, 1[$ and $m > 0$,, consider

$$E(m) := \inf \left\{ \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + u^\beta \right) dx \ : \ u \in H_0^1(\mathbb{R}^d), u \geq 0, \int_{\mathbb{R}^d} u(x)dx = m \right\}.$$

1. Prove that we have $E(m) > 0$ for every $m > 0$.
2. Prove that the value $E(m)$ satisfies $E(m) = m^\alpha E(1)$ for an exponent α to be found.
3. Prove that the infimum in $E(m)$ can be restricted to radially decreasing functions.
4. Prove that the infimum in $E(m)$ is attained.
5. Prove that the minimizers in $E(m)$ are compactly supported.

Exercice 5 (6 points)**.** Let $\Omega_n \subset D$ be a sequence of open domains contained in a large ball B, converging to a domain Ω in the following sense : the indicator functions I_{Ω_n} converge to I_Ω in $L^1(B)$, and the complement $\bar{B} \setminus \Omega_n$ converge to $\bar{D} \setminus \Omega$ in the Hausforff sense. Also suppose $d(\partial B, \Omega) > 0$.

Given $f \in L^2(B)$ and $\omega \subset B$, consider the functional $F_\omega : H_0^1(B) \to \mathbb{R}$ given by

$$F_\omega(u) = \begin{cases} \frac{1}{2}\int_\omega |\nabla u|^2 + \int_\omega fu & \text{if } u = 0 \text{ a.e. on } \bar{B} \setminus \omega, \\ +\infty & \text{if not.} \end{cases}$$

Prove that F_{Ω_n} Γ-converges to F_Ω in $L^2(B)$. Deduce the limit of the solutions of the equation

$$\begin{cases} \Delta u = f & \text{in } \Omega_n, \\ u = 0 & \text{on} \partial\Omega_n. \end{cases}$$