ENS Lyon and Université Claude Bernard Lyon 1, M2A
November 2nd, 2020

Calculus of Variations and Elliptic PDEs

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Mid-Term Examination

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All kind of documents (notes, books...) are authorized, but you cannot collaborate with anyone else. The total number of points is much larger than 20, which means that attacking two or three exercises could be a reasonable option.

Exercice 1 (7 points)**.** Find the minimal value of the following variational problem

$$\min\left\{\int_0^1 e^t\left(u'(t)^2+2u(t)^2+4u(t)\right)dt \quad : \quad u\in C^1([0,1]),\ u(0)=0\right\}.$$

Exercice 2 (6 points)**.** Let Ω be a bounded, open, and connected subset of $\mathbb{R}^d$ and $f:\Omega\to\mathbb{R}$ a given L^1 function. Prove that the following minimization problem admits a solution

$$\min\left\{\int_\Omega \sqrt{(1+|u|)(1+|v|^2)}+\sqrt{1+|\nabla u|^3}+\sqrt{1+|v|^6}\ :\ u\in W^{1,1}(\Omega), v\in L^1(\Omega), uv\geq f\right\}.$$

Exercice 3 (6 points)**.** Let $\Omega=\mathbb{T}^d$ be the d-dimensional torus and $f\in H^1(\Omega)$ a given function with zero mean. Consider the function $H:\mathbb{R}^d\to\mathbb{R}$ given by $H(z):=\sqrt{1+|z|^4}$ and the variational problem

$$\min\left\{\int_\Omega H^*(v)\ :\ v\in L^2(\Omega),\ \nabla\cdot v=f\right\}.$$

Prove that it admits a unique minimizer v_0 and that we have $v_0\in H^1(\Omega)$.

Exercice 4 (9 points)**.** Let $\Omega=[0,1]^d$ be the unit cube in $\mathbb{R}^d$ and $f:\Omega\to\mathbb{R}$ a given L^∞ function. Consider the following minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|\nabla u|^2+u(u^3+f)\ :\ u\in H^1(\Omega)\right\}.$$

1. Prove that it admits a unique minimizer u_0.
2. In dimension $d=1,2,3$, prove that we have $u_0\in W^{2,p}(\Omega)$ for every $p<\infty$.
3. In higher dimension, prove $u_0\in W^{2,\frac{4}{3}}$.
4. Supposing $d\leq 3$ and $f\in W^{1,\infty}(\Omega)$, also prove $u_0\in W^{3,p}(\Omega)$ for every $p<\infty$.
5. In higher dimension and still supposing $f\in W^{1,\infty}(\Omega)$, prove $u_0\in W^{3,r}(\Omega)$ for $r=4d/(3d-4)$.

Note that the required regularity results in this exercise are meant to be global in Ω and not only local.