ENS Lyon and Université Claude Bernard Lyon 1, M2A
November 2nd, 2020

Calculus of Variations and Elliptic PDEs
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Mid-Term Examination
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All kind of documents (notes, books...) are authorized, but you cannot collaborate with anyone else. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

Exercice 1 (7 points)**.** Find the minimal value of the following variational problem

$$\min\left\{\int_0^1 e^t\left(u'(t)^2+2u(t)^2+4u(t)\right)dt \quad : \quad u\in C^1([0,1]),\ u(0)=0\right\}.$$

Solution :
The integrand $L(t,u,v):=e^t(|v|^2+2|u|^2+4u)$ is convex in u and v, hence any solution to the E-L equation with the suitable boundary condition is a minimizer. Since we have $\partial_v L=2e^tv, \partial_u L=4e^t(u+1)$ the equation to be solved is

$$\begin{cases}(2e^tu'(t))'=4e^t(u(t)+1),\\ u(0)=0,\\ 2e^1u'(1)=0.\end{cases}$$

the first equation can be re-written as $2u''+2u'=4u+4$, i.e. $u''+u'-2u-2=0$. The function $u+1$ must be of the form Ae^t+Be^{-2t}. The boundary conditions impose $A+B=1$ and $Ae-2Be^{-2}=0$, hence $A=\frac{2}{e^3+2}, B=\frac{e^3}{e^3+2}$. The solution of the minimization problem is hence

$$u(t)=\frac{2}{e^3+2}e^t+\frac{e^3}{e^3+2}e^{-2t}-1.$$

We can re-write the quantity

$$\int_0^1 e^t|u'(t)|^2dt=-\int_0^1\left(e^tu'(t)\right)'u(t)dt,$$

thanks to an integration by parts and without boundary terms because of the boundary conditions. Using the equation we get

$$\int_0^1 e^t\left(u'(t)^2+2u(t)^2+4u(t)\right)dt=\int_0^1\left(-4e^t(u(t)+1)u(t)+2u(t)^2+4u(t)\right)dt=-2\int_0^1 e^tu(t)^2dt.$$

Then we must compute the integral, which I won't do :-).

Exercice 2 (6 points)**.** Let Ω be a bounded, open, and connected subset of $\mathbb{R}^d$ and $f:\Omega\to\mathbb{R}$ a given L^1 function. Prove that the following minimization problem admits a solution

$$\min\left\{\int_\Omega\sqrt{(1+|u|)(1+|v|^2)}+\sqrt{1+|\nabla u|^3}+\sqrt{1+|v|^6}\ :\ u\in W^{1,1}(\Omega), v\in L^1(\Omega), uv\geq f\right\}.$$

Solution :
Take a minimizing sequence (u_n, v_n). We see that v_n is bounded in L^3 and $|\nabla u_n|$ in $L^{3/2}$. Moreover, $\int \sqrt{1+|u_n|}$ is also bounded. Set $m_n = \frac{1}{|\Omega|}\int u_n$ the average of u_n. Since the function $s \mapsto \sqrt{1+|s|}$ is Lipschitz continuous, we also have a bound from above on $\int \sqrt{1+|m_n|} - C\int |u_n - m_n|$. Thanks to a Poincaré-Wirtinger inequality, $||u_n - m_n||_{L^1} \le C||u_n - m_n||_{L^{3/2}} \le C||\nabla u_n||_{L^{3/2}}$, hence this term is bounded. We obtain a bound on $\int \sqrt{1+|m_n|} = |\Omega|\sqrt{1+|m_n|}$, hence on m_n. Then, u_n is bounded in $W^{1,3/2}$ (before we only had a bound on the gradient, now we obtain a bound on the full norm in the Sobolev space). We can then extract a subsequence where v_n weakly converges in L^3 to v and u_n weakly converges in $W^{1,3/2}$ to u (and hence strongly in $L^{3/2}$).

The integrand in convex in v and ∇u (the terms which converge weakly) and continuous in u (which converges strongly). Then, the functional is lower semicontinuous for this convergence. The constraint $uv \ge f$ can be written as "for every $\varphi \ge 0$ we have $\int uv\varphi \ge \int f\varphi$ and passes to the limit since we have $u_n \to u$ strongly in $L^{3/2}$ and $v_n \rightharpoonup v$ weakly in L^3 and these two spaces are in duality. The pair (u, v) is a minimizer.

Exercice 3 (6 points)**.** Let $\Omega = \mathbb{T}^d$ be the d-dimensional torus and $f \in H^1(\Omega)$ a given function with zero mean. Consider the function $H : \mathbb{R}^d \to \mathbb{R}$ given by $H(z) := \sqrt{1+|z|^4}$ and the variational problem

$$\min\left\{\int_\Omega H^*(v)\ :\ v \in L^2(\Omega),\ \nabla\cdot v = f\right\}.$$

Prove that it admits a unique minimizer v_0 and that we have $v_0 \in H^1(\Omega)$.

Solution :
We can compute the gradient and the Hessian of H:

$$\nabla H(z) = \frac{2|z|^2 z}{H(z)},\ D^2H(z) = \frac{4z\otimes z}{H(z)} + \frac{2|z|^2 I}{H(z)} - \frac{2|z|^2 z}{H^2(z)} \otimes \nabla H(z) = \frac{4z\otimes z + 2|z|^2 I}{H(z)} - \frac{4|z|^4 z\otimes z}{H^3(z)}.$$

Using $H(z) \ge |z|^2$ we see that D^2H is bounded, hence H is $C^{1,1}$. We can also see that it is convex, since the Hessian cn be re-written as

$$D^2H(z) = z\otimes z\frac{4H^2 - 4|z|^4}{H^3(z)} + 2\frac{|z|^2 I}{H(z)},$$

which is positive definite since, again, $H(z) \ge |z|^2$.

Hence, H^* is an elliptic function (i.e. D^2H^* is bounded from below by a positive constant times the identity matrix I). In particula it is strictly convex, which proves the uniqueness of the minimizer. Since H^* is elliptic, we also have $H^*(v) \ge c|v|^2 - C$, which allows to obtain an L^2 bound on any minimizing sequence v_n. If we extract a weakly converging subsequence, the limit will be a minimizer since it satisfies the constraint (which passes to the limit for weak convergence, tested against H^1 test functions) and the functional is l.s.c. because of the convexity of H^*.

For the regularity, we use the ellipticity of H^* to write

$$H(w) + H(v) \ge w\cdot v + c|v - \nabla H(w)|^2.$$

The duality theory we developed, using $f \in H^1$, hence provides $v = \nabla H(\nabla u) \in H^1$ (where u is the solution of the dual problem to this one).

Exercice 4 (9 points)**.** Let $\Omega = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$ a given L^∞ function. Consider the following minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|\nabla u|^2 + u(u^3 + f)\ :\ u \in H^1(\Omega)\right\}.$$

1. Prove that it admits a unique minimizer u_0.
2. In dimension $d = 1, 2, 3$, prove that we have $u_0 \in W^{2,p}(\Omega)$ for every $p < \infty$.

3. In higher dimension, prove $u_0 \in W^{2,\frac{4}{3}}$.
4. Supposing $d \leq 3$ and $f \in W^{1,\infty}(\Omega)$, also prove $u_0 \in W^{3,p}(\Omega)$ for every $p < \infty$.
5. In higher dimension and still supposing $f \in W^{1,\infty}(\Omega)$, prove $u_0 \in W^{3,r}(\Omega)$ for $r = 4d/(3d-4)$.

Note that the required regularity results in this exercise are meant to be global in Ω and not only local.

Solution :

1. The integrand is strictly convex in u and in ∇u, which provides uniqueness of a possible minimizer. Calling J the functional to be minimized, we have
$$J(u) \geq c||\nabla u||_{L^2}^2 + ||u||_{L^4}^4 - c||u||_{L^4},$$
which provides at the same time a bound on $||\nabla u_n||_{L^2}$ and $||u_n||_{L^4}$ for any minimizing sequence u_n. The sequence u_n is then bounded in H^1 and we can extract a weakly converging subsequence. The functional is obviously l.s.c. for this convergence.

2. The minimizer u_0 solves the Euler-Lagrange equation $\Delta u_0 = 4u_0^3 + f$ with Neumann boundary conditions. By reflection (and without changing the signs of the functions u_0 or f, we can assume that the same equation is solved on the the whole space, and that we only look at the restriction to the cube. In dimension $d = 1, 2$ we have $u_0 \in H^1$ whch implies that it belongs to any L^p space. The same is true for $4u_0^3 + f$ so the Calderon-Zygmund regularity result provides $u_0 \in W^{2,p}$. The case of dimension 3 is trickier. In this case we have $u_0 \in L^6$, if we use the Sobolev injection (we also know $u_0 \in L^4$ but L^6 is better). We know $4u_0^3 + f \in L^2$ since $u_0 \in L^6$ and we deduce $u_\in W^{2,2} \subset W^{1,6} \subset L^\infty$. Hence, the right-hand side of the equation is in all the L^p spaces, and we deduce again $u_0 \in W^{2,p}$ for every p.

3. In higher dimension, it is better to use $u_0 \in L^4$ than the Sobolev injection (the result is the same for $d = 4$). This provides $4u_0^3 + f \in L^{4/3}$ and $u_0 \in W^{2,4/3}$. Unofrtunately, the Sobolev injection does not improve the summability of u_0 and we should stop here.

4. Supposing $d \leq 3$ we now know $u_0 \in W^{2,p}$ for every p. We can differentiate the equation, and write the equation for $v = \partial_i u$. The equation is $\Delta v = 12u_0^2 v + \partial_i f$. It is satisfied on the whole space because of the reflection trick, and luckily the reflection does not destroy the property $f \in W^{1,\infty}$. By assumption, $\partial_i f$ is L^∞. Moreover, $u_0 \in W^{2,p}$ implies u_0 bounded and $v \in W^{1,p}$, so that v is also bounded (using $p > d$). Hence the right hand side is bounded and we obtain $v \in W^{2,p}$ for every p, i.e. $u_0 \in W^{3,p}(\Omega)$ for every $p < \infty$.

5. In higher dimension the strategy is the same, and the only missing point is to prove $u_0^2 v \in L^r$. This is unfortunately quite difficult, since we do have $v \in L^r$ (using $u \in W^{2,4/3}$ which gives $v \in W^{1,4/3} \subset L^r$), but we do not have yet $u_0 \in L^\infty$. Even worse, even in the simplest case $d = 4$, we have $u_0 \in L^4$ and $r = 2$, i.e. the term $u_0^2 v$ is only in L^1 and we cannot apply any elliptic regulairty theory. We should definitely obtain $u_0 \in L^\infty$ in another way. This is possible (and could actually allow to improve the previous results for $d \geq 4$) using a truncation argument. Let M be a constant such that $4M^3 + |f(x)| > 0$ for every x (M exists since $f \in L^\infty$).

   We can see that $u_M := \min\{u, M\}$ is a better competitor than u, i.e. $J(u_M) \leq J(u)$. Indeed, the truncation reduces the gradient part of J, and we have
$$(u(x)^4 + u(x)f(x)) - (u_M(x)^4 + u_M(x)f(x)) = \alpha(x)(u(x) - u_M(x))$$
   where $\alpha(x)$ is the derivative of $s \mapsto s^4 + sf(x)$ computed at a certain value $s \in (u_M(x), u(x))$, in particular, unless $u_M(x) = u(x)$, we have $s \geq M$ and $\alpha > 0$, which proves that the integral decreases when passing from u to u_M. By uniqueness of the minimizer, this shows $u_0 \leq M$. A similar argument with $\max\{u, -M\}$ also shows $u_0 \geq -M$ and $u_0 \in L^\infty$.

   Actually, this argument shows $u_0 \in W^{2,p}$ for every p and every d, and (with the assumption $f \in W^{1,\infty}$), $u_0 \in W^{3,p}$ for every p and every d.